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On orders of approximation of the generalized Nikol'skiy-Besov class in Lorentz spaces

In this paper Lorentz space is considered with mixed norm. Definition of the generalized Nikol'skiy-Besov classes is given. The main results of the paper are the estimates of the order of approximation of functions of the generalized Nikol'skiy-Besov class by partial sums of multiply series Fourier constructed with harmonics from the hyperbolic cross.

Key words: Lorentz space, Nikol'skiy-Besov class, approximation of function, hyperbolic cross.

§1. Introduction

Let $\bar{x} = (x_1, ..., x_m) \in I^m = [0, 2\pi]^m$ and let $\theta_j, p_j \in [1, +\infty], j = 1, ..., m,$ N is the set of natural numbers.

We shall denote by $L_{\bar{p},\bar{\theta}}(I^m)$ Lorentz the spaces with mixed norm of Lebesque-measurable functions $f(\bar{x})$ of period 2π in each variable such that the quantity

$$\|f\|_{\bar{p},\bar{\theta}} = \|\ldots\|f\|_{p_1,\theta_1}\ldots\|_{p_m,\theta_m} < +\infty,$$

where

$$\|g\|_{p,\theta} = \left\{\int_0^{2\pi} (g^*(t))^{\theta} t^{\frac{\theta}{p}-1} dt\right\}^{\frac{1}{\theta}},$$

where g^* — is the non-increasing of the function $|g|$ (see [1]).

As we know, that in case when the $p_j = \theta_j$, $j = 1, ..., m$ space $L_{\bar{p},\bar{\theta}}(I^m)$ coincide with the space of Lebesque $L_{\bar{p}}(I^m)$ with mixed norm (definition see in [2; 128]).

$$\|f\|_{\bar{p}} = \left[\int_0^{2\pi}\left[\ldots\left[\int_0^{2\pi}\left|f(\bar{x})\right|^{p_1} dx_1\right]^{\frac{p_2}{p_1}}\ldots\right]^{\frac{p_m}{p_m-1}} dx_m\right]^{\frac{1}{p_m}}.$$

Let $\overset{\circ}{L}_{\bar{q},\bar{\theta}}(I^m)$ will be set of functions $f \in L_{\bar{q},\bar{\theta}}(I^m)$ such that

$$\int_0^{2\pi} f(\bar{x}) dx_j = 0, \ \forall j = 1, ..., m.$$

and let $a_{\bar{n}}(f)$ will be the Fourier coefficients $f \in L_1(I^m)$ with respect to the multiple trigonometric system. Then we set

$$\delta_{\bar s}(f,\bar x)=\sum_{\bar n\in\rho(\bar s)} a_{\bar n}(f)e^{i\langle \bar n,\bar x\rangle},$$

where

$$\left\langle \bar y,\bar x\right\rangle=\sum_{j=1}^{m} y_j x_j,\ \rho(\bar s)=\left\{\bar k=(k_1,\dots,k_m)\in Z^m: 2^{s_j-1}\le |k_j|<2^{s_j}, j=1,\dots,m\right\}.$$

A function $\Omega(\bar t)=\Omega(t_1,\dots,t_m)$ is a function of mixed module continuity type of an order $l\in N$ if it satisfies the following conditions:

1) $\Omega(\bar t)>0,\ t_j>0,\ j=1,\dots,m,\ \Omega(\bar t)=0,$ if $\Pi_{j=1}^{m}t_j=0;$

2) $\Omega(\bar t)$ increases on each variable;

3) $\Omega(k_1t_1,\dots,k_mt_m)\le(\Pi_{j=1}^{m}k_j)^{l}\,\Omega(t_1,\dots,t_m),\ \ k_j\in N, j=1,\dots,m;$

4) $\Omega(\bar t)$ is continuous for $t_j>0,\ j=1,\dots,m.$

Let us consider the following sets

$$\Gamma(N)=\Gamma(\Omega,N)=\left\{\bar s=(s_1,\dots,s_m)\in Z_+^m:\Omega(2^{-s_1},\dots,2^{-s_m})\ge\frac{1}{N}\right\};$$

$$Q(N)=U_{(\bar s\in\Gamma(\Omega,N)}\rho(\bar s);$$

$$\Gamma^{\perp}(N)=\Gamma^{\perp}(\Omega,N)=Z_+^m/\Gamma(\Omega,N); \tag{1}$$

$$\aleph(N)=\Gamma^{\perp}(N)/\Gamma^{\perp}(2^{l}N). \tag{2}$$

It follows from (1), (2) that $\aleph(N)\subset\Gamma^{\perp}(N)$ and

$$\frac{1}{2^{l}N}\le\Omega(2^{-\bar s})<\frac{1}{N} \tag{3}$$

for $\bar s\in\aleph(N)$. In [3] N.N. Pustovoitov proved, that $\aleph(N)\ne\varphi$ and

$$|\aleph(N)|\succ\prec(\log_2 N)^{m-1}, \tag*{4)}$$

where $|F|$ — is the number of elements of the set F.

$S_{Q(N)}(f,\bar x)=\Sigma_{\bar k\in Q(N)}a_{\bar k}(f)\cdot e^{i\langle\bar k,\bar x\rangle}$ is a partial sum of the Fourier series of function f.

For a number sequence we write $\{a_{\bar n}\}_{\bar n\in Z^m,}\in l_{\bar p}$ if

$$\left\|\{a_{\bar n}\}_{\bar n\in Z^m}\right\|_{l_{\bar p}}=\left\{\sum_{n_m=-\infty}^{\infty}\left[\dots\left[\sum_{n_1=-\infty}^{\infty}|a_{\bar n}|^{p_1}\right]^{\frac{p_2}{p_1}}\right]^{\frac{p_m}{p_{m-1}}}\right\}^{\frac{1}{p_m}}<+\infty,$$

where $\bar p=(p_1,\dots,p_m),\ \ 1\le p_j<+\infty,\ \ j=1,2,\dots,m.$

For given function of mixed module smoothness $\Omega(\bar t)$ type we consider the generalized Nikol'skiy-Besov class

$$S^{\Omega}_{\bar p,\bar\theta,\bar\tau}B=\left\{f\in\mathring{L}_{\bar p,\bar\theta}(I^m):\left\|\left\{\Omega^{-1}(2^{-\bar s})\left\|\delta_{\bar s}(f)\right\|_{\bar p,\bar\theta}\right\}_{\bar n\in Z_+^m}\right\|_{l_{\bar\tau}}\le 1\right\},$$

where $\bar p=(p_1,\dots,p_m),\ \ \bar\theta=(\theta_1,\dots,\theta_m),\ \ \bar\tau=(\tau_1,\dots,\tau_m),\ \ 1<p_j<+\infty,\ \ 1\le\theta_j;\ \ \tau_j<+\infty,\ \ j=1,\dots,m$ and $\Omega(2^{-\bar s})=\Omega(2^{-s_1},\dots,2^{-s_m}).$

If $\Omega(\bar t)=\Pi_{j=1}^{m}t_j^{r_j}$, then this class is denoted by $S^{\bar r}_{\bar p,\bar\theta,\bar\tau}B.$

In case $p_j=\theta_j=p$ and $\Omega(\bar{t})=\Pi_{j=1}^{m}t_j^{r_j}$, $r_j<l$, $\tau_j=+\infty, j=1,...,m$, $S^{\Omega}_{\bar{p},\bar{\theta},\bar{\tau}}B$ was defined first time by S.M. Nikol'skiy [4], and for $1\le\tau_j=+\infty, j=1,...,m,$ by T.I. Amanov [5]. The generalized Besov class was considered by M.L. Gol'dman [6].

As pointed out in [7], one difficulty in the theory approximation of a function of several variables is the choice of the harmonics of the approximating polynomials. The first author suggesting approximation of functions of several variables by polynomials with harmonics in hyperbolic crosses was K.I. Badenko (see [7]). After that, approximation of various classes of smooth functions by this method was considered by S.A. Telyakovskii, B.S. Mityagin, Ya.S. Bugrov, N.S. Nikol'skaya, E.M. Galeev, V.N. Temlyakov, Dinh Dung (see [7]), A.R. DeVore, S.V. Konyagin and V.N. Temlyakov [8], H.-J. Schmeisser and W. Sickel [9], W. Sickel and T. Ullrich [10], A.S. Romanyuk [11, 12], N.N. Pustovoitov [3].

For generalized Besov class of this topic was considered by Sun Youbgsheng and Wang Heping [13], S.A. Stasyuk [14].

Exact orders of approximation of Nikol'skiy-Besov classes in the metric of the Lorentz space were found by the author [15, 16] and K.A. Bekmaganbetov [17].

An order of approximation of the class $S^{\bar{r}}_{\bar{p},\bar{\theta},\bar{\tau}}B$ by partial Fourier sums $S_n^{\bar{\gamma}}(f,x)=\sum_{<\bar{s},\bar{\gamma}><n}\delta_{\bar{s}}(f,\bar{x})$ was found in [15]. It is stated in the following theorem.

Theorem (see [15]). Let $\bar{\theta}^{(1)}=(\theta_1^{(1)},...,\theta_m^{(1)})$, $\bar{\theta}^{(2)}=(\theta_1^{(2)},...,\theta_m^{(2)})$, $\bar{\tau}=(\tau_1,...,\tau_m)$, $\bar{p}=(p_1,...,p_m)$, $\bar{q}=(q_1,...,q_m)$, $\bar{r}=(r_1,...,r_m)$, $\bar{\gamma}=(\gamma_1,...,\gamma_m)$, $\gamma_j=\frac{r_j}{r_1}$, and assume that $1\le\theta_j^{(1)},\theta_j^{(2)},\tau_j<+\infty$, $1\le p_j<q_j<+\infty$, $\max_{j=1,...,m-1}\left\{\theta_j^{(2)}\right\}<\min_{j=2,...,m}\left\{q_j\right\}$ $\frac{1}{p_j}-\frac{1}{q_j}<r_j, j=1,...,m$, $0<r_1=...=r_\nu<r_{\nu+1}\le...\le r_m$, $\frac{1}{p_1}-\frac{1}{q_1}=...=\frac{1}{p_\nu}-\frac{1}{q_\nu}$, $r_1\left(\frac{1}{p_j}-\frac{1}{q_j}\right)<r_j\left(\frac{1}{p_1}-\frac{1}{q_1}\right)$ $j=\nu+1,...,m.$ Then the following relation hold:

$$\sup_{f\in s^{\bar{r}}_{\bar{p}},\bar{\theta}^{(1)},\bar{\tau}B}\left\|f-S_n^{\bar{\tau}}(f)\right\|_{\bar{q},\bar{\theta}^2}\succ\prec\begin{cases}2^{-n\left(r_1+\frac{1}{q_1}-\frac{1}{p_1}\right)}\cdot n^{\sum_{j=2}^{\nu}\left(\frac{1}{\theta_j^{(2)}}-\frac{1}{\tau_j}\right)},\theta_j^{(2)}<\tau_j, j=1,...,m;\\ 2^{-n\left(r_1+\frac{1}{q_1}-\frac{1}{p_1}\right)},\tau_j\le\theta_j^{(2)}, j=1,...,m.\end{cases}$$

The main aim of the present paper is an estimate of the order of the quantity

$$\sup_{f\in s^{\Omega}_{\bar{p},\bar{\theta},\bar{\tau}}B}\left\|f-S_{Q(N)}(f)\right\|_{\bar{q},\bar{\theta}}.$$

This paper is organized as follows. In second section some auxiliary suppositions are given. In third section the estimate of order approximation Nikol'skiy-Besov classes is established in space of Lorentz with mixed norm.

We shall denote by $C(p,q,y,...)$ positive quantities depending only on the parameter in the parentheses and not necessarily the same in distinct formulae. The notation $A(y)\succ\prec B(y)$ means that there exit positive constants C_1,C_2 such that $C_1\cdot A(y)\le B(y)\le C_2\cdot A(y)$.

§ 2. Auxiliary suppositions

In that follows we denote by $\chi_{\aleph(n)}(\bar{s})$ the characteristic function of the set $\aleph(n)=\left\{\bar{s}=(s_1,...,s_m)\in Z_+^m:\left\langle\bar{s},\bar{\gamma}\right\rangle=n\right\}$.

Lemma 2.1. Let $\bar{\tau}=(\tau_1,...,\tau_m)$, $1\le\tau_j<+\infty$, $j=1,...,m.$ *Then the following relation holds:*

$$\left\|\left\{\chi_{\aleph(n)}(\bar{s})\right\}_{\bar{s}\in\aleph(n)}\right\|_{l_{\bar{\tau}}}\succ\prec C(\tau,m)\cdot n^{\sum_{j=2}^{m}\frac{1}{r_j}}.$$

Lemma 2.2. Let $\bar{\gamma}=(\gamma_1,...,\gamma_m)$, $\bar{\gamma}'=(\gamma'_1,...,\gamma'_m)$, $\gamma'_j=\lambda_j, j=1,...,v$, $1<\gamma_j<\gamma'_j, j=v+1,...,m$ *let* $\bar{\tau}=(\tau_{l+1},...,\tau_m)$, *with* $1\le\tau_j<+\infty)$ $j=1,...,m$, *and* $\alpha>0$. *Then the following relation holds:*

$$I_n^{(l)}=\left\| \left\{ 2^{-\alpha\left\langle \bar{s},\bar{\gamma}' \right\rangle} \right\}_{\bar{s}\in\aleph(n)} \right\|_{l_{\bar{\tau}}} \succ\prec 2^{-n\alpha}\cdot n^{\sum_{j=2}^{v}\frac{1}{\tau_j}}.$$

Lemma 2.1, 2.2 are proved in [16].

Let us recall definitions of the conditions $(S),(S_l)$ given by N.K. Bary and S.B. Stechkin [18].

Definition 2.3. A function $g(t)$ satisfies the condition (S) if for some $\alpha\in(0,1)$ function $t^{-\alpha}g(t)$ almost increases on (0, 1].

We say that a function $\Omega(\bar{t})$ satisfies the condition (S) on $(0,1]^m$, if it satisfies this condition on each variable.

Definition 2.4. A function $g(t)$ satisfies the condition (S_l), if for some $\alpha\in(0,l)$ function $t^{-\alpha}g(t)$ almost decreases on (0, 1].

We say that a function $\Omega(\bar{t})$ satisfies the condition (S_l) on $(0,1]^m$, if it satisfies this condition on each variable.

In what follows we use the denotation: $\chi_{\aleph(n)}(\bar{s})$ — characteristic function of set

$$\aleph(n)=\left\{ \bar{s}=(s_1,...,s_m)\in Z_+^m : \left\langle \bar{s},\bar{\gamma} \right\rangle = n \right\}.$$

Lemma 2.5 (see [19].) *Let* $1\le\theta_j<+\infty, j=1,2$ *and* $\Omega(\bar{t})$ is *a function of mixed module continuity type of an order* l *which satisfies* (S) — *conditions for* $\bar{\alpha}=(\alpha_1,...,\alpha_m)$, $\alpha_j>\beta_j\ge0, j=1,...,m$. *Then for* $0<\theta_j<+\infty, j=1,...,m$ *the text relation holds*

$$\left\| \left\{ \Omega(2^{-s_1},...,2^{-s_m})\prod_{j=1}^{m}2^{s_j\beta_j} \right\}_{\bar{s}\in\Gamma^{\perp}(\Omega,N)} \right\|_{l_{\bar{\theta}}} \succ\prec \left\| \left\{ \Omega(2^{-s_1},...,2^{-s_m})\prod_{j=1}^{m}2^{s_j\beta_j} \right\}_{\bar{s}\in\aleph(N)} \right\|_{l_{\bar{\theta}}}.$$

Lemma 2.6. Let a function $\Omega(\bar{t})$ *will be a function of mixed continuity type of an order l which satisfies the conditions* (S) *and* (S_l), $1\le\tau_j<+\infty, j=1,...,m$ *and* $\aleph(N)=\Gamma^{\perp}(N)/\Gamma^{\perp}(2^lN)$.

Then

$$\left\| \left\{ X_{\aleph(N)}(\bar{s}) \right\}_{\bar{s}\in\aleph(N)} \right\|_{l_{\bar{\tau}}} \succ\prec (\log_2 N)^{\sum_{j=2}^{m}\frac{1}{\tau_j}}.$$

Proof. It is known (see [3; 114]), that the see $\aleph(N)$ can be one-to-one mapped onto some subset of the set $A=\prod_{n=m_1}^{m_1+m_2}\theta_n$, where

$$\theta_n=\left\{ \bar{s}=(s_1,...,s_m)\in Z_+^m : \sum_{j=1}^{m}s_j=n \right\},$$

$m_1=\left[\frac{1}{\alpha}\log_2(C_2 2^l N)\right]+1$, $m_2=\frac{l+\log_2 C_1}{\alpha}$. Here $[y]$ — is the integer part of y. By the property of norm

$$\left\| \left\{ X_{\aleph(N)}(\bar{s}) \right\}_{\bar{s}\in\aleph(N)} \right\|_{l_{\bar{\tau}}} \le \left\| \left\{ X_A(\bar{s}) \right\}_{\bar{s}\in A} \right\|_{l_{\bar{\tau}}} = \left\| \sum_{n=m_1}^{m_1+m_2} \left\{ X_{\theta(n)}(\bar{s}) \right\}_{\bar{s}\in\theta(n)} \right\|_{l_{\bar{\theta}}} = \sum_{n=m_1}^{m_1+m_2} \left\| \left\{ X_{\theta(n)}(\bar{s}) \right\}_{\bar{s}\in\theta(n)} \right\|_{l_{\bar{\theta}}}. \quad (5)$$

By lemma 2.2

$$\left\| \left\{ X_{\theta(n)}(\bar{s}) \right\}_{\bar{s}\in\theta(n)} \right\|_{l_{\bar{\theta}}} \le Cn^{\sum_{j=2}^{m}\frac{1}{\tau_j}}.$$

Therefore from the inequality (5) taking into account the definitions of m_1, m_2 we get

$$\left\| \left\{ X_{\aleph(N)}(\bar{s}) \right\}_{\bar{s}\in\aleph(N)} \right\|_{l_{\bar{\theta}}} \le C \sum_{n=m_1}^{m_1+m_2} n^{\sum_{j=2}^{m}\frac{1}{\tau_j}} \le C m_2 \left(\frac{1}{\alpha}\log_2(C_2 2^l N) + 1 + m^2 \right)^{\sum_{j=2}^{m}\frac{1}{\tau_j}}. \tag{6}$$

Since m_2 doesn't depend on N, by the property of logarifmic functions from the from the estimation (6) for some sufficiently big N we get

$$\left\| \left\{ X_{\aleph(N)}(\bar{s}) \right\}_{\bar{s}\in\aleph(N)} \right\|_{l_{\bar{\theta}}} \le C(\log_2 N)^{\sum_{j=2}^{m}\frac{1}{\tau_j}}.$$

This proves the upper estimation of the lemma. Let us give the lower estimation. By Holder inequality we have

$$\sum_{\bar{s}\in\aleph(N)} 1 \le \left\| \{1\}_{\bar{s}\in\aleph(N)} \right\|_{l_{\bar{\tau}}} \left\| \{1\}_{\bar{s}\in\aleph(N)} \right\|_{l_{\bar{\tau}'}}, \tag{7}$$

where $\frac{1}{\tau_j} + \frac{1}{\tau_j'} = 1, j = 1,...,m.$ By the proved fact

$$\left\| \{1\}_{\bar{s}\in\aleph(N)} \right\|_{l_{\bar{\tau}'}} \le C(\log_2 N)^{\sum_{j=2}^{m}\frac{1}{\tau_j'}}$$

and (4) therefore from (7) we get

$$C_1(\log_2 N)^{\sum_{j=2}^{m}\frac{1}{\tau_j}} \le \left\| \left\{ X_{\aleph(N)}(\bar{s}) \right\}_{\bar{s}\in\aleph(N)} \right\|_{l_{\bar{\tau}}}.$$

The lemma is proved.

Remark. We note that for the case $\tau_1 = ... = \tau_m = 1$ lemma 2.6 was proved N.N. Pustovoitov [3].

Theorem 2.7. Let $\bar{q} = (q_1,...,q_m),\ 1 < q_j < \infty, j = 1,...,m,\ \beta = \min(q_1,...,q_m,2).$ *Then for any function* $f \in L_{\bar{q}}(I^m)$ *the following inequality holds*

$$\|f\|_{\bar{q}} \le C(q,m) \left\{ \sum_{\bar{s}\in Z_+^m} \left\| \delta_{\bar{s}}(f) \right\|_{\bar{q}}^{\beta} \right\}^{\frac{1}{\beta}}.$$

This theorem 2.7 was proved in [20].

Theorem 2.8 (see [15]). *Let* $\bar{p} = (p_1,...,p_m), \bar{q} = (q_1,...,q_m),\ \bar{\theta}^{(1)} = (\theta_1^{(1)},...,\theta_m^{(1)}),\ \bar{\theta}^{(2)} = (\theta_1^{(2)},...,\theta_m^{(2)}).$ *Assume that* $1 \le p_j < q_j < +\infty, 1 \le \theta_j^{(1)}, \theta_j^{(2)} < +\infty\ \ j = 1,...,m.$ *If* $f \in \overset{\circ}{L}_{\bar{p},\bar{\theta}^{(1)}}(I^m),\ \max_{j=1,...,m-1}\theta_j^{(2)} < \min_{j=2,...,m} q_j$ *and the quantity*

$$\sigma(f) \equiv \left\{ \sum_{s_m=1}^{\infty} 2^{s_m\theta_m^{(2)}\left(\frac{1}{p_m}-\frac{1}{q_m}\right)} \left[\cdots \left[\sum_{s_1=1}^{\infty} 2^{s_1\theta_1^{(2)}\left(\frac{1}{p_1}-\frac{1}{q_1}\right)} \left\| \delta_{\bar{s}}(f) \right\|_{\bar{p},\bar{\theta}^{(1)}}^{\theta_1^{(2)}} \right] \cdots \right]^{\frac{\theta_m^{(2)}}{\theta_{m-1}^{(2)}}} \right\}^{\frac{1}{\theta_m^{(2)}}}.$$

If it is finite, then $f \in \overset{\circ}{L}_{\bar{q},\bar{\theta}^{(2)}} \le C(p,q,\theta)\cdot\sigma(f).$

Theorem 2.9 (see [15]). *Let* $\bar{q} = (q_1,...,q_m), \bar{\theta} = (\theta_1,...,\theta_m)\ \ \bar{\lambda} = (\lambda_1,...,\lambda_m).$ *We assume that* $1 < q_j < \tau_j < +\infty,\ 1 < \theta_j < +\infty,\ \ j = 1,...,m.$ *If* $f \in \overset{\circ}{L}_{\bar{q},\bar{\theta}}(I^m)$ *and* $f(\bar{x}) = \sum_{\bar{s}\in Z_+^m} b_{\bar{s}} \sum_{\bar{k}\in\rho(\bar{s})} e^{i\langle\bar{k},\bar{x}\rangle},$ *then*

$$\|f\|_{\bar{q},\bar{\theta}} \ge C(q,\theta,\lambda,m) \left\{ \sum_{s_m=1}^{\infty} 2^{s_m\theta_m\left(\frac{1}{\lambda_m}-\frac{1}{q_m}\right)} \left[\cdots \left[\sum_{s_1=1}^{\infty} 2^{s_1\theta_1\left(\frac{1}{\lambda_1}-\frac{1}{q_1}\right)} \left(\left\| \delta_{\bar{s}}(f) \right\|_{\bar{\lambda},\bar{\theta}} \right)^{\theta_1} \right]^{\frac{\theta_2}{\theta_1}} \cdots \right]^{\frac{\theta_m}{\theta_{m-1}}} \right\}^{\frac{1}{\theta_m}}.$$

§3. Main results

Let us prove the main results of the present paper.

Theorem 3.1. Let $1 \le \theta_j^{(1)}, \theta_j^{(2)}, \tau_j < +\infty$, $1 < p_j < q_j < \infty, j = 1,...m,$ *and a function* $\Omega(\bar{t})$ *will be function of mixed module continuity type of an order* l *which satisfies the conditions* (S) *and* (S_l), $\alpha_j > \frac{1}{p_j} - \frac{1}{q_j}, j = 1,...,m.$

1) If $1 \le \theta_j^{(2)} < \tau_j < +\infty, j = 1,...,m,$ *then*

$$\frac{C_1}{N}(\log_2 N)^{-\sum_{j=2}^{m}\frac{1}{\tau_j}} \left\| \left\{ \prod_{j=1}^{m} 2^{s_j\left(\frac{1}{p_j}-\frac{1}{q_j}\right)} \right\}_{\bar{s}\in\aleph(N)} \right\|_{l_{\bar{\theta}(2)}} \le \sup_{f\in S^{\Omega}_{\bar{p},\bar{\theta}^{(1)},\bar{\tau}}B} \left\| f - S_{Q(N)}(f) \right\|_{\bar{q},\bar{\theta}^{(2)}} \le \frac{C_2}{N} \left\| \left\{ \prod_{j=1}^{m} 2^{s_j\left(\frac{1}{p_j}-\frac{1}{q_j}\right)} \right\}_{\bar{s}\in\aleph(N)} \right\|_{l_{\bar{\varepsilon}}},$$

where $\bar{\varepsilon} = (\varepsilon_1,...,\varepsilon_m)$; $\varepsilon_j = \frac{\tau_j\theta_j^{(2)}}{\tau_j - \theta_j^{(2)}}, j = 1,...,m.$

2) If $\tau_j \le \theta_j^{(2)}, j = 1,...,m,$ *then*

$$C_1 \sup_{\bar{s}\in\aleph(N)} \Omega(2^{-\bar{s}}) \prod_{j=1}^{m} 2^{s_j\left(\frac{1}{p_j}-\frac{1}{q_j}\right)} \le \sup_{f\in S^{\Omega}_{\bar{p},\bar{\theta}^{(1)},\bar{\tau}}B} \left\| f - S_{Q(N)}(f) \right\|_{\bar{q},\bar{\theta}^{(2)}} \le C_2 \sup_{\bar{s}\in\Gamma^{\perp}(N)} \Omega(2^{-\bar{s}}) \prod_{j=1}^{m} 2^{s_j\left(\frac{1}{p_j}-\frac{1}{q_j}\right)}.$$

Proof. By theorem 2.8 we have

$$\sup_{f\in S^{\Omega}_{\bar{p},\bar{\theta}^{(1)},\bar{\tau}}B} \left\| f - S_{Q(N)}(f) \right\|_{\bar{q},\bar{\theta}^{(2)}} \le C \left\| \left\{ \prod_{j=1}^{m} 2^{s_j\left(\frac{1}{p_j}-\frac{1}{q_j}\right)} \left\| \delta_{\bar{s}}(f - S_{Q(N)}(f)) \right\|_{\bar{q},\bar{\theta}^{(1)}} \right\}_{\bar{s}\in Z_+^m} \right\|_{l_{\bar{\theta}^{(2)}}} =$$

$$= C \left\| \left\{ \prod_{j=1}^{m} 2^{s_j\left(\frac{1}{p_j}-\frac{1}{q_j}\right)} \left\| \delta_{\bar{s}}(f - S_{Q(N)}(f)) \right\|_{\bar{p},\bar{\theta}^{(1)}} \right\}_{\bar{s}\in\Gamma^{\perp}(N)} \right\|_{l_{\bar{\theta}^{(2)}}}.$$

Since $\beta_j = \frac{\tau_j}{\theta_j^{(2)}} > 1, j = 1,...,m,$ applying Holder inequality we get from it that

$$\sup_{f\in S^{\Omega}_{\bar{p},\bar{\theta}^{(1)},\bar{\tau}}B} \left\| f - S_{Q(N)}(f) \right\|_{\bar{q},\bar{\theta}^{(2)}} \le C \left\| \left\{ \Omega^{-1}(2^{\bar{s}}) \left\| \delta_{\bar{s}}(f) \right\|_{\bar{q},\bar{\theta}^{(1)}} \right\}_{\bar{s}\in Z_+^m} \right\|_{l_{\bar{\tau}}} \times C \left\| \left\{ \Omega(2^{-\bar{s}}) \prod_{j=1}^{m} 2^{s_j\left(\frac{1}{p_j}-\frac{1}{q_j}\right)} \right\}_{\bar{s}\in\Gamma^{\perp}(N)} \right\|_{l_{\bar{\varepsilon}}}, \qquad (8)$$

where $\bar{\varepsilon} = (\varepsilon_1,...,\varepsilon_m)$; $\varepsilon_j = \frac{\tau_j\theta_j^{(2)}}{\tau_j - \theta_j^{(2)}}.$

By lemma 2.5 and the definition of the set $\Gamma^{\perp}(N)$ in (8) we have

$$\sup_{f\in S^{\Omega}_{\bar{p},\bar{\theta}^{(1)},\bar{\tau}}B} \left\| f - S_{Q(N)}(f) \right\|_{\bar{q},\bar{\theta}^{(2)}} \le \frac{C}{N} \left\| \left\{ \prod_{j=1}^{m} 2^{s_j\left(\frac{1}{p_j}-\frac{1}{q_j}\right)} \right\}_{\bar{s}\in\aleph(N)} \right\|_{l_{\bar{\varepsilon}}}.$$

In item 1 of the theorem the upper estimation has been proved.

Let us prove the lower estimation. Consider the function

$$f_{\circ}(\bar{x}) = (\log_2 N)^{-\sum_{j=2}^{m}\frac{1}{\tau_j}} \sum_{\bar{s}\in\aleph(N)} \prod_{j=1}^{m} \Omega(2^{-\bar{s}}) 2^{-s_j\left(1-\frac{1}{p_j}\right)} \sum_{\bar{k}\in\rho(\bar{s})} e^{i\langle\bar{k},\bar{x}\rangle}.$$

In one-dimensional case for Dirichlet kernel $D_n(x)=\frac{1}{2}+\sum_{k=1}^{n}e^{ikx}$ the following statement holds

$$\left\|D_n\right\|_{p,\theta}\succ\prec n^{1-\frac{1}{p}},1<p<+\infty,1<\theta<+\infty.$$

It implies that

$$\left\|\sum_{\bar{k}\in\rho(\bar{s})}e^{i\langle\bar{k},\bar{x}\rangle}\right\|_{\bar{p},\bar{\theta}^{(1)}}\succ\prec C\sum_{j=1}^{m}2^{s_j\left(1-\frac{1}{p_j}\right)}. \tag{9}$$

Therefore by lemma 2.6 and by this estimation we have

$$\left\|\left\{\Omega^{-1}(2^{\bar{s}})\left\|\delta_{\bar{s}}(f_0)\right\|_{\bar{q},\bar{\theta}^{(1)}}\right\}_{\bar{s}\in Z_+^m}\right\|_{l_{\bar{\tau}}}=(\log_2 N)^{-\sum_{j=2}^{m}\frac{1}{\tau_j}}\left\|\left\{X_{\aleph(N)}(\bar{s})\right\}_{\bar{s}\in\aleph(N)}\right\|_{l_{\bar{\tau}}}\le C_0.$$

Hence $C_0^{-1}f_0\in S_{\bar{p},\bar{\tau}}^{\Omega}B.$ Now taking into account that $S_{Q(N)}^{\bar{\gamma}}(f_0,\bar{x})=0$ theorem 2.9 and (9) we have

$$\left\|f_0-S_{Q(N)}(f_0)\right\|_{\bar{q},\bar{\theta}^{(2)}}=\left\|f_0\right\|_{\bar{q},\bar{\theta}^{(2)}}\ge C\left\|\left\{\prod_{j=1}^{m}2^{s_j\left(\frac{1}{\lambda_j}-\frac{1}{q_j}\right)}\left\|\delta_{\bar{s}}(f_0)\right\|_{\bar{\lambda},\bar{\theta}^{(1)}}\right\}_{\bar{s}\in Z_+^m}\right\|_{l_{\bar{\theta}^{(2)}}}=$$

$$=C\left\|\left\{\prod_{j=1}^{m}2^{s_j\left(\frac{1}{\lambda_j}-\frac{1}{q_j}\right)}(\log_2 N)^{-\sum_{j=2}^{m}\frac{1}{\tau_j}}\Omega(2^{-\bar{s}})\prod_{j=1}^{m}2^{-s_j\left(1-\frac{1}{p_j}\right)}\left\|\sum_{\bar{k}\in\rho(\bar{s})}e^{i\langle\bar{k},\bar{x}\rangle}\right\|_{\bar{\lambda},\bar{\theta}^{(1)}}\right\}_{\bar{s}\in\aleph(N)}\right\|_{l_{\bar{\theta}^{(2)}}}\ge$$

$$\ge\frac{C}{N}(\log_2 N)^{-\sum_{j=2}^{m}\frac{1}{\tau_j}}\left\|\left\{\prod_{j=1}^{m}2^{s_j\left(\frac{1}{\lambda_j}-\frac{1}{q_j}\right)}\Omega(2^{-\bar{s}})\right\}_{\bar{s}\in\aleph(N)}\right\|_{l_{\bar{\theta}^{(2)}}}.$$

Thus

$$\sup_{f\in S_{\bar{p},\bar{\theta}^{(1)}\bar{\tau}}^{\Omega}B}\left\|f-S_{Q(N)}(f)\right\|_{\bar{q},\bar{\theta}^{(2)}}\ge\frac{C}{N}(\log_2 N)^{-\sum_{j=2}^{m}\frac{1}{\tau_j}}\left\|\left\{\prod_{j=1}^{m}2^{s_j\left(\frac{1}{p_j}-\frac{1}{q_j}\right)}\right\}_{\bar{s}\in\aleph(N)}\right\|_{l_{\bar{\varepsilon}}^{(2)}}.$$

Item 1) has been proved.

Let us prove the second item of the theorem. Since $\tau_j\le\theta_j^{(2)},j=1,...,m,$ applying Jensen inequality (see [2; 125]) we obtain

$$\sup_{f\in S_{\bar{p},\bar{\theta}^{(1)}\bar{\tau}}^{\Omega}B}\left\|f-S_{Q(N)}(f)\right\|_{\bar{q},\bar{\theta}^{(2)}}\le C\left\|\left\{\prod_{j=1}^{m}2^{s_j\left(\frac{1}{p_j}-\frac{1}{q_j}\right)}\left\|\delta_{\bar{s}}(f)\right\|_{\bar{p},\bar{\theta}^{(1)}}\right\}_{\bar{s}\in\Gamma^{\perp}(N)}\right\|_{l_{\bar{\varepsilon}}}\le$$

$$\le C\left\|\left\{\Omega^{-1}(2^{-\bar{s}})\left\|\delta_{\bar{s}}(f)\right\|_{\bar{p},\bar{\theta}^{(1)}}\right\}_{\bar{s}\in Z_+^m}\right\|_{l_{\bar{\theta}}}\sup_{\bar{s}\in\Gamma^{\perp}(N)}\Omega(2^{-\bar{s}})\prod_{j=1}^{m}2^{s_j\left(\frac{1}{p_j}-\frac{1}{q_j}\right)}$$

for any function $f\in S_{\bar{p},\bar{\theta}^{(1)},\bar{\tau}}^{\Omega}B.$ In item 2) the upper estimation has been proved. For the lower estimation consider the function

$$f_1(\bar{x})=\Omega(2^{-\bar{\tilde{s}}})2^{-\sum_{j=1}^{m}\tilde{s}_j\left(1-\frac{1}{p_j}\right)}\sum_{\bar{k}\in\rho(\bar{\tilde{s}})}e^{i\langle\bar{k},\bar{x}\rangle},$$

where $\bar{\tilde{s}}\in\aleph(N).$ Then $f_1\in S_{\bar{p},\bar{\theta}^{(1)},\bar{\tau}}^{\Omega}B.$ Next by (9) we have

$$\left\| f_1 - S_{Q(N)}(f_1) \right\|_{\bar{q},\bar{\theta}^{(2)}} = \left\| f_1 \right\|_{\bar{q},\bar{\theta}^{(2)}} \geq$$

$$\geq C\Omega(2^{-\tilde{\tilde{s}}}) 2^{-\sum_{j=1}^{m} \tilde{s}_j \left(1-\frac{1}{p_j}\right)} \prod_{j=1}^{m} 2^{\tilde{s}_j \left(1-\frac{1}{q_j}\right)} = C\Omega(2^{-\tilde{\tilde{s}}}) \prod_{j=1}^{m} 2^{\tilde{s}_j \left(\frac{1}{p_j}-\frac{1}{q_j}\right)} \quad \forall \bar{\tilde{s}} \in \aleph(N).$$

This proves the lower estimation.

Theorem 3.2. Let a function $\Omega(\bar{t})$ *will be a function of mixed continuity type of an order* l *which satisfies the conditions* (S) *and* (S_l).

1) *If* $1 < q_j < p_j < \infty,\ p_j \geq 2,\ 1 \leq \theta_j, \tau_j < +\infty, j = 1,...,m,$ *then*

$$\sup_{f \in S^{\Omega}_{\bar{p},\bar{\tau}}B} \left\| f - S_{Q(N)}(f) \right\|_{\bar{q},\bar{\theta}} \succ\prec \frac{1}{N} (\log_2 N)^{\sum_{j=2}^{m}\left(\frac{1}{2}-\frac{1}{\tau_j}\right)}$$

for $2 < \tau_j < +\infty, j = 1,...,m.$

2) *If* $\tau_j \leq 2, j = 1,...,m,$ then

$$\sup_{f \in S^{\Omega}_{\bar{p},\bar{\tau}}B} \left\| f - S_{Q(N)}(f) \right\|_{\bar{q},\bar{\theta}} \succ\prec \frac{1}{N}.$$

3) *If* $\tau_j \leq \beta = \max\{p_1,...,p_m,2\},$ then

$$\sup_{f \in S^{\Omega}_{\bar{p},\bar{\tau}}B} \left\| f - S_{Q(N)}(f) \right\|_{\bar{q},\bar{\theta}} \leq \frac{C}{N}.$$

Proof. Since $q_j < p_j, j = 1,...,m,$ we have

$$\|f\|_{\bar{q},\bar{\theta}} \leq \|f\|_{\bar{p}}, f \in L_{\bar{p}}(I^m).$$

Therefore

$$\left\| f - S_{Q(N)}(f) \right\|_{\bar{q},\bar{\theta}} \leq \left\| f - S_{Q(N)}(f) \right\|_{\bar{p}} = \left\| \sum_{\bar{s} \in k^{\perp}(N)} \delta_{\bar{s}}(f) \right\|_{\bar{p}}. \tag{10}$$

Now, since $2 \leq p_j < +\infty, j = 1,...,m$ by theorem 1 from (10) we obtain

$$\left\| f - S_{Q(N)}(f) \right\|_{\bar{q},\bar{\theta}} \leq \left\{ \sum_{\bar{s} \in \Gamma^{\perp}(N)} \left\| \delta_{\bar{s}}(f) \right\|_{\bar{p}}^2 \right\}^{\frac{1}{2}} = \left\{ \sum_{\bar{s} \in \Gamma^{\perp}(N)} \Omega^2(2^{-\bar{s}}) (\Omega^{-1}(2^{-\bar{s}}) \left\| \delta_{\bar{s}}(f) \right\|_{\bar{p}})^2 \right\}^{\frac{1}{2}}. \tag{11}$$

If $2 < \tau_j < +\infty, j = 1,...,m,$ then applying Holder inequality from (11) we get

$$\left\| f - S_{Q(N)}(f) \right\|_{\bar{q},\bar{\theta}} \leq \left\| \left\{ \Omega^{-1}(2^{-\bar{s}}) \left\| \delta_{\bar{s}}(f) \right\|_{\bar{p}} \right\}_{\bar{s} \in Z_+^m} \right\|_{l_{\bar{\tau}}} \times \left\| \left\{ \Omega(2^{-\bar{s}}) \right\}_{\bar{s} \in \Gamma^{\perp}(N)} \right\|_{l_{\bar{\varepsilon}}}, \tag{12}$$

where $\bar{\varepsilon} = (\varepsilon_1,...,\varepsilon_m),\ \varepsilon_j = 2\beta_j', \frac{1}{\beta_j} + \frac{1}{\beta_j'} = 1, \beta_j = \frac{\tau_j}{2}, j = 1,...,m.$

Now by lemma 2.5 for $\beta_j = 0, j = 1,...,m$ and by lemma 2.6 from (12) we obtain

$$\sup_{f \in S^{\Omega}_{\bar{p},\bar{\tau}}B} \left\| f - S_{Q(N)}(f) \right\|_{\bar{q},\bar{\theta}} \leq \left\| \left\{ \Omega^{-1}(2^{-\bar{s}}) \left\| \delta_{\bar{s}}(f) \right\|_{\bar{p}} \right\}_{\bar{s} \in Z_+^m} \right\|_{l_{\bar{\tau}}} \times \left\| \left\{ \Omega(2^{-\bar{s}}) \right\}_{\bar{s} \in \aleph(N)} \right\|_{l_{\bar{\varepsilon}}} =$$

$$= \frac{1}{N} \left\| \left\{ X_{\aleph(N)}(\bar{s}) \right\}_{\bar{s} \in \aleph(N)} \right\|_{l_{\bar{\varepsilon}}} \leq C \frac{1}{N} (\log_2 N)^{\sum_{j=2}^{m} (\frac{1}{2}-\frac{1}{\tau_j})},$$

where $X_{\aleph(N)}$ — is the characteristic function of the set $\aleph(N).$ This proved the upper estimation in item 1).

If $\tau_j \le 2, j=1,...,m,$ then using Jensen inequality from (16) we have

$$\left\{\sum_{\bar{s}\in\Gamma^{\perp}(N)} \left\|\delta_{\bar{s}}(f)\right\|_{\bar{p}}^{2}\right\}^{\frac{1}{2}} \le \left\{\sum_{\bar{s}\in\Gamma^{\perp}(N)} \left(\Omega^{-1}(2^{-\bar{s}})\left\|\delta_{\bar{s}}(f)\right\|_{\bar{p}}\right)^{2}\right\}^{\frac{1}{2}} \sup_{\bar{s}\in\Gamma^{\perp}(N)} \Omega(2^{-\bar{s}}).$$

Therefore from inequality (11) we obtain

$$\sup_{f\in S^{\Omega}_{\bar{p},\bar{\tau}}B} \left\|f - S_{Q(N)}(f)\right\|_{\bar{q},\bar{\theta}} \le \sup_{\bar{s}\in\Gamma^{\perp}(N)} \Omega(2^{-\bar{s}}) \le \frac{C}{N},$$

In the case $2 < p_j < +\infty, \tau_j \le 2, j=1,...,m.$

Let us prove item 3). Let $1 < p_j < +\infty, j=1,...,m,$ $\beta = \min\{p_1,...,p_m,2\}.$ Then by theorem 2.7 from (10) we have

$$\left\|f - S_{Q(N)}(f)\right\|_{\bar{q},\bar{\theta}} \le C\left\{\sum_{\bar{s}\in\Gamma^{\perp}(N)} \left\|\delta_{\bar{s}}(f)\right\|_{\bar{p}}^{\beta}\right\}^{\frac{1}{\beta}}. \tag{13}$$

If $\tau_j \le \beta, j=1,...,m,$ then using Jensen inequality we obtain

$$\left\{\sum_{\bar{s}\in\Gamma^{\perp}(N)} \left\|\delta_{\bar{s}}(f)\right\|_{\bar{p}}^{\beta}\right\}^{\frac{1}{\beta}} \le \left\{\sum_{\bar{s}\in\Gamma^{\perp}(N)} (\Omega^{-1}(2^{-\bar{s}})\left\|\delta_{\bar{s}}(f)\right\|_{\bar{p}})^{\beta}\right\}^{\frac{1}{\beta}} \sup_{\bar{s}\in\Gamma^{\perp}(N)} \Omega(2^{-\bar{s}}).$$

Therefore from (13) we get

$$\sup_{f\in S^{\Omega}_{\bar{p},\bar{\tau}}B} \left\|f - S_{Q(N)}(f)\right\|_{\bar{q},\bar{\theta}} \le \sup_{\bar{s}\in\Gamma^{\perp}(N)} \Omega(2^{-\bar{s}}) \le \frac{C}{N}, \tag{14}$$

in case $\tau_j \le \beta, 1 < p_j < +\infty, j=1,...,m.$ This proves the upper estimation.

Let us prove the lower estimation. Firstly, we consider item 1). Let $2 \le q_j < p_j < +\infty,$ $2 < \tau_j < \infty, j=1,...,m.$ We consider the function

$$f_{N,\tau}(\bar{x}) = \left\|\left\{X_{\aleph(N)}(\bar{s})\right\}_{\bar{s}\in\aleph(N)}\right\|_{l_{\bar{\tau}}}^{-1} \sum_{\bar{s}\in\aleph(N)} \Omega(2^{-\bar{s}}) e^{i\langle\bar{k},\bar{x}\rangle},$$

where $\bar{k}_{\bar{s}} \in \rho(\bar{s})$ is a some fixed element. By the definition if the space $S^{\Omega}_{\bar{p},\bar{\tau}}B$ we have

$$\left\|\left\{\Omega^{-1}(2^{-\bar{s}})\left\|\delta_{\bar{s}}(f_{N,\tau})\right\|_{\bar{p}}\right\}_{\bar{s}\in Z_{+}^{m}}\right\|_{l_{\bar{\tau}}} = \left\|\left\{X_{\aleph(N)}(\bar{s})\right\}_{\bar{s}\in\aleph(N)}\right\|_{l_{\bar{\tau}}}^{-1} \left\|\left\{X_{\aleph(N)}(\bar{s})\right\}_{\bar{s}\in\aleph(N)}\right\|_{l_{\bar{\tau}}} = 1,$$

i.e. the function $f_{N,\tau} \in S^{\Omega}_{\bar{p},\bar{\tau}}B.$ Since by the assumption of the theorem $2 \le q_j < +\infty, j=1,...,m,$ then

$$\|f\|_2 \le \|f\|_{\bar{q},\bar{\theta}}.$$

Therefore taking into account $S_{Q(N)}(f_{N,\tau},\bar{x}) = 0$ by Parcevalle equality we get

$$\left\|f_{N,\tau} - S_{Q(N)}(f_{N,\tau})\right\|_{\bar{q},\bar{\theta}} = \left\|f_{N,\tau}\right\|_{\bar{q},\bar{\theta}} \ge \left\|f_{N,\tau}\right\|_{2} =$$

$$= \left\|\left\{X_{\aleph(N)}(\bar{s})\right\}_{\bar{s}\in\aleph(N)}\right\|_{l_{\bar{\tau}}}^{-1} \left\{\sum_{\bar{s}\in\aleph(N)} \Omega^{2}(2^{-\bar{s}})\right\}^{\frac{1}{2}} \ge$$

$$\ge \frac{C}{N}\left\|\left\{X_{\aleph(N)}(\bar{s})\right\}_{\bar{s}\in\aleph(N)}\right\|_{l_{\bar{\tau}}}^{-1} \left\|\left\{X_{\aleph(N)}(\bar{s})\right\}_{\bar{s}\in\aleph(N)}\right\|_{l_{2}}.$$

Thus

$$\sup_{f\in S^{\Omega}_{\bar{p},\bar{\tau}}B} \left\|f - S_{Q(N)}(f)\right\|_{\bar{q},\bar{\theta}} \ge \frac{C}{N}\left\|\left\{X_{\aleph(N)}(\bar{s})\right\}_{\bar{s}\in\aleph(N)}\right\|_{l_{\bar{\tau}}}^{-1} \left\|\left\{X_{\aleph(N)}(\bar{s})\right\}_{\bar{s}\in\aleph(N)}\right\|_{l_{2}}; \tag{15}$$

$$2 \le q_j < p_j < +\infty,\ 2 < \tau_j < \infty, j=1,...,m.$$

Next by lemma 2.6

$$\left\|\left\{X_{\aleph(N)}(\bar{s})\right\}_{\bar{s}\in\aleph(N)}\right\|_{l_{\bar{\tau}}}\le C_2(\log_2 N)^{\sum_{j=2}^{m}\frac{1}{\tau_j}}$$

and for $\tau_j=2, j=1,...,m$

$$C_1(\log_2 N)^{\sum_{j=2}^{m}\frac{1}{2}}\le\left\|\left\{X_{\aleph(N)}(\bar{s})\right\}_{\bar{s}\in\aleph(N)}\right\|_{l_2}.$$

Therefore from (15) follows that

$$\sup_{f\in S^{\Omega}_{\bar{p},\bar{\tau}}B}\left\|f-S_{Q(N)}(f)\right\|_{\bar{q},\bar{\theta}}\ge C_3\frac{1}{N}(\log_2 N)^{\sum_{j=2}^{m}(\frac{1}{2}-\frac{1}{\tau_j})}.$$

Let us prove the lower estimation in item 2). Let $2\le p_j<+\infty, j=1,...,m,$ then we consider the function

$$g_{\Omega}(\bar{x})=C_0\Omega(2^{-\bar{s}})e^{i\langle\bar{k}_{\bar{s}},\bar{x}\rangle}, C_0>0,$$

where $\bar{k}_s\in\rho(\bar{x}), \bar{s}=(s_1,...,s_m)\in\aleph(N)$. Then

$$\left\|\left\{\Omega^{-1}(2^{-\bar{s}})\left\|\delta_{\bar{s}}(g\Omega)\right\|_{\bar{p}}\right\}_{\bar{s}\in Z^m_+}\right\|_{l_{\bar{\tau}}}=C_0\Omega^{-1}(2^{-\overset{\cong}{s}})\left\|e^{i\langle\bar{k}_{\bar{s}},\bar{x}\rangle}\right\|_{\bar{p}}\Omega(2^{-\bar{s}})=C_0.$$

Hence the function $g_{\Omega}\in S^{\Omega}_{\bar{p},\bar{\tau}}B$. Since $S_{Q(N)}(g_{\Omega},\bar{x})=0,$ then

$$\left\|g_{\Omega}-S_{Q(N)}(g_{\Omega})\right\|_{\bar{q},\bar{\theta}}=\left\|g_{\Omega}\right\|_{\bar{q},\bar{\theta}}=C_0\Omega(2^{-\overset{\cong}{s}})\left\|e^{i\langle\bar{k}_{\bar{s}},\bar{x}\rangle}\right\|_{\bar{q},\bar{\theta}}=C_1\Omega(2^{-\bar{s}}).$$

This means that

$$\sup_{f\in S^{\Omega}_{\bar{p},\bar{\tau}}B}\left\|f-S_{Q(N)}(f)\right\|_{\bar{q},\bar{\theta}}\ge C_1\Omega(2^{-\overset{\approx}{s}}),\ \forall\overset{\approx}{s}\in\aleph(N), \tag{16}$$

in the case $2\le p_j<+\infty, \tau_j\le 2, j=1,...,m.$

It is known that $\frac{1}{2N}\le\Omega(2^{-\bar{s}})$ from $\bar{s}\in\aleph(N)$ (see (3)). Therefore from (16) it follows that

$$\sup_{f\in S^{\Omega}_{\bar{p},\bar{\tau}}B}\left\|f-S_{Q(N)}(f)\right\|_{\bar{q},\bar{\theta}}\ge C_1\frac{1}{N},$$

in case $2\le p_j<+\infty, \tau_j\le 2, j=1,...,m.$

Remark. We note that for the case $q_j=\theta_j=q,\ \ p_j=p,\ \ \tau_j=\tau, j=1,...,m$ theorem 3.2 was proved S.A. Stasyuk [14]. For the case $p_j=\theta^{(1)}_j=p,\ q_j=\theta^{(2)}_j=q,\ \ \tau_j=+\infty, j=1,...,m$ theorem 3.1 was proved N.N. Pustovoitov [3].

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Ғ.Ақышев

Лоренц кеңістігінде жалпыланған Никольский–Бесов класын жуықтаудың реті жайлы

Мақалада аралас нормалы Лоренц кеңістігі қарастырылған. Жалпыланған Никольский–Бесов класының анықтамасы берілген. Мақаланың негізгі нәтижелері Никольский–Бесов класының функцияларын гармоникалары гиперболалық кресте жататын еселі Фурье қатарының дербес қосындыларымен жуықтауының реттерін бағалау болып табылады.

Г.Акишев

О порядках приближения обобщенного класса Никольского–Бесова в пространстве Лоренца

В статье рассмотрено пространство Лоренца со смешанной нормой. Дано определение обобщенного класса Никольского–Бесова. Основными результатами статьи являются оценки порядка приближения функции класса Никольского–Бесова частичными суммами кратного ряда Фурье с гармониками из гиперболического креста.

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Adaptation of GIS Open Source Maps to Lanes Graph

This article describes the algorithm for automatic adaptation of a graph of streets taken from open source maps to a graph of traffic lanes. During the work, we encountered several problems, some of which have been left open. They key idea is to generate new edges «to the right» of the edges given in a street map and to connect them together. All solutions are described with a text followed by some programming code.

Key words: GIS, maps, streets, lanes, graphs, traffic simulation.

Introduction

Given a set of vertices and edges that represent streets with their properties like «one-way», «number of lanes» etc., our goal is to produce a proper graph of traffic lanes with connected edges so that it would be possible to find the exact ways of traffic movement.

What are the general problems with the graph? First, all neighbor edges must be connected in a sense that they really need to have a common vertex. Second, the graph must obey some basic traffic rules like it is not allowed to turn right from the second lane or to turn left from the right lane.

The basic approach is as follows:

1. If the street is not one-way, we copy it and swap the direction. Both streets become one-way.
2. Having only one-way streets, we build vertices and edges to the right of this street, as many as the stated number of lanes. By default, we build three lanes.
3. Correcting the graph.

The last step is the most difficult and hence the most interesting part.

Data from Openstreetmap.org. The data available on openstreetmap.org is divided into two basic parts:

1. Node. This object represents a vertex with geographic coordinates.
2. Way — a sequence of nodes and some properties. The ordered sequence of nodes can represent a road (street), building or a park depending on its properties. Moreover, each type of these objects has its own unique properties. For instance, a street has a property «One-way»; a building has a property «Levels».

Reading a way from the database we build a set of edges in our graph. Of course, all the original data is stored in each edge so we can find the corresponding original way in the database in just one step. We skip the simple function that reads an XML file downloaded from openstreetmap.org and creates an array of