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EXISTENCE OF SOLUTIONS IN BOUNDARY VALUE PROBLEMS FOR NONLINEAR EQUATIONS INVOLVING Q-ANALOGS OF FRACTIONAL DERIVATIVES

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This study begins by presenting the key definitions and preliminary concepts, including the fundamental principles of q -calculus, which form the theoretical foundation of the work. Detailed discussions can be found in (1), (2) and (3). This work operates under the standing assumption that $0 < q < 1$.

Let $\alpha \in \mathbb{R}$. Then a q -real number $[\alpha]_q$ is defined by

$$[\alpha]_q = \frac{1 - q^\alpha}{1 - q},$$

where $\lim_{q \rightarrow 1} \frac{1 - q^\alpha}{1 - q} = \alpha$.

We introduce for $k \in \mathbb{N}$:

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - q^k a), \quad (a; q)_\infty = \lim_{n \rightarrow \infty} (a; q)_n, \quad (a; q)_\alpha = \frac{(a; q)_\infty}{(q^\alpha a; q)_\infty}.$$

For any two real numbers α and β , we have

$$(a - b)_q^\alpha (a - q^\alpha b)_q^\beta = (a - b)_q^{\alpha + \beta}.$$

The gamma function $\Gamma_q(x)$ is defined by

$$\Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty} (1 - q)^{1-x},$$

for any $x > 0$. Moreover, it yields that

$$\Gamma_q(x)[x]_q = \Gamma_q(x + 1).$$

The q -analogue differential operator $D_q f(x)$ is

$$D_q f(x) = \frac{f(x) - f(qx)}{x(1 - q)},$$

and the q -derivatives $D_q^n(f(x))$ of higher order are defined inductively as follows:

$$D_q^0(f(x)) = f(x), \quad D_q^n(f(x)) = D_q(D_q^{n-1}f(x)), \quad (n = 1, 2, 3, \dots).$$

The q -integral of a function f defined in the interval $[0, b]$ is defined by the expression

$$(I_q f)(x) = \int_0^x f(x) d_q x = x(1 - q) \sum_{n=0}^{\infty} f(xq^n) q^n, \quad x \in [0, b].$$

The q -Beta function is defined for any $\alpha, \beta > 0$ as follows:

$$B_q(\alpha, \beta) = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha + \beta)} = \int_0^1 x^{\alpha-1}(1 - qx)_q^{\beta-1} d_q x.$$

Definition 1. (4). Let $\alpha \geq 0$ and f be a function defined on $[0, 1]$. The Riemann-Liouville fractional q -integral is expressed as $(I_q^\alpha f)(x)$, and are defined by

$$(I_{q,0+}^\alpha f)(x) = \frac{1}{\Gamma_q(\alpha)} \int_0^x (x - qs)_q^{(\alpha-1)} f(s) d_q s.$$

Definition 2. The Riemann-Liouville fractional q -derivative of order $\alpha \geq 0$ is defined as $(D_q^\alpha f)(x)$, and

$$(D_q^\alpha f)(x) = (D_q^{[\alpha]} I_q^{[\alpha]-\alpha} f)(x), \quad \alpha > 0,$$

where $[\alpha]$ is the smallest integer greater than or equal to α .

Lemma 1. (5). Let $g(x) \in L[0, 1]$ and $1 < \alpha \leq 2$, then there exists a unique solution for

$$D_{q,0+}^\alpha y(x) + f(x, y(x)) = 0, \quad 0 < x < 1, \quad 0 < q < 1, \tag{1}$$

$$y(0) = 0, \quad D_{q,0+}^\beta y(1) = a D_{q,0+}^\beta y(\tau) + \lambda \tag{2}$$

is

$$y(x) = \int_0^1 G_q(x, s) f(x, y(x)) d_q s + \lambda x^{\alpha-1} \frac{\Gamma_q(\alpha + \beta)}{\Gamma_q(\alpha)},$$

where

$$G_q(x, s) = \begin{cases} \frac{dx^{\alpha-1}(1-qs)_q^{(\alpha-\beta-1)} - adx^{\alpha-1}(\tau-qs)_q^{(\alpha-\beta-1)} - (x-qs)_q^{(\alpha-1)}}{\Gamma_q(\alpha)}, & 0 \leq s \leq \min\{x, \tau\} < 1, \\ \frac{dx^{\alpha-1}(1-qs)_q^{(\alpha-\beta-1)} - (x-qs)_q^{(\alpha-1)}}{\Gamma_q(\alpha)}, & 0 \leq \tau \leq s \leq x \leq 1, \\ \frac{dx^{\alpha-1}(1-qs)_q^{(\alpha-\beta-1)} - adx^{\alpha-1}(\tau-qs)_q^{(\alpha-\beta-1)}}{\Gamma_q(\alpha)}, & 0 \leq x \leq s \leq \tau \leq 1, \\ \frac{dx^{\alpha-1}(1-qs)_q^{(\alpha-\beta-1)}}{\Gamma_q(\alpha)}, & \max\{x, \tau\} \leq s \leq 1, \end{cases}$$

for which $d = (1 - a\tau^{\alpha-\beta-1})^{-1}$, $0 \leq \beta \leq 1$, $0 \leq a \leq 1$, $\lambda \geq \lambda_0 > 0$, $\tau \in (0, 1)$, $a\tau^{\alpha-\beta-2} \leq 1 - \beta$, $0 \leq \alpha - \beta - 1$ and $f : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$.

In this work, we assume that the function

$$f : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$$

the following Caratheodory conditions are satisfied

(H1):

- (i) For every fixed $t \in [0, \infty)$, the mapping $x \mapsto f(x, t)$ is Lebesgue measurable on $[0, 1]$;
(ii) For every fixed $x \in [0, 1]$, the mapping $t \mapsto f(x, t)$ is continuous on $[0, \infty)$.

Theorem 1. Assume that (H1) is satisfied and that there exists a real-valued function $h(x) \in L[0, 1]$ such that

$$|f(x, y) - f(x, z)| \leq h(x)|y - z|,$$

for almost every $x \in [0, 1]$ and all $y, z \in [0, \infty)$. If

$$0 < \int_0^1 G_q(s, s) h(s) d_q s < 1,$$

then, the boundary value problem (BVP) defined by (1) and (2) on $[0, 1]$ admits a unique positive solution.

Lemma 2. Assume that condition (H1) holds and that there exist two nonnegative, real-valued functions m and n in $L[0, 1]$ such that

$$f(x, t) \leq n(x) + m(x)t,$$

for almost all $x \in [0, 1]$ and all $t \in [0, \infty)$. Hence, the operator $T : P \rightarrow P$, as defined in (Lemma 1), is completely continuous.

Theorem 2. Assume that all the conditions of lemma 2 are satisfied. If

$$\int_0^1 G_q(s, s) m(s) d_q s < 1,$$

then the boundary value problem (1) and (2) admits at least one solution.

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