

A boundary value problem for the fourth-order degenerate equation of the mixed type

J.A. Otarova

*Karakalpak State University named after Berdakh, Nukus, Uzbekistan
(E-mail: j.otarova@mail.ru)*

Many problems in mechanics, physics, and geophysics lead to solving partial differential equations that are not included in the known classes of elliptic, parabolic or hyperbolic equations. Such equations, as a rule, began to be called non-classical equations of mathematical physics. The theory of degenerate equations is one of the central branches of the modern theory of partial differential equations. This is primarily due to the identification of a variety of applied problems, the mathematical modeling of which serves the study of various types of degenerate equations. The study of boundary value problems for mixed type's equations of the fourth-order with power-law degeneration remains relevant. In this work, a boundary value problem in a rectangular domain for a degenerate equation of the fourth-order mixed-type is posed and investigated. Well-posedness of the boundary value problem for a fourth-order partial differential equation is established by proving the existence and uniqueness of the solution. Under sufficient conditions, a solution to the problem under consideration was explicitly found by the variable separation method.

Keywords: fourth-order mixed type equation, Bessel functions, Fourier series, completeness, regular solution.

2020 Mathematics Subject Classification: 35M12.

Introduction

In the modern theory of partial differential equations, the studies of degenerate equations and equations of a mixed type occupy an important place, which is explained both by the theoretical significance of the results obtained and the presence of their practical applications in the gas dynamics of transonic flows, magnetic hydrodynamics, in the theory of infinitesimal bending of surfaces, in various sections of continuum mechanics and other branches of knowledge.

The fundamental results for second-order degenerate equations of elliptic type were obtained by academician M.V. Keldysh (1951), where first the cases were indicated in which the characteristic part of the domain boundary can be freed from boundary conditions, which are then replaced by the condition of boundedness of solutions. The work of M.V. Keldysh spurred numerous further studies in the direction he indicated. Later, A.V. Bitsadze noted in his work that the boundedness condition can be replaced by a boundary condition with a certain weight function. The results he obtained were then developed and generalized by O.A. Oleinik. It is also worth noting the works of V.P. Glushko, Yu.B. Savchenko [1], S.A. Iskhokov [2], S.N. Sidorov [3]. In particular, the work [2] was focused on the study of the unique solvability of a variational problem for an elliptic equation with a "non-power" degeneration. It is also noteworthy that a "power" degeneration was initiated in the research of M.I. Vishik and V.V. Grushin. Degenerate elliptic equations of high order, including their connection with pseudodifferential operators, were studied in the works of A.D. Baev [4–6].

K.B. Sabitov [7] investigated the Dirichlet problem for the second-order degenerate equation of the mixed type of first kind in a rectangular domain. By the methods of spectral analysis, the criteria of uniqueness of a solution that is constructed in the form of the sum of a Fourier series was established. The question of the correctness of the formulation of the Dirichlet problem depending on

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the degree of degeneracy was investigated for a mixed type equation of second kind by K.B. Sabitov and A.Kh. Tregubova (Suleimanova) [8, 9]. A boundary-value problem with nonlocal boundary conditions for a mixed type equation was studied by M.E. Lerner and O.A. Repin in work [10]. The uniqueness of solutions of the problem was proved by using the principle of extremum, the existence of solutions of the problem was proved by methods of integral transformations and equations. Nonlocal boundary value problems of Bitsadze-Samarsky type for a fourth-order mixed type equation were studied by L.R. Rustamova in [11]. Many authors also studied boundary value problems for degenerate equations [12–14]. In the present paper, the boundary value problem is studied for the fourth-order degenerate equation in a rectangular domain.

1 Problem formulation

In the domain

$$\Omega = \{(x, t) : 0 < x < b, -T < t < T, T > 0\}$$

we consider the equation

$$Lu \equiv \operatorname{sgn} t \cdot |t|^m u_{xxxx} - u_{tt} + a^2 \operatorname{sgn} t \cdot |t|^m u = 0, \tag{1}$$

where $m = \operatorname{const} > 0$, $a = \operatorname{const} \geq 0$. The equation $Lu = 0$ for $t > 0$ has the form

$$|t|^m (u_{xxxx} + a^2 u) - u_{tt} = 0, \tag{2}$$

for $t < 0$

$$|t|^m (u_{xxxx} + a^2 u) + u_{tt} = 0. \tag{3}$$

We denote $\Omega^+ = \Omega \cap (t > 0)$, $\Omega^- = \Omega \cap (t < 0)$.

Problem A. Find in the domain Ω a bounded function $u(x, t)$ satisfying the conditions

$$u(x, t) \in C(\bar{\Omega}) \cap C_{x,t}^{2,1}(\Omega) \cap C_{x,t}^{4,2}(\Omega^+ \cup \Omega^-), \tag{4}$$

$$Lu = 0, \quad (x, t) \in \Omega, \tag{5}$$

$$\frac{\partial^k u}{\partial t^k}(x, +0) = \frac{\partial^k u}{\partial t^k}(x, -0), \quad k = 0, 1 \tag{6}$$

$$\left. \begin{aligned} u(0, t) = u(b, t) = 0, & \quad -T \leq t \leq T, \\ u_{xx}(0, t) = u_{xx}(b, t) = 0, & \quad -T \leq t \leq T, \end{aligned} \right\} \tag{7}$$

$$u(x, T) = \varphi(x), \quad 0 \leq x \leq b, \tag{8}$$

$$u(x, -T) = \psi(x), \quad 0 \leq x \leq b, \tag{9}$$

$\varphi(x)$ and $\psi(x)$ are given sufficiently smooth functions, moreover $\varphi(0) = \varphi(b) = 0$, $\psi(0) = \psi(b) = 0$.

2 The existence of a solution

To prove the existence of a solution to the problem, we use the method of separation of variables, i.e. particular solutions of equation (1) that are not equal to zero in the domain Ω , will be sought in the form of a product $u(x, t) = X(x) \cdot T(t)$, satisfying zero boundary conditions (7). The following theorem holds:

Theorem 1.

$$\begin{cases} \gamma(k) = K_{1/(2q)}(p_k T^q) \neq 0, \\ \delta(k) = \tilde{Y}_{1/(2q)}(p_k T^q) \neq 0. \end{cases} \tag{10}$$

Proof. By substituting this product into equation (1), we obtain

$$X^{IV}(X) - \lambda^4 X(X) = 0, \quad 0 < x < b, \tag{11}$$

we solve equation (11) with conditions (7), which change to the following

$$X(0) = X(b) = X''(0) = X''(b) = 0, \tag{12}$$

$$T''(t) - (\lambda^4 + a^2) \operatorname{sgn} \cdot |t|^m T(t) = 0, \quad -T < t < T, \tag{13}$$

where λ is the separation constant.

The solution to problem (11), (12) has the form

$$X_k(x) = \sqrt{\frac{2}{b}} \sin \lambda_k x, \quad \lambda_k = \frac{k\pi}{b}, \quad k = 1, 2, \dots \tag{14}$$

From equation (13), following [15], (with $\lambda = \lambda_k$) for $t > 0$ we obtain

$$T(t) = W(p_k t^q) \sqrt{t} = \sqrt{t} W(z), \tag{15}$$

in which $q = (m + 2)/2$, $p_k^2 = (a^2 + \lambda_k)/q^2$. Then we obtain the modified Bessel equation [16]

$$W''(z) + \frac{1}{z} W'(z) - \left(1 + \frac{\nu^2}{z^2}\right) W(z) = 0,$$

where $z = p_k t^q$, $\nu = 1/(2q) = 1/(m + 2) \in (0, 1/2)$, the general solution of which is determined by the formula as follows

$$W(z) = C_1 I_{1/(2q)}(z) + C_2 K_{1/(2q)}(z), \quad z > 0, \tag{16}$$

where $I_{1/(2q)}(z)$ and $K_{1/(2q)}(z)$ are the modified Bessel functions and C_1, C_2 are arbitrary constants. Taking into account (15), (16), the general solution (13) for $t > 0$ can be written as

$$T_k^+(t) = A_k \sqrt{t} I_{1/(2q)}(p_k t^q) + B_k \sqrt{t} K_{1/(2q)}(p_k t^q), \quad t > 0, \tag{17}$$

where A_k, B_k are arbitrary constants.

In the same way, from equation (13) for $t < 0$ we get

$$T(t) = Z(p_k (-t)^q) \sqrt{-t} = \sqrt{-t} Z(z), \tag{18}$$

and the Bessel equation

$$Z''(z) + \frac{1}{z} Z'(z) + \left(1 - \frac{\nu^2}{z^2}\right) Z(z) = 0.$$

The general solution is written as

$$Z(z) = C_1 J_{1/(2q)}(z) + C_2 Y_{1/(2q)}(z), \quad z > 0, \tag{19}$$

where $J_{1/(2q)}(z), Y_{1/(2q)}(z)$ are the Bessel functions. Taking into account (18), (19), the general solution of equation (13) for $t < 0$ can be written as

$$T_k^-(t) = C_k \sqrt{-t} J_{1/(2q)}(p_k (-t)^q) + D_k \sqrt{-t} Y_{1/(2q)}(p_k (-t)^q), \quad t < 0,$$

where C_k, D_k are arbitrary constants.

Therefore, the solutions of equation (13) for $t > 0$ have the form of (17), and for $t < 0$ have the form of (21). To find the unknown constants A_k, B_k, C_k, D_k , we use the gluing conditions (6), that respectively change to the following conditions

$$T_k(0+0) = T_k(0-0), \tag{20}$$

$$T_k'(0+0) = T_k'(0-0). \tag{21}$$

Condition (20) is satisfied for any A_k, B_k if $D_k = -\pi B_k/2$, and condition (21) is satisfied for $C_k = \pi B_k ctg(\pi/(4q))/2 - A_k$ and when $D_k = -\pi B_k/2$. Considering all of these, the solution of equation (13) can be written as

$$T_k(t) = \begin{cases} T_k^+(t) = A_k \sqrt{t} I_{1/(2q)}(p_k t^q) + B_k \sqrt{t} K_{1/(2q)}(p_k t^q), & t > 0, \\ T_k^-(t) = -A_k \sqrt{-t} J_{1/(2q)}(p_k (-t)^q) - \frac{1}{2} \pi B_k \tilde{Y}_{1/(2q)}(p_k (-t)^q), & t < 0, \end{cases} \tag{22}$$

where

$$\tilde{Y}_{1/(2q)}(p_k (-t)^q) = \frac{\pi}{2 \sin(\pi/2q)} [J_{1/(2q)}(p_k (-t)^q) + J_{-1/(2q)}(p_k (-t)^q)].$$

For function (22), the equality, $T_k''(0+0) = T_k''(0-0)$, holds i.e. functions (22) belong to the class $C^2[-T; T]$ and satisfy equation (13). Functions (22) are not limited, because $\sqrt{t} I_{1/(2q)}(p_k t^q) \rightarrow \infty$, therefore we assume $A_k = 0, \forall k \in N$, then

$$T_k(t) = \begin{cases} T_k^+(t) = B_k \sqrt{t} K_{1/(2q)}(p_k t^q), & t > 0, \\ T_k^-(t) = B_k \tilde{Y}_{1/(2q)}(p_k (-t)^q), & t < 0. \end{cases} \tag{23}$$

3 The uniqueness of the solution

Theorem 2. If there is a solution to problem A, then it is unique when

$$\lim_{x \rightarrow 0+0} u_{xx}(x, t) \sin \frac{\pi k}{p} x = \lim_{x \rightarrow p-0} u_{xx}(x, t) \sin \frac{\pi k}{p} x = 0, \quad T \leq t \leq -T \tag{24}$$

and if condition (10) is satisfied for all $k \in N$.

Proof. Let $u(x, t)$ be the solution of problem (4)–(9). Consider the functions (14) which form into $L_2(0, b)$ a complete orthonormal system.

We denote

$$u(x, t) = \begin{cases} u^+(x, t), & (x, t) \in \Omega^+, \\ u^-(x, t), & (x, t) \in \Omega^-. \end{cases} \tag{25}$$

We consider the integral

$$\int_0^b u^+(x, t) X_k(x) dx = \alpha_k(t), \quad k = 1, 2, \dots \tag{26}$$

Suppose that the partial derivative $u_{xx}(x, t)$ satisfies conditions (24).

Differentiating (26) with respect to t twice, taking into account equation (2) and conditions (7) we have

$$\alpha_k''(t) - |t|^m \left(a^2 + \left(\frac{\pi k}{b} \right)^4 \right) \alpha_k(t) = 0, \quad k = 1, 2, \dots \tag{27}$$

For negative values of t we denote the integral

$$\int_0^b u^-(x, t) X_k(x) dx = \beta_k(t), \quad k = 1, 2, \dots \tag{28}$$

By a similar transformation from (28) and (3) we obtain

$$\int_0^b u^-(x, t) X_k(x) dx = \beta_k(t), \quad k = 1, 2, \dots \tag{29}$$

Equations (27) and (29) for $\lambda = \lambda_k$ coincide with equation (13), i.e. for $t > 0$, $\alpha_k(t) = T_k^+(t)$, and for $t < 0$, there will be $\beta_k(t) = T_k^-(t)$, which means that functions $\alpha_k(t)$ and $\beta_k(t)$ are determined by functions (25). To find the coefficients B_k , we use the boundary conditions (8), (9), i.e. $\alpha_k(T) = \varphi_k$, $\beta_k(-T) = \psi_k$ (8), (9) and formulas (26), (28), then:

$$\alpha_k(T) = \int_0^b u(x, T) \sin \frac{\pi k}{b} x dx = \int_0^b \varphi(x) \sin \frac{\pi k}{b} x dx = \varphi_k, \tag{30}$$

$$\beta_k(-T) = \int_0^b u(x, -T) \sin \frac{\pi k}{b} x dx = \int_0^b \psi(x) \sin \frac{\pi k}{b} x dx = \psi_k, \tag{31}$$

then from (23), (30) and (31), taking into account condition (10), we have

$$\begin{cases} B_k = \frac{\varphi_k}{\sqrt{T\gamma(k)}}, & t > 0, \\ B_k = \frac{\psi_k}{\sqrt{T\delta(k)}}, & t < 0. \end{cases} \tag{32}$$

Substituting (32) into (23), we find the functions $T_k(t)$:

$$T_k(t) = \begin{cases} \varphi_k \sqrt{\frac{t}{T}} \frac{K_{1/(2q)}(p_k t^q)}{\gamma(k)}, & t > 0, \\ \psi_k \sqrt{\frac{-t}{T}} \frac{\tilde{Y}_{1/(2q)}(p_k (-t)^q)}{\delta(k)}, & t < 0. \end{cases} \tag{33}$$

Let now $\varphi(x) \equiv 0$ and $\psi(x) \equiv 0$. Then, from equalities (30), (31) and solution (33), it follows that $T_k(t) = 0, \forall k \in N$. Therefore, by virtue of (26) and (28)

$$\int_0^b u^+(x, t) X_k(x) dx = 0, \quad k = 1, 2, \dots$$

$$\int_0^b u^-(x, t) X_k(x) dx = 0, \quad k = 1, 2, \dots$$

Hence follows that $u(x, t) \equiv 0$ for all $x \in [0, b]$ and $t \in [-T; T]$, due to the completeness of system (24) in $L_2(0, b)$.

Based on the Bessel asymptotic formula [16], for large k , we have estimate

$$\begin{cases} \left| \sqrt{k} \delta(k) \right| \geq C > 0, \\ \left| \sqrt{k} \gamma(k) \right| \geq C_0 > 0. \end{cases} \tag{34}$$

Under conditions (10) and (34), taking into account (25) and (33), the solution to problem (1), (4)–(9) can be written as

$$u(x, t) = \sum_{k=1}^{\infty} T_k(t) \cdot \sin \frac{\pi k}{b} x. \quad (35)$$

Given (34) and if the functions $\varphi(x) \in C^{3+\gamma}$, $0 < \gamma < T$; $\varphi''(0) = \varphi''(b) = 0$, $\varphi(0) = \varphi(b) = 0$ and $\psi(x) \in C^{3+\delta}$, $-T < \delta < 0$; $\psi(0) = \psi(b) = \psi''(0) = \psi''(b) = 0$, then for φ_k and ψ_k the estimates $|\varphi_k| \leq C_3/k^{3+\gamma}$; $|\psi_k| \leq C_4/k^{3+\delta}$; $C_3, C_4 = \text{const} > 0$ are valid. Then the series (35) converges uniformly in the domain $\bar{\Omega}$ and it can be differentiated term-by-term twice with respect to, t and 4 times with respect to, x . Therefore, for the solution of problem A we have $u(x, t) \in C_{x,t}^{4,2}(\bar{\Omega})$.

Since the constructed solution is $u(x, t) \in C_{x,t}^{4,2}(\bar{\Omega})$, then the condition (24) of Theorem 2 is always satisfied.

Conflict of Interest

The author declares no conflict of interest.

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Төртінші ретті аралас типті өзгешеленетін теңдеу үшін шекаралық есеп

Ж.А. Отарова

Бердах атындағы Қарақалтақ мемлекеттік университеті, Нүкіс, Өзбекстан

Механика, физика және геофизикадағы көптеген есептер эллиптикалық, параболалық немесе гиперболалық теңдеулердің белгілі кластарына кірмейтін дербес туындылы дифференциалдық теңдеулерді шешуге әкеледі. Мұндай теңдеулер әдетте, математикалық физиканың классикалық емес теңдеулері деп аталады. Өзгешеленетін теңдеулер теориясы қазіргі дербес дифференциалдық теңдеулер теориясының негізгі бөлімдерінің бірі. Бұл, ең алдымен, әртүрлі қолданбалы есептерді анықтаумен түсіндіріледі, олардың математикалық модельдеуі өзгешеленетін теңдеулердің әртүрлі түрлерін зерттеуге қолданылады. Осы уақытқа дейін өзгешеленетін теңдеулерге арналған есептер негізінен модельдік теңдеулер мен төменгі мүшелерінің жеткілікті тегіс коэффициенттері бар төменгі мүшелері бар теңдеулер үшін зерттелді. Төртінші ретті аралас типті дәрежелері өзгешеленетін теңдеулер үшін шекаралық есептерді зерттеу өзекті болып қала береді. Мақалада төртінші ретті аралас типті бір өзгешеленетін теңдеу үшін тікбұрышты облыста шекаралық есеп қойылып, зерттелді. Төртінші ретті дербес туындылы дифференциалдық теңдеу үшін шекаралық есептің корректілігі шешімнің бар болуы мен жалғыздығын дәлелдеу арқылы белгіленеді. Жеткілікті шарттар қойылған жағдайда қарастырылған есептің шешімі айнымалыларды бөлу әдісімен табылды.

Кілт сөздер: төртінші ретті аралас теңдеу, Бессель функциялары, Фурье қатары, толықтық, регулярлы шешім.

Краевая задача для вырождающегося уравнения смешанного типа четвёртого порядка

Ж.А. Отарова

Каракалтакский государственный университет имени Бердаха, Нукус, Узбекистан

Многие задачи механики, физики, геофизики приводят к решению уравнений в частных производных, которые не входят в известные классы эллиптических, параболических или гиперболических уравнений. Такие уравнения, как правило, стали называть неклассическими уравнениями математической физики. Теория вырождающихся уравнений является одним из центральных разделов современной теории уравнений с частными производными. Это объясняется, прежде всего, выявлением множества прикладных задач, математическое моделирование которых обслуживает изучение различных типов вырождающихся уравнений. До настоящего времени задачи для вырождающихся уравнений, в основном, исследованы для модельных уравнений и уравнений с младшими членами с достаточно гладкими коэффициентами при младших членах. Исследование краевых задач для уравнений смешанного типа четвертого порядка со степенным вырождением остаётся актуальным. В этой работе поставлена и исследована краевая задача в прямоугольной области для одного вырождающегося уравнения смешанного типа четвертого порядка. Корректность краевой задачи для уравнения с частными производными четвертого порядка устанавливается доказательством существования и единственности решения. При достаточных условиях найдено решение рассматриваемой задачи в явном виде методом разделения переменных.

Ключевые слова: смешанное уравнение четвёртого порядка, функции Бесселя, ряд Фурье, полнота, регулярное решение.

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*Author Information**

Jamila Amanbayevna Otarova — Candidate of physical and mathematical sciences, Associate Professor, Head of the Department of Differential equations, Karakalpak State University named after Berdakh, 1 Chardzhou Abdirov Street, Nukus, 230112, Uzbekistan / Karakalpakstan; e-mail: j.otarova@mail.ru; <https://orcid.org/0000-0003-3661-7937>.

*The author's name is presented in the order: First, Middle and Last Names.