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Asymptotic estimates of the solution for a singularly perturbed Cauchy problem

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The article focuses on the initial problem for a third-order linear integro-differential equation with a small parameter at the higher derivatives, assuming that the roots of the additional characteristic equation have opposite signs. This paper presents a fundamental set of solutions and initial functions for a singularly perturbed homogeneous differential equation. The solution to the singularly perturbed initial integro-differential problem employs analytical formulas. A theorem concerning asymptotic estimates of the solution is established.

Keywords: singularly perturbed integro-differential equation, asymptotic estimates, Cauchy functions, fundamental solutions, small parameter.

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Introduction

Vasil'eva A.B. and Butuzov V.F. introduced the theory of singularly perturbed equations in their work [1]. Kassymov K.A. [2] investigated the most common cases of the Cauchy problem for singularly perturbed nonlinear systems of ordinary differential and integro-differential equations, as well as partial differential equations of hyperbolic type. Subsequently, singularly perturbed initial and boundary value problems with initial jumps were studied in [3,4]. Mirzakulova A.E. [5] extensively examines boundary value problems, particularly when the roots of the additional characteristic equation have opposing signs. While linear integro-differential equations are presented in [6], numerous papers have been dedicated to singularly perturbed integro-differential equations [7–12]. This article also provides an asymptotic solution for a singularly perturbed differential equation in a boundary value problem where the roots of the characteristic equation are opposite [13]. In addition, in recent years, significant work has been done on the numerical solution of integro-differential problems [14–17].

In this paper, we consider the initial problem for third-order linear integral-differential equations with a small parameter, where the roots of the corresponding characteristic equation have opposite signs. It is well known that there is no solution to a third-order linear differential equation with a small parameter (where the roots of the corresponding characteristic equation have opposite signs). However, we demonstrate that adding an integral term to the right-hand side yields an asymptotic formula for the solution. This article presents the findings of this research, which include an analytical formula and asymptotic estimates for solving a singularly perturbed integral-differential equation with a small parameter and initial conditions. Furthermore, a theorem on asymptotic estimation of these equations' solutions is proven.

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1 Statement of the problem

Consider the singularly perturbed integro-differential equation with a small parameter

$$L_\varepsilon y \equiv \varepsilon^2 y''' + \varepsilon A_2(t) y'' + A_1(t) y' + A_0(t) y = F(t) + \int\limits_0^1 \sum_{i=0}^{2} H_i(t,x) y^{(i)}(x,\varepsilon)\, dx, \tag{1}$$

in the initial conditions

$$y^{(i)}(0,\varepsilon) = \alpha_i, \quad i = \overline{0,2}. \tag{2}$$

ε is a small parameter, while α_i are known constants.

Assume the following conditions hold:

I. Functions $A_i(t),\ i = \overline{0,2}$ and $F(t)$ are continuously differentiable on a segment $0 \le t \le 1$, and functions $H_i(t,x),\ i = \overline{0,2}$ are continuously differentiable on a domain $D = \{0 \le t \le 1,\ 0 \le x \le 1\}$.

II. Roots of "additional characteristic equation" $\mu^2 + A_2(t)\mu + A_1(t) = 0$ satisfy the following inequalities $\mu_1(t) \ne \mu_2(t)$ and $\mu_1(t) < -\gamma_1 < 0$, $\mu_2(t) > \gamma_2 > 0$.

To construct the solution, we first identify auxiliary functions. This article [13] serves as a reference. From the third formula to the seventh, we obtained the required results.

We look for the solution to equation (1) with conditions (2) in the form

$$y(t,\varepsilon) = \sum_{i=1}^{3} C_i y_i(t,\varepsilon) + \frac{1}{\varepsilon^2}\int\limits_0^t K_0(t,s,\varepsilon) z(s,\varepsilon) ds - \frac{1}{\varepsilon^2}\int\limits_t^1 K_1(t,s,\varepsilon) z(s,\varepsilon) ds; \tag{3}$$

here the fundamental solutions are $y_i(t,\varepsilon),\ i = \overline{1,3}$. $K_0(t,s,\varepsilon),\ K_1(t,s,\varepsilon)$ are auxiliary functions expressed by article [13], $C_i,\ i = \overline{1,3}$ are unknown constants.

We now denote the right side of equation (1) as follows

$$z(t,\varepsilon) = F(t) + \int\limits_0^1 \sum_{i=0}^{2} H_i(t,x) y^{(i)}(x,\varepsilon) dx.$$

Instead of $y(x,\varepsilon)$, we apply formula (3) to obtain the formula for $z(t,\varepsilon)$

$$z(t,\varepsilon) = f(t,\varepsilon) + \frac{1}{\varepsilon^2}\int\limits_0^1 \left[\int\limits_s^1 \Big(\sum_{i=0}^{2} H_i(t,x) K_0^{(i)}(x,s,\varepsilon) dx\right] z(s,\varepsilon) ds -$$

$$-\frac{1}{\varepsilon^2}\int\limits_0^1 \left[\int\limits_0^s \Big(\sum_{i=0}^{2} H_i(t,x) K_1^{(i)}(x,s,\varepsilon) dx\right] z(s,\varepsilon) ds,$$

here

$$f(t,\varepsilon) = F(t) + C_1 \int\limits_0^1 \Big(\sum_{i=0}^{2} H_i(t,x) y_1(x,\varepsilon)\Big) dx + C_2 \int\limits_0^1 \Big(\sum_{i=0}^{2} H_i(t,x) y_2(x,\varepsilon)\Big) dx +$$

$$+ C_3 \int\limits_0^1 \Big(\sum_{i=0}^{2} H_i(t,x) y_3(x,\varepsilon)\Big) dx = F(t) + C_1\phi_1(t,\varepsilon) + C_2\phi_2(t,\varepsilon) + C_3\phi_3(t,\varepsilon),$$

$$\phi_i(t,\varepsilon) = \int_0^1 \Big(H_0(t,x) y_i(x,\varepsilon) + H_1(t,x) y_i'(x,\varepsilon) + H_2(t,x) y_i''(x,\varepsilon) \Big) dx, \quad i = \overline{1,3}. \tag{4}$$

Now, we provide the following notation

$$H(t,s,\varepsilon) = \frac{1}{\varepsilon^2} \int_s^1 \Big(\sum_{i=0}^{2} H_i(t,x) K_0^{(i)}(x,s,\varepsilon) \Big) dx - \frac{1}{\varepsilon^2} \int_0^s \Big(\sum_{i=0}^{2} H_i(t,x) K_1^{(i)}(x,s,\varepsilon) \Big) dx.$$

III. The kernel $H(t,s,\varepsilon)$ does not have an eigenvalue with the value 1.

We get the Fredholm equation of the second type

$$z(t,\varepsilon) = f(t,\varepsilon) + \int_0^1 H(t,s,\varepsilon) z(s,\varepsilon) ds. \tag{5}$$

If condition III is satisfied, the solution of integral equation (5) is the only one and it is written

$$z(t,\varepsilon) = f(t,\varepsilon) + \int_0^1 R(t,s,\varepsilon) z(s,\varepsilon) ds, \tag{6}$$

where $R(t,s,\varepsilon)$ represents the resolvent of the kernel $H(t,s,\varepsilon)$.

Formula (6) is substituted for $z(s,\varepsilon)$ in formula (3), then formula (3) is written

$$y(t,\varepsilon) = \sum_{i=1}^{3} C_i y_i(t,\varepsilon) + \frac{1}{\varepsilon^2} \int_0^t K_0(t,s,\varepsilon) \Big[F(s) + \sum_{i=1}^{3} C_i \phi_i(t,\varepsilon) + \int_0^1 R(s,p,\varepsilon) \Big(F(p) + \sum_{i=1}^{3} C_i \phi_i(p,\varepsilon) \Big) dp \Big] ds -$$

$$- \frac{1}{\varepsilon^2} \int_t^1 K_1(t,s,\varepsilon) \Big[F(s) + \sum_{i=1}^{3} C_i \phi_i(t,\varepsilon) + \int_0^1 R(s,p,\varepsilon) \Big(F(p) + \sum_{i=1}^{3} C_i \phi_i(p,\varepsilon) \Big) dp \Big] ds.$$

Now, we provide the following notations:

$$Q_i(t,\varepsilon) = y_i(t,\varepsilon) + \frac{1}{\varepsilon^2} \int_0^t K_0(t,s,\varepsilon) \overline{\phi_i}(s,\varepsilon) ds - \frac{1}{\varepsilon^2} \int_t^1 K_1(t,s,\varepsilon) \overline{\phi_i}(s,\varepsilon) ds, \quad i = \overline{1,3}, \tag{7}$$

$$P(t,\varepsilon) = \frac{1}{\varepsilon^2} \int_0^t K_0(t,s,\varepsilon) \overline{F}(s,\varepsilon) ds - \frac{1}{\varepsilon^2} \int_t^1 K_1(t,s,\varepsilon) \overline{F}(s,\varepsilon) ds, \tag{8}$$

then the formulas for $y(t,\varepsilon)$ are abbreviated

$$y^{(j)}(t,\varepsilon) = C_1 Q_1^{(j)}(t,\varepsilon) + C_2 Q_2^{(j)}(t,\varepsilon) + C_3 Q_3^{(j)}(t,\varepsilon) + P^{(j)}(t,\varepsilon), \quad j = \overline{0,2}. \tag{9}$$

Substituting equation (9) into condition (2) yields the algebraic system for calculating C_i, where $i = \overline{1,3}$:

$$\begin{cases} C_1 Q_1(0,\varepsilon) + C_2 Q_2(0,\varepsilon) + C_3 Q_3(0,\varepsilon) + P(0,\varepsilon) = \alpha_0, \\ C_1 Q_1'(0,\varepsilon) + C_2 Q_2'(0,\varepsilon) + C_3 Q_3'(0,\varepsilon) + P'(0,\varepsilon) = \alpha_1, \\ C_1 Q''1(0,\varepsilon) + C_2 Q_2''(0,\varepsilon) + C_3 Q_3''(0,\varepsilon) + P''(0,\varepsilon) = \alpha_2. \end{cases} \tag{10}$$

As $\varepsilon \to 0$, the asymptotic behavior of $C_i,\ i=\overline{1,3}$, is obtained when the main determinant of system (10) is $\delta(\varepsilon)\neq 0$.

$$C_1(\varepsilon)=-\varepsilon\frac{\mu_2(0)(\alpha_0 y_{30}'(0)-\alpha_1)}{\mu_1(0)(\mu_2(0)-\mu_1(0))}+O(\varepsilon^2),$$
$$C_2(\varepsilon)=-\varepsilon\frac{\alpha_0}{\overline{\phi_2}(0)}\Big(\overline{\phi_3}(0)+y_{30}'(0)\mu_1(0)\mu_2(0)- \tag{11}$$
$$-y_{30}'(0)\frac{\mu_2(0)\overline{\phi_1}(0)}{\mu_1(0)(\mu_2(0)-\mu_1(0))}\Big)+\varepsilon\frac{\alpha_1}{\overline{\phi_2}(0)}\Big(\mu_1(0)\mu_2(0)-$$
$$-\frac{\mu_2(0)\overline{\phi_1}(0)}{\mu_1(0)(\mu_2(0)-\mu_1(0))}\Big)-\varepsilon\frac{\overline{F}(0)}{\overline{\phi_2}(0)}+O(\varepsilon^2),$$
$$C_3(\varepsilon)=\alpha_0+O(\varepsilon).$$

By formula (4) $\phi_i(t,\varepsilon)$, $i=\overline{1,3}$ are found

$$\overline{\phi}_1(s,\varepsilon)=\frac{1}{\varepsilon}\bigg(-\overline{H}_2(s,0)\mu_1(0)y_{10}(0)+O(\varepsilon)\bigg)=\frac{1}{\varepsilon}\Big(\overline{\phi}_1(s)+O(\varepsilon)\Big),$$
$$\overline{\phi}_2(s,\varepsilon)=\frac{1}{\varepsilon}\bigg(\overline{H}_2(s,1)\mu_2(1)y_{20}(1)+O(\varepsilon)\bigg)=\frac{1}{\varepsilon}\Big(\overline{\phi}_2(s)+O(\varepsilon)\Big),$$
$$\overline{\phi}_3(s,\varepsilon)=\int_0^1\sum_{j=0}^{2}\overline{H}_j(s,x)y_{30}^{(j)}(x)dx+O(\varepsilon)=\overline{\phi}_3(s)+O(\varepsilon).$$

Given formulas (7) and (8), we obtain the asymptotic behaviors of $Q_i^{(j)}(t,\varepsilon)$ and $P^{(j)}(t,\varepsilon)$, $j=\overline{0,2}$, $i=\overline{1,3}$:

$$Q_1^{(j)}(t,\varepsilon)=\frac{1}{\varepsilon}\int_0^t\frac{y_{30}^{(j)}(t)\overline{\phi}_1(s)}{\mu_1(s)\mu_2(s)y_{30}(s)}ds+\frac{1}{\varepsilon^j}e^{\frac{1}{\varepsilon}\int_0^t\mu_1(x)dx}\bigg(y_{10}(t)\mu_1^j(t)-\frac{\overline{\phi}_1(0)y_{10}(t)\mu_1^j(t)}{y_{10}(0)\mu_1^2(0)(\mu_2(0)-\mu_1(0))}\bigg)+$$
$$+\frac{1}{\varepsilon^j}e^{-\frac{1}{\varepsilon}\int_t^1\mu_2(x)dx}\frac{\overline{\phi}_1(1)y_{20}(t)\mu_2^j(t)}{y_{20}(1)\mu_2^2(1)(\mu_2(1)-\mu_1(1))}+\frac{1}{\varepsilon^j}\frac{\overline{\phi}_1(t)}{(\mu_2(t)-\mu_1(t))}\bigg(\frac{\mu_1^j(t)}{\mu_1^2(t)}-\frac{\mu_2^j(t)}{\mu_2^2(t)}\bigg),\quad j=\overline{0,2}, \tag{12.a}$$

$$Q_2^{(j)}(t,\varepsilon)=\frac{1}{\varepsilon}\int_0^t\frac{y_{30}^{(j)}(t)\overline{\phi}_2(s)}{\mu_1(s)\mu_2(s)y_{30}(s)}ds-\frac{1}{\varepsilon^j}e^{\frac{1}{\varepsilon}\int_0^t\mu_1(x)dx}\frac{\overline{\phi}_2(0)y_{10}(t)\mu_1^j(t)}{y_{10}(0)\mu_1^2(0)(\mu_2(0)-\mu_1(0))}+$$
$$+\frac{1}{\varepsilon^j}e^{-\frac{1}{\varepsilon}\int_t^1\mu_2(x)dx}\bigg(y_{20}(t)\mu_2^{(j)}(t)+\frac{\overline{\phi}_2(1)y_{20}(t)\mu_2^j(t)}{y_{20}(1)\mu_2^2(1)(\mu_2(1)-\mu_1(1))}\bigg)+$$
$$+\frac{1}{\varepsilon^j}\frac{\overline{\phi}_2(t)}{(\mu_2(t)-\mu_1(t))}\bigg(\frac{\mu_1^j(t)}{\mu_1^2(t)}-\frac{\mu_2^j(t)}{\mu_2^2(t)}\bigg),\quad j=\overline{0,2}, \tag{12.b}$$

$$Q_3^{(j)}(t,\varepsilon)=y_{30}^{(j)}(t)+\int_0^t\frac{y_{30}^{(j)}(t)\overline{\phi}_3(s)}{\mu_1(s)\mu_2(s)y_{30}(s)}ds-\frac{1}{\varepsilon^{j-1}}e^{\frac{1}{\varepsilon}\int_0^t\mu_1(x)dx}\frac{\overline{\phi}_3(0)y_{10}(t)\mu_1^j(t)}{y_{10}(0)\mu_1^2(0)(\mu_2(0)-\mu_1(0))}+$$

$$+\frac{1}{\varepsilon^{j-1}}e^{-\frac{1}{\varepsilon}\int_t^1\mu_2(x)dx}\frac{\overline{\phi}_3(1)y_{20}(t)\mu_2^j(t)}{y_{20}(1)\mu_2^2(1)(\mu_2(1)-\mu_1(1))}+\frac{1}{\varepsilon^{j-1}}\frac{\overline{\phi}_3(t)}{(\mu_2(t)-\mu_1(t))}\left(\frac{\mu_1^j(t)}{\mu_1^2(t)}-\frac{\mu_2^j(t)}{\mu_2^2(t)}\right),\ j=\overline{0,2}, \tag{12.c}$$

$$P^{(j)}(t,\varepsilon)=\int_0^t\frac{y_{30}^{(j)}(t)\overline{F}(s)}{\mu_1(s)\mu_2(s)y_{30}(s)}ds-\frac{1}{\varepsilon^{j-1}}e^{\frac{1}{\varepsilon}\int_0^t\mu_1(x)dx}\frac{\overline{F}(0)y_{10}(t)\mu_1^j(t)}{y_{10}(0)\mu_1^2(0)(\mu_2(0)-\mu_1(0))}+$$

$$+\frac{1}{\varepsilon^{j-1}}e^{-\frac{1}{\varepsilon}\int_t^1\mu_2(x)dx}\frac{\overline{F}(s)y_{20}(t)\mu_2^j(t)}{y_{20}(1)\mu_2^2(1)(\mu_2(1)-\mu_1(1))}+\frac{1}{\varepsilon^{j-1}}\frac{\overline{F}(t)}{(\mu_2(t)-\mu_1(t))}\left(\frac{\mu_1^j(t)}{\mu_1^2(t)}-\frac{\mu_2^j(t)}{\mu_2^2(t)}\right),\ j=\overline{0,2}. \tag{12.d}$$

Theorem 1. If conditions I-III are valid, then the solution for integro-differential equation (1) and (2) holds the following asymptotic estimates as $\varepsilon\to 0$:

$$|y^{(j)}(t,\varepsilon)|\le C\Big(|\alpha_0|+\varepsilon|\alpha_1|+\varepsilon^2|\alpha_2|+\max_{0\le t\le 1}|F(t)|\Big)+$$

$$+\frac{C}{\varepsilon^{j-1}}e^{-\frac{t}{\varepsilon}\gamma_1}\Big(|\alpha_0|+|\alpha_1|+\varepsilon^2|\alpha_2|+\max_{0\le t\le 1}|F(t)|\Big)+ \tag{13}$$

$$+\frac{C}{\varepsilon^{j-1}}e^{-\frac{1-t}{\varepsilon}\gamma_2}\Big(|\alpha_0|+|\alpha_1|+\varepsilon^2|\alpha_2|+\max_{0\le t\le 1}|F(t)|\Big),\quad j=0,1,2,$$

where $C>0$ is a constant independent of ε.

Proof. Asymptotic estimates of $C_i,\ i=\overline{1,3}$ and $Q_i^{(j)}(t,\varepsilon),\ P^{(j)}(t,\varepsilon)$ are obtained by applying formulas (11)-(12.a-12.d):

$$|C_i|\le C\varepsilon\Big(|\alpha_0|+|\alpha_1|\Big),\quad i=1,2,$$

$$|C_3|\le C\Big(|\alpha_0|+\varepsilon^2|\alpha_2|\Big),$$

$$|Q_1^{(j)}(t,\varepsilon)|\le C\Big(1+\frac{1}{\varepsilon^j}e^{-\frac{t}{\varepsilon}\gamma_1}+\frac{1}{\varepsilon^j}e^{-\frac{1-t}{\varepsilon}\gamma_2}\Big),\quad j=\overline{0,2},$$

$$|Q_2^{(j)}(t,\varepsilon)|\le C\Big(1+\frac{1}{\varepsilon^j}e^{-\frac{t}{\varepsilon}\gamma_1}+\frac{1}{\varepsilon^j}e^{-\frac{1-t}{\varepsilon}\gamma_2}\Big),\quad j=\overline{0,2}, \tag{14}$$

$$|Q_3^{(j)}(t,\varepsilon)|\le C\Big(1+\frac{1}{\varepsilon^{j-1}}e^{-\frac{t}{\varepsilon}\gamma_1}+\frac{1}{\varepsilon^{j-1}}e^{-\frac{1-t}{\varepsilon}\gamma_2}\Big),\quad j=\overline{0,2},$$

$$|P^{(j)}(t,\varepsilon)\le C\max_{0\le t\le 1}|F(t)|\Big(1+\frac{1}{\varepsilon^{j-1}}e^{-\frac{t}{\varepsilon}\gamma_1}+\frac{1}{\varepsilon^{j-1}}e^{-\frac{1-t}{\varepsilon}\gamma_2}\Big),\quad j=\overline{0,2}.$$

We derive asymptotic estimations (13) from asymptotic behaviors (14). Theorem 1 has been proved.

Conclusion

This article examines the initial problem for a third-order linear integro-differential equation with a small parameter at the higher derivatives, assuming that the roots of the additional characteristic equation have opposite signs. This paper presents the construction of a fundamental system of solutions and a Cauchy function for a singularly perturbed homogeneous differential equation. The functions $Q_i(t,\varepsilon)$, $P(t,\varepsilon)$, $i=\overline{1,3}$, and constants C_i, $i=\overline{1,3}$ exhibit asymptotic behaviors and estimates. Furthermore, the article provides an analytical formula for solving this singularly perturbed initial problem. A theorem on asymptotic estimates of the solution is proven.

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Author Contributions

N.U. Bukanay used the fundamental solutions of the homogeneous equation to calculate the Cauchy function and derive the asymptotic formula. A.E. Mirzakulova checked all the calculations and wrote the asymptotic behavior of the solutions. A.T. Assanova conducted a comprehensive verification of the calculations. All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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