

Hessian measures in the class of m -convex ($m - cv$) functions

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The theory of m -convex ($m - cv$) functions is a new direction in the real geometry. In this work, by using the connection $m - cv$ functions with strongly m -subharmonic (sh_m) functions and using well-known and rich properties of sh_m functions, we show a number of important properties of the class of $m - cv$ functions, in particular, we study Hessians $H^k(u)$, $k = 1, 2, \dots, n - m + 1$, in the class of bounded $m - cv$ functions.

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Introduction

It is well known that m -convex functions are a real analogue in \mathbb{R}^n strongly m -subharmonic (sh_m) functions in the complex space \mathbb{C}^n . Let us recall the definition of the class sh_m of functions, which at this time has become the subject of research by many authors (Z. Błocki [1], S. Dinew and S. Kolodziej [2–4], S. Li [5], H.C. Lu [6, 7], H.C. Lu and V.D. Nguyen [8], A. Sadullaev and his students [9–11], etc.).

A twice differentiable function $u(z) \in C^2(D)$, $D \subset \mathbb{C}^n$, is said to be strongly m -subharmonic, if at each point of the domain D it holds inequalities

$$(dd^c u)^k \wedge \beta^{n-k} \geq 0, \quad k = 1, 2, \dots, n - m + 1,$$

where $\beta = dd^c \|z\|^2$ is the standard volume form in \mathbb{C}^n .

It's clear that $psh = sh_1 \subset sh_2 \subset \dots \subset sh_n = sh$. Operators $(dd^c u)^k \wedge \beta^{n-k}$ are closely related to the Hessians. For a twice differentiable function $u \in C^2(D)$, the second-order differential $dd^c u = \frac{i}{2} \sum_{j,t} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_t} dz_j \wedge d\bar{z}_t$ (at a fixed point $o \in D$) is a Hermitian quadratic form. After a suitable unitary coordinate transform, it is reduced to the diagonal form $dd^c u = \frac{i}{2} [\lambda_1 dz_1 \wedge d\bar{z}_1 + \dots + \lambda_n dz_n \wedge d\bar{z}_n]$, where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of the Hermitian matrix $\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_t} \right)$, which are real: $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$. Note that the unitary transformation does not change the differential form. $\beta = dd^c \|z\|^2$. Therefore, it is not difficult to see that

$$(dd^c u)^k \wedge \beta^{n-k} = k!(n-k)! H_o^k(u) \beta^n,$$

where $H_o^k(u) = \sum_{1 \leq j_1 < \dots < j_k \leq n} \lambda_{j_1} \dots \lambda_{j_k}$ is the Hessian of dimension k of the vector $\lambda = \lambda(u) \in \mathbb{R}^n$.

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Consequently, the twice differentiable function $u(z) \in C^2(D)$, $D \subset \mathbb{C}^n$, is strongly m -subharmonic, if at each point $o \in D$ the next inequalities hold

$$H^k(u) = H_o^k(u) \geq 0, \quad k = 1, 2, \dots, n - m + 1. \tag{1}$$

The following theorem is important

Theorem 1. (see [1]). For any twice differentiable $sh_m \cap C^2(D)$ functions $v_1, \dots, v_k \in sh_m(D) \cap C^2(D)$, $1 \leq k \leq n - m + 1$, the relation

$$dd^c v_1 \wedge \dots \wedge dd^c v_k \wedge \beta^{m-1} \geq 0$$

is valid. In particular, for $u \in sh_m(D) \cap C^2(D)$ and for any $v_1, \dots, v_{n-m} \in sh_m(D) \cap C^2(D)$ it holds

$$dd^c u \wedge dd^c v_1 \wedge \dots \wedge dd^c v_{n-m} \wedge \beta^{m-1} \geq 0. \tag{2}$$

The last property has dual character: if a twice differentiable function u , it satisfies (2) for all $v_1, \dots, v_{n-m} \in sh_m(D) \cap C^2(D)$, then the function u is certainly sh_m function. Moreover, the class of second-order polynomials of the form is sufficient here (see [1, 2])

$$v_j = \sum_{k=1}^n c_{j,k} |z_k|^2 \in sh_m(\mathbb{C}^n), \quad c_{j,k} \in \mathbb{R} \text{ is const.} \tag{3}$$

Theorem 1 allows us to define sh_m functions in the class L_{loc}^1 .

Definition 1. A function $u \in L_{loc}^1(D)$ is called sh_m in the domain $D \subset \mathbb{C}^n$, if it is upper semi-continuous and for any twice differentiable sh_m functions v_1, \dots, v_{n-m} of the form (3), the current $dd^c u \wedge dd^c v_1 \wedge \dots \wedge dd^c v_{n-m} \wedge \beta^{m-1}$ defined as

$$\begin{aligned} & [dd^c u \wedge dd^c v_1 \wedge \dots \wedge dd^c v_{n-m} \wedge \beta^{m-1}] (\omega) = \\ & = \int u dd^c v_1 \wedge \dots \wedge dd^c v_{n-m} \wedge \beta^{m-1} \wedge dd^c \omega, \quad \omega \in F^{0,0} \end{aligned}$$

is positive, $\int u dd^c v_1 \wedge \dots \wedge dd^c v_{n-m} \wedge \beta^{m-1} \wedge dd^c \omega \geq 0, \quad \forall \omega \in F^{0,0}, \quad \omega \geq 0.$

1 m -convex functions and associated measures

In this section, similarly to (1), we define Hessians $H^k(u)$, $k = 1, 2, \dots, n - m + 1$, in the class of bounded m -convex functions as Borel measures. This method of defining $H^k(u)$ as a measure belongs to A. Sadullaev.

Let $D \subset \mathbb{R}^n$ and $u(x) \in C^2(D)$. Then matrix $\left(\frac{\partial^2 u}{\partial x_j \partial x_t}\right)$ is orthogonal, $\frac{\partial^2 u}{\partial x_j \partial x_t} = \frac{\partial^2 u}{\partial x_t \partial x_j}$. Therefore, after a suitable orthonormal transformation, it is transformed into a diagonal form,

$$\left(\frac{\partial^2 u}{\partial x_j \partial x_t}\right) \rightarrow \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix},$$

where $\lambda_j = \lambda_j(x) \in \mathbb{R}$ are the eigenvalues of the matrix $\left(\frac{\partial^2 u}{\partial x_j \partial x_t}\right)$. Let

$$H^k(u) = H^k(\lambda) = \sum_{1 \leq j_1 < \dots < j_k \leq n} \lambda_{j_1} \dots \lambda_{j_k}$$

be Hessian of the dimension k of the vector $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$.

Definition 2. A twice differentiable function $u \in C^2(D)$ is called m -convex in $D \subset \mathbb{R}^n$, $u \in m - cv(D)$, if its eigenvalue vector $\lambda = \lambda(x) = (\lambda_1(x), \lambda_2(x), \dots, \lambda_n(x))$ satisfies the conditions

$$m - cv \cap C^2(D) = \left\{ H^k(u) = H^k(\lambda(x)) \geq 0, \quad \forall x \in D, \quad k = 1, \dots, n - m + 1 \right\}.$$

Potential theory of $m - cv$ functions is poorly-studied and is a new direction in the theory of real geometry. However, when $m = 1$, this class $1 - cv \cap C^2(D) = \{H^1(\lambda) \geq 0\} = \{\lambda_1 \geq 0, \lambda_2 \geq 0, \dots, \lambda_n \geq 0\}$ coincides with the convex functions in \mathbb{R}^n , and when $m = n$, the class $n - cv \cap C^2(D) = \{\lambda_1 + \lambda_2 + \dots + \lambda_n \geq 0\}$ coincides with the class of subharmonic functions in \mathbb{R}^n , $cv = 1 - cv \subset 2 - cv \subset \dots \subset n - cv = sh$. The class of convex functions is well studied A. Aleksandrov [12], I. Bakelman [13], A. Pogorelov [14], A. Artykbaev [15] and others. When $m > 1$ this class has been studied in a series of works N. Trudinger, H. Wang, N. Ivochkina and other mathematicians (see [16–22]).

Principal difficulties in the theory of $m - cv$ are the introduction of the class $m - cv \cap L^1_{loc}$, i.e. definition $m - cv(D)$ of functions in the class of upper semicontinuous, locally integrable or bounded functions and the definition of Hessians $H^k(u)$, $u \in m - cv \cap L^1_{loc}$. So for $m = n$ (the case of subharmonic functions) in the class of upper semicontinuous, locally integrable functions $u(x) \in n - cv(D)$ are defined as a distribution and the Laplace operator $\Delta u = dd^c u \wedge \beta^{n-1}$ is a Borel measure.

To define operators $(dd^c u)^k \wedge \beta^{n-k} \geq 0$, $k = 1, 2, \dots, n - m + 1$ for the function $u(z) \in sh_m(D)$ in a domain $D \subset \mathbb{C}^n$ the function $u(z)$ must be locally bounded, i.e. $u(z) \in L^\infty_{loc}(D)$. In this case, the operators $(dd^c u)^k \wedge \beta^{n-k} \geq 0$, $k = 1, 2, \dots, n - m + 1$ are also positive Borel measures (see [10]).

In this work, by using the connection of $m - cv$ functions with strongly m -subharmonic functions and using well-known and rich properties sh_m of functions, we show a number of important properties of the class of $m - cv$ functions, in particular, of the Hessians $H^k(u)$, $k = 1, 2, \dots, n - m + 1$, in the class of bounded $m - cv$ functions.

We embed \mathbb{R}^n_x into \mathbb{C}^n , by $\mathbb{R}^n_x \subset \mathbb{C}^n_z = \mathbb{R}^n_x + i\mathbb{R}^n_y$ ($z = x + iy$), as a real n -dimensional subspace of the complex space \mathbb{C}^n .

Proposition 1. (see [23]). A twice differentiable function $u(x) \in C^2(D)$, $D \subset \mathbb{R}^n_x$, is $m - cv$ in D , if and only if a function $u^c(z) = u^c(x + iy) = u(x)$ that does not depend on variables $y \in \mathbb{R}^n_y$, is sh_m in the domain $D \times \mathbb{R}^n_y$.

Proof. We establish a connection between the Hessians $H^k(u)$ and $H^k(u^c)$. We have,

$$\frac{\partial u^c}{\partial z_j} = \frac{1}{2} \left[\frac{\partial u^c}{\partial x_j} - \frac{\partial u^c}{\partial y_j} \right] = \frac{1}{2} \frac{\partial u^c}{\partial x_j};$$

$$\frac{\partial^2 u^c}{\partial z_j \partial \bar{z}_t} = \frac{1}{2} \frac{\partial}{\partial \bar{z}_t} \left[\frac{\partial u^c}{\partial x_j} \right] = \frac{1}{4} \left[\frac{\partial^2 u^c}{\partial x_j \partial x_t} + \frac{\partial^2 u^c}{\partial x_j \partial y_t} \right] = \frac{1}{4} \frac{\partial^2 u^c}{\partial x_j \partial x_t}.$$

Thus, $\left(\frac{\partial^2 u^c}{\partial z_j \partial \bar{z}_t} \right) = \frac{1}{4} \left(\frac{\partial^2 u}{\partial x_j \partial x_t} \right)$ and therefore, $H^k(u) = 4^k H^k(u^c)$, that is the proof of the proposition.

Let now $u(x)$ be an upper semicontinuous function in the domain $D \subset \mathbb{R}^n_x$. Then $u^c(z)$ also will be upper semicontinuous function in the domain $D \times \mathbb{R}^n_y \subset \mathbb{C}^n_z$.

Definition 3. An upper semicontinuous function $u(x)$ in a domain $D \subset \mathbb{R}^n_x$ is called m -convex in D , if the corresponding function $u^c(z)$ is strongly m -subharmonic, $u^c(z) \in sh_m(D \times \mathbb{R}^n_y)$.

Let $u(x)$ be a locally bounded m -convex function in the domain $D \subset \mathbb{R}^n_x$. Then $u^c(z)$ will be also locally bounded, strongly m -subharmonic function in the domain $D \times \mathbb{R}^n_y \subset \mathbb{C}^n_z$. Therefore, the operators

$$(dd^c u^c)^k \wedge \beta^{n-k}, \quad k = 1, 2, \dots, n - m + 1$$

are defined as Borel measures in the domain $D \times \mathbb{R}^n_y \subset \mathbb{C}^n_z$, $\mu_k = (dd^c u^c)^k \wedge \beta^{n-k}$.

Since for a twice differentiable function $(dd^c u^c)^k \wedge \beta^{n-k} = k!(n-k)!H^k(u^c)\beta^n$, then for a bounded, strongly m -subharmonic function in the domain $D \times \mathbb{R}_y^n \subset \mathbb{C}_z^n$, it is natural to determine its Hessians, equating them to the measure

$$H^k(u^c) = \frac{\mu_k}{k!(n-k)!} = \frac{1}{k!(n-k)!} (dd^c u^c)^k \wedge \beta^{n-k}.$$

We can now define Hessians H^k , $k = 1, 2, \dots, n - m + 1$ in the class of locally bounded, m -convex domain $D \subset \mathbb{R}_x^n$ functions.

Definition 4. Let a function $u(x)$ be locally bounded and m -convex in the domain $D \subset \mathbb{R}_x^n$. Let us define Borel measures in the domain $D \times \mathbb{R}_y^n \subset \mathbb{C}_z^n$,

$$\mu_k = (dd^c u^c)^k \wedge \beta^{n-k}, k = 1, 2, \dots, n - m + 1.$$

Since $u^c \in sh_m(D \times \mathbb{R}_y^n)$ does not depend on $y \in \mathbb{R}_y^n$, then for any Borel sets $E_x \subset D$, $E_y \subset \mathbb{R}_y^n$, the measures $\frac{4^k}{mes E_y} \mu_k(E_x \times E_y)$ do not depend on the set $E_y \subset \mathbb{R}_y^n$, i.e. $\frac{4^k}{mes E_y} \mu_k(E_x \times E_y) = \nu_k(E_x)$. The Borel measures

$$\nu_k : \nu_k(E_x) = \frac{4^k}{mes E_y} \mu_k(E_x \times E_y), \quad k = 1, 2, \dots, n - m + 1,$$

we call by Hessians H^k , $k = 1, 2, \dots, n - m + 1$, for a locally bounded, m -convex function $u(x) \in m - cv(D)$ in the domain $D \subset \mathbb{R}_x^n$.

For twice differentiable function $u(x) \in m - cv(D) \cap C^2(D)$ the Hessians are ordinary functions, however, for a non-twice differentiable, bounded semicontinuous function $u(x) \in m - cv(D) \cap L^\infty(D)$, the Hessians H^k , $k = 1, 2, \dots, n - m + 1$ are positive Borel measures.

Using Theorem 1 and Proposition 1 (see also Definition 3) $m - cv$ functions are defined as

Definition 5. A function $u(x) \in L_{loc}^1(D)$ is called m -convex function in the domain $D \subset \mathbb{R}_x^n$, $u(x) \in m - cv(D)$, if it is upper semicontinuous and for any twice differentiable $m - cv(D)$ functions v_1, \dots, v_{n-m} , the current $dd^c u^c \wedge dd^c v_1^c \wedge \dots \wedge dd^c v_{n-m}^c \wedge \beta^{m-1}$ defined as

$$\begin{aligned} & [dd^c u^c \wedge dd^c v_1^c \wedge \dots \wedge dd^c v_{n-m}^c \wedge \beta^{m-1}] (\omega) = \\ & = \int u^c dd^c v_1^c \wedge \dots \wedge dd^c v_{n-m}^c \wedge \beta^{m-1} \wedge dd^c \omega, \omega \in F^{0,0}(D \times \mathbb{R}_y^n) \end{aligned}$$

is positive.

2 General definitions of m -convex functions

In various works (see, for example, [18, 19]) m -convex functions in the class of bounded upper semicontinuous $m - cv(D)$ functions define using the “viscosity” definition: an upper semicontinuous function $u(x)$ is called $m - cv(D)$, $u(x) \in m - cv(D)$, if any quadratic polynomial $q(x)$ for which the difference $u(x) - q(x)$ achieves a local maximum only at a finite number of points $x^1, \dots, x^q \in D$, is $m - cv(D)$, $q(x) \in m - cv(D)$.

The following important proposition belongs to Trudinger-Wang [19]

Lemma 1. A semicontinuous function $u(x)$ is in $m - cv(D)$, if for each domain $G \subset\subset D$ and each function $v(x) \in C^2(D) : H_m(v) \leq 0$ from $u|_{\partial G} \leq v|_{\partial G} \Rightarrow u_G \leq v|_G$.

Lemma 2. A semicontinuous function $u(x)$ is in $m - cv(D)$, if and only if for any domain $G \subset\subset D$ there exists $u_j(x) \in C^2(G) \cap m - cv(G) : u_j(x) \downarrow u(x)$.

Lemma 3. If $m < \frac{n}{2} + 1$, then $m - cv(D) \subset C^{0,\gamma} = Lip_\gamma$, where $\gamma = 2 - \frac{n}{n-m+1}$, $0 < \gamma \leq 1$.

Corollary 1. If $m < \frac{n}{2} + 1$, then $u(x) \in m - cv(D)$ continuous.

For our purpose, it is convenient to use the Trudinger-Wang's definition based on Lemma 2:

Definition 6. An upper semicontinuous function $u(x)$ is called m -convex $m - cv(D)$, if for any domain $G \subset\subset D$ there exists a sequence of functions $u_j(x) \in C^2(G) \cap m - cv(G) : u_j(x) \downarrow u(x)$.

In fact, the two main ones, Definition 3 and Definition 6, are equivalent.

Theorem 2. A function $u(x)$ is $m - cv(D)$ in the sense of Definition 3, if and only if it is $m - cv(D)$ in the sense of Definition 6.

Proof. Let the function $u(x)$ have a monotonically decreasing sequence of functions $u_j(x) \in m - cv(G) : u_j(x) \downarrow u(x)$. Let us put \mathbb{R}_x^n in \mathbb{C}_z^n , $\mathbb{R}_x^n \subset \mathbb{C}_z^n = \mathbb{R}_x^n + i\mathbb{R}_y^n$ ($z = x + iy$), and construct a monotonically decreasing sequence $u_j^c(z) = u_j(x) \in sh_m(G \times \mathbb{R}_y^n)$. Then $\lim_{j \rightarrow \infty} u_j^c(z) = u^c(z) \in sh_m(G \times \mathbb{R}_y^n)$ and $u(x) = u^c(x)$ is $m - cv(G)$.

On the other side, let the function $u(x)$ be such that $u^c(z) = u(x) \in sh_m(D \times \mathbb{R}_y^n)$. Let us construct a standard approximation $u_j^c(z) = u^c \circ K_{\frac{1}{j}}(w - z)$, $j = 1, 2, \dots$ (see [10]). For any compact domain $G \subset\subset D$, starting from a certain number $j \geq j_0$, they are defined, infinitely smooth functions $u_j^c(z) \in sh_m(G) : u_j^c(z) \downarrow u^c(z)$. Moreover, it is easy to see that $u_j^c(z)$ do not depend on $y \in \mathbb{R}_y^n$. Therefore, $u_j^c(x) = u_j(x) \downarrow u(x)$, $u_j(x) \in m - cv(G) \cap C^\infty(G)$.

3 Example (fundamental solution)

$$\chi_m(x, 0) = \begin{cases} |x|^{2 - \frac{n}{n-m+1}} & \text{if } m < \frac{n}{2} + 1, \\ \ln |x| & \text{if } m = \frac{n}{2} + 1, \\ -|x|^{2 - \frac{n}{n-m+1}} & \text{if } m > \frac{n}{2} + 1. \end{cases}$$

Thus, when $m < \frac{n}{2} + 1$, the fundamental solution is bounded and Lipschitz, when $m \geq \frac{n}{2} + 1$, it is equal $-\infty$ at the point $x = 0$. Note that at $m = n$, i.e. for the subharmonic case it coincides with fundamental solution of the Laplace operator Δ .

4 Weakly convergence of m -convex functions

We will continue our study of Borel measures

$$\left\{ H^k(u) \geq 0, \forall x \in D, k = 1, 2, \dots, n - m + 1 \right\}$$

in the class $u(x) \in m - cv(D) \cap L_{loc}^\infty(D)$.

Theorem 3. If $u(x) \in m - cv(D) \cap L_{loc}^\infty(D)$ and $u_j(x) \in m - cv(D)$ are sequences of monotonically decreasing functions, converging to $u(x)$, $u_j(x) \downarrow u(x)$, then there is weakly convergence of measures $H^k(u_j) \mapsto H^k(u)$, $k = 1, 2, \dots, n - m + 1$.

Proof. Let us continue the functions $u(x)$, $u_j(x)$ from $D \subset \mathbb{R}_x^n$ to $D \times \mathbb{R}_y^n$, as sh_m - functions $u^c(z)$, $u_j^c(z) \in sh_m(D \times \mathbb{R}_y^n)$. Then $u^c(z) \in sh_m(D \times \mathbb{R}_y^n) \cap L_{loc}^\infty(D \times \mathbb{R}_y^n)$ and $u_j^c(z) \downarrow u^c(z)$. According to Theorem Sadullaev-Abdullaev (see. [10]), Borel measures

$$H^k(u_j^c) = \frac{\mu^k}{k!(n-k)!} = \frac{1}{k!(n-k)!} (dd^c u_j^c)^k \wedge \beta^{n-k}$$

weakly converges: $H^k(u_j^c) \mapsto H^k(u^c)$, $k = 1, 2, \dots, n - m + 1$. This implies weakly convergence $H^k(u_j) \mapsto H^k(u)$, $k = 1, 2, \dots, n - m + 1$.

As is known, if $\{u_\alpha(z)\} \subset sh_m(D \times \mathbb{R}_y^n)$, $D \times \mathbb{R}_y^n \subset \mathbb{C}^n$, a family of uniformly bounded, strongly m -subharmonic functions, then for any compact set $K \subset\subset D$ there exists a constant $C(K)$, such that the integral averages

$$\int_K (dd^c u_\alpha)^k \wedge \beta^{n-k} \leq C(K), \quad k = 1, 2, \dots, n - m + 1$$

(see. [10]). From this it follows that the Hessians

$$H^k(u_\alpha) = \frac{1}{k!(n-k)!} (dd^c u_\alpha)^k \wedge \beta^{n-k},$$

which are Borel measures, are uniformly bounded on average on compact subsets of the domain D . This fact, discovered by Chern-Levine-Nirenberg [24] for a class of psh functions, then it played a main role in the construction of the theory of potential in the class psh and sh_m functions.

Here we will prove a similar fact for Hessians $H^k(u)$, $k = 1, 2, \dots, n - m + 1$, in the class of $m - cv(D)$, $D \subset \mathbb{R}^n$, functions. At the same time, we note that, if in a class $sh_m(D \times \mathbb{R}_y^n)$, $D \times \mathbb{R}_y^n \subset \mathbb{C}^n$, the proof is based on differential forms and Stokes' Theorem, then for the estimate $H^k(u)$, $k = 1, 2, \dots, n - m + 1$, in the class of $m - cv(D)$, $D \subset \mathbb{R}^n$, we do not have this technique.

Theorem 4. If $\{u_\alpha(x)\} \subset m - cv(D)$, $D \subset \mathbb{R}_x^n$, is a family of locally uniformly bounded m -convex functions, then the family of measures $\{H^k(u_\alpha)\}$, $k = 1, 2, \dots, n - m + 1$, in Hessians are uniformly bounded on average on compact subsets of the domain D . In other words, for any compact set $K \subset\subset D$ there is a constant $C(K)$ that is upper bound for integral averages

$$\int_K H^k(u_\alpha) \leq C(K), \quad k = 1, 2, \dots, n - m + 1.$$

Proof. Let us use Proposition 1 and Definition 3. We put \mathbb{R}_x^n in \mathbb{C}^n , $\mathbb{R}_x^n \subset \mathbb{C}_z^n = \mathbb{R}_x^n + i\mathbb{R}_y^n$ ($z = x + iy$), as a real n -dimensional subspace of a complex space \mathbb{C}^n and construct a family of locally uniformly bounded functions. $\{u_\alpha^c(z)\} \subset sh_m(D \times \mathbb{R}_y^n)$. For this family Borel measures $\{H^k(u_\alpha^c)\}$, $k = 1, 2, \dots, n - m + 1$ is uniformly bounded on average on compact subsets of the domain. $D \times \mathbb{R}_y^n$. From the definition of measures $\{H^k(u_\alpha)\}$ in Hessians it follows that the family of measures $\{H^k(u_\alpha)\}$, $k = 1, 2, \dots, n - m + 1$ is uniformly bounded on average on compact subsets of the domain D .

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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