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A different look at the soft topological polygroups

Soft topological polygroups are defined in two different ways. First, it is defined as a usual topology. In the usual topology, there are five equivalent definitions for continuity, but not all of them are necessarily established in soft continuity. Second it is defined as a soft topology including concepts such as soft neighborhood, soft continuity, soft compact, soft connected, soft Hausdorff space and their relationship with soft continuous functions in soft topological polygroups.

Keywords: soft set, soft continuous, soft topological polygroups, soft Hausdorff space, soft open covering, soft compact, soft connected.

1 Introduction

The real world is full of uncertainties. To support these uncertainties, we insert soft sets into mathematical structures. As polygroups have the closest properties to groups among all hyperstructures, we combine polygroups with soft sets and usual topologies, then introduce soft topological polygroups and provide their examples. We are interested in making connections between complete parts in polygroups with closure of soft topological polygroups, continuous function, and usual topology. Then we enter the soft topology and present a combination of the polygroups and the soft sets with the soft topology and another definition of the soft topological polygroups.

The efforts of many scientists were used in this direction, including G. Oguz [1], Heidari et al. [2], Cagman et al. [3], Wang et al. [4], Shah and Shaheen [5], Davvaz [6], Maji [7], Mousarezaei and Davvaz [8], Nuzmul [9], and Hida [10]. Figure 1 shows the relations between polygroups, topology and soft sets, where each item is studied and investigated by many authors.

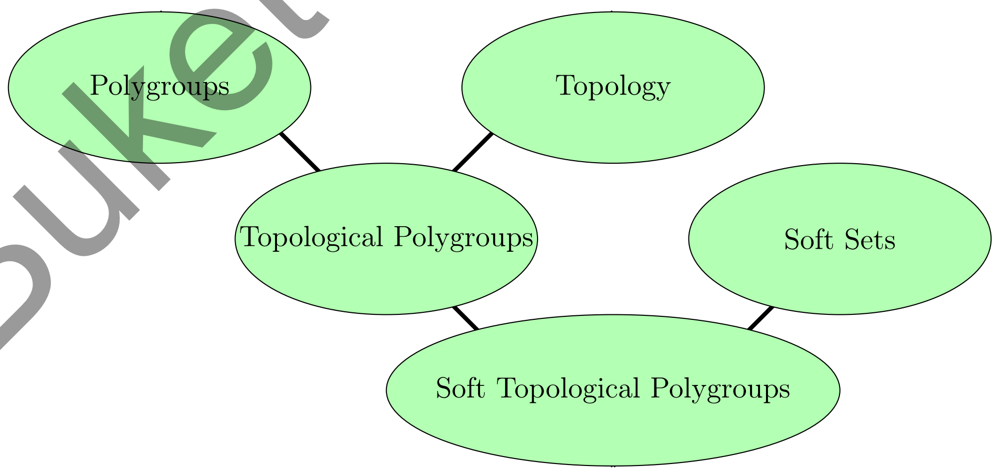

Figure 1. Relations between polygroups, topology, and soft sets

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In [8], R. Mousarezaei and B. Davvaz made a soft topological polygroup over a polygroup. The ideas presented in this article can be used to build more polygroups and more soft topological polygroup.

This paper aims to combine soft sets, topology, and polygroups from different point of view. Also, the concepts of soft neighborhood, soft continuity, soft compact, soft connected, soft Hausdorff space appear and their relationship with soft continuous functions in soft topological polygroups are studied.

To consider soft topological polygroups which represent a generalization of topological polygroups, this paper is constructed as follows: after an introduction, Section 2 contains a brief review of basic definitions related to soft sets and polygroups that are used throughout the paper. Section 3 studies two different definitions of the soft topological polygroup. Attributes are given for each definition along with examples. In continuing the connection between the complete parts and the soft continuous function, soft Hausdorff space, soft $\mathcal{T}_0$ space, soft $\mathcal{T}_1$ space, soft open covering, soft compact, soft connected in soft topological polygroups are studied.

2 Basic definitions

2.1 Soft Sets

Let U be an initial universe, $\mathcal{P}(U)$ denote the power set of U, and $\mathcal{P}^*(U)$ be power set without $\varnothing$. Suppose that E is a set of parameters and A is a non-empty subset of E. A pair $(\mathbb{F}, A)$ is said to be a soft set over U, if $\mathbb{F} : A \to \mathcal{P}(U)$ is a function.

Let $(\mathbb{F}, A)$ and $(\mathbb{G}, B)$ be soft sets over U. In this case, we have the following compliments:

• $(\mathbb{F}, A)$ is a soft subset of $(\mathbb{G}, B)$ and denoted by $(\mathbb{F}, A)\widehat{\subset}(\mathbb{G}, B))$ if $A \subseteq B$ and $\mathbb{F}(a) \subseteq \mathbb{G}(a)$ for all $a \in A$.

• $(\mathbb{F}, A)$ is soft super set of $(\mathbb{G}, B)$ and denoted by $(\mathbb{F}, A)\widehat{\supset}(\mathbb{G}, B)$ If $(\mathbb{G}, B)\widehat{\subset}(\mathbb{F}, A)$.

• $(\mathbb{F}, A)$ is soft equal $(\mathbb{G}, B)$ and denoted by $(\mathbb{F}, A)\widehat{=}(\mathbb{G}, B)$ if $(\mathbb{F}, A)\widehat{\subset}(\mathbb{G}, B)$ and $(\mathbb{G}, B)\widehat{\subset}(\mathbb{F}, A)$.

• $(\mathbb{F}, A)$ is a absolute soft set and denoted by $\widehat{U}$ If $\mathbb{F}(a) = U$ for all $a \in A$. The set $Supp(\mathbb{F}, A) = \{a \in A : \mathbb{F}(a) \neq \varnothing\}$ said to be the support of the soft set $(\mathbb{F}, A)$. A soft set is called non-null if its support is not equal to the empty set.

• $(\mathbb{F}, A)$ is a null soft set and denoted by $\widehat{\varnothing}$ if $\mathbb{F}(a) = \varnothing$ (null set) for all $a \in A$. If A is equal to E we write $\mathbb{F}$ instead of $(\mathbb{F}, A)$.

• Let $\theta : U \longmapsto U'$ be a function and $\mathbb{F}(resp.\mathbb{F}')$ be a soft set over $U(resp.U')$ with a parameter set E. Then $\theta(\mathbb{F})(resp.\theta^{-1}(\mathbb{F}'))$ is the soft set on $U'(resp.U)$ defined by $(\theta(\mathbb{F}))(e) = \theta(\mathbb{F}(e))(resp.(\theta^{-1}(\mathbb{F}'))(e) = \theta^{-1}(\mathbb{F}'(e)))$.

• Use hat $\widehat{(.)}$ to distinguish "soft"objects from usual ones. For example, for a subset X of U, $\widehat{X}$ denotes the soft set satisfying that $\widehat{X}(e) = X$ for all $e \in E$.

• We write $\mathbb{F}_1\widehat{\cap}\mathbb{F}_2$ for the soft intersection of $\mathbb{F}_1$ and $\mathbb{F}_2$, where it is defined by $(\mathbb{F}_1\widehat{\cap}\mathbb{F}_2)(e) = \mathbb{F}_1(e) \cap \mathbb{F}_2(e)$ for every $e \in E$.

• The soft union of $\mathbb{F}_1$ and $\mathbb{F}_2$, will be denoted by $\mathbb{F}_1\widehat{\cup}\mathbb{F}_2$, is defined by $(\mathbb{F}_1\widehat{\cup}\mathbb{F}_2)(e) = \mathbb{F}_1(e) \cup \mathbb{F}_2(e)$ for all $e \in E$.

• We will use the symbol $\mathbb{F}^{\widehat{c}}$ to denote soft complement of $\mathbb{F}$ and is defined by $\mathbb{F}^{\widehat{c}}(e) = U \setminus \mathbb{F}(e)(e \in E)$.

• Let $\mathbb{F}$ be a soft set over U and x be an element of U. We call x is a soft element of $\mathbb{F}$, if $x \in \mathbb{F}(e)$ for all parameters $e \in E$ and denoted by $x\widehat{\in}\mathbb{F}$.

2.2 Polygroups

• Let H be a non-empty set, the couple $(H, \circ)$ is called a hypergroupoid if

$\circ : H \times H \longmapsto \mathcal{P}(H^*)$ be a function, the combination of two subset A and B of H is defined as $A \circ B = \bigcup\limits_{a \in A} a \circ B$ and $a \circ B = \bigcup\limits_{b \in B} a \circ b$.

• A hypergroupoid $(H, \circ)$ is called a quasihypergroup if for every $h \in H$, $h \circ H = H = H \circ h$ and is called a semihypergroup if for every $t, u, w \in H$, $t \circ (u \circ w) = (t \circ u) \circ w$. The pair $(H, \circ)$ is called a hypergroup if it is a quasihypergroup and a semihypergroup [11, 12].

• Let $(H, \circ)$ be a semihypergroup and A be a subset of H. Say that A is a complete part of H if for any $n \in \mathbb{N}$ and for all $a_1, \ldots, a_n$ of H, the following implication true:

$$A \cap \prod_{i=1}^{n} a_i \neq \varnothing \Longrightarrow \prod_{i=1}^{n} a_i \subseteq A.$$

The complete parts were introduced for the first time by Koskas [13].

• Let $(P, \circ)$ be hypergroup and have other additional features. If there exist unitary operation $^{-1}$ on P and $e \in P$ with the property that for all $p, q, r \in P$, the following items be true;

(A) $(p \circ q) \circ r = p \circ (q \circ r)$,

(B) $e \circ p = p \circ e = p$,

(C) If $p \in q \circ r$, then $q \in p \circ r^{-1}$ and $r \in q^{-1} \circ p$.

In this case, hypergroup P is called polygroup.

• The following results follow from the above axioms:

$e \in p \circ p^{-1} \cap p^{-1} \circ p, e^{-1} = e, (p^{-1})^{-1} = p$, and $(p \circ q)^{-1} = q^{-1} \circ p^{-1}$. A nonempty subset Q of a polygroup P is called a subpolygroup of P if and only if for all $x, y \in Q$ follows that $x \circ y \subseteq Q$ and for all $x \in Q$ follows that $x^{-1} \in Q$.

• Let P be polygroup and $(\mathbb{F}, A)$ be a soft set on P. Then $(\mathbb{F}, A)$ is called a (normal)soft polygroup on P if $\mathbb{F}(x)$ is a (normal)subpolygroup of P for all $x \in Supp(\mathbb{F}, A)$.

Example 1. Let P be $\{e, a, b, c\}$ and multiplication table be:

$\circ$	e	a	b	c
e	e	a	b	c
a	a	$\{e, a\}$	c	$\{b, c\}$
b	b	c	e	a
c	c	$\{b, c\}$	a	$\{e, a\}$

Then P is a polygroup. Let $\mathbb{F}, A)$ be a soft set over P, where A equal with P and define $\mathbb{F} : A \longmapsto \mathcal{P}(P)$ by $\mathbb{F}(x) = \{y \in P | xRy \Leftrightarrow y \in x^2\}$ for all $x \in A$. In this case, we will have $\mathbb{F}(e) = \mathbb{F}(b) = \{e\}$ and $\mathbb{F}(a) = \mathbb{F}(c) = \{e, a\}$ are subpolygroups of P. In conclusion, $(\mathbb{F}, A)$ is a soft polygroup over P [4].

3 Soft Topological polygroups

Let $(P, \mathcal{T})$ be a topological space, where $(P, \circ, e, ^{-1})$ is a polygroup. Then the $(P, \mathcal{T})$ is called a topological polygroup if the following axioms hold:

(1) The mapping $\circ : P \times P \longmapsto \mathcal{P}(P)$ is continuous, where $\circ(x, y) = x \circ y$,

(2) The mapping $^{-1} : P \longmapsto P$ is continuous, where $^{-1}(x) = -x$.

Definition 1. [8] Let $\mathcal{T}$ be a topology on a polygroup P. Let $(\mathbb{F}, A)$ be a soft set over P. Then the system $(\mathbb{F}, A, \mathcal{T})$ said to be soft topological polygroup over P if the following axioms hold:

(a) $\mathbb{F}(a)$ is a subpolygroup of P for all $a \in A$,

(b) The mapping $(x, y) \longmapsto x \circ y$ of the topological space $\mathbb{F}(a) \times \mathbb{F}(a)$ onto $\mathcal{P}^*(\mathbb{F}(a))$ and mapping $x \longmapsto x^{-1}$ of the topological space $\mathbb{F}(a)$ onto $\mathbb{F}(a)$ are continuous for all $a \in A$.

Topology $\mathcal{T}$ on P induces topolgies on $\mathbb{F}(a)$, $\mathbb{F}(a) \times \mathbb{F}(a)$ and $\mathcal{P}^*(\mathbb{F}(a))$.

Example 2. Let P be $\{1, 2\}$ and its hyperoperation be as follows:

⋇	1	2
1	1	2
2	2	$\{1,2\}$

Hyperoperation $\circledast : P \times P \longmapsto \mathcal{P}(P)$ and inverse operation $^{-1} : P \longmapsto P$ are continuous with topology $\mathcal{T}_1 = \{\varnothing, P, \{1\}\}$ but $\circledast : P \times P \longmapsto \mathcal{P}(P)$ is not continuous with topology $\mathcal{T}_2 = \{\varnothing, P, \{2\}\}$. Therefore, P with $\mathcal{T}_1$, $\mathcal{T}_{dis}$,$\mathcal{T}_{ndis}$ is topological polygroup. Subpolygroups of P are $\varnothing, P, \{1\}$. Let A be arbitrary set and $a_1, a_2 \in A$ and define soft set $\mathbb{F}$:

$$\mathbb{F}(x) = \begin{cases} \{1\} & \text{if } \ x = a_1, \\ P & \text{if } \ x = a_2, \\ \varnothing & \text{otherwise.} \end{cases}$$

Therefore, $(\mathbb{F}, A, \mathcal{T}_1)$ is a soft topological polygroup.

Example 3. Let $\widehat{D_4}$ be $\{1,2,3,4,5\}$ and its hyperoperation be as follows:

$*$	1	2	3	4	5
1	1	2	3	4	5
2	2	1	4	3	5
3	3	4	1	2	5
4	4	3	2	1	5
5	5	5	5	5	$\{1,2,3,4\}$

Hyperoperation $* : \widehat{D_4} \times \widehat{D_4} \longmapsto \mathcal{P}(\widehat{D_4})$ is not continuous with the following topologies:

$$\begin{array}{ll} \mathcal{T}_1 = \{\varnothing, \widehat{D_4}, \{1\}\} & \mathcal{T}_2 = \{\varnothing, \widehat{D_4}, \{2\}\} \\ \mathcal{T}_3 = \{\varnothing, \widehat{D_4}, \{3\}\} & \mathcal{T}_4 = \{\varnothing, \widehat{D_4}, \{4\}\} \\ \mathcal{T}_5 = \{\varnothing, \widehat{D_4}, \{5\}\} & \mathcal{T}_6 = \{\varnothing, \widehat{D_4}, \{1,2\}\} \\ \mathcal{T}_7 = \{\varnothing, \widehat{D_4}, \{1,3\}\} & \mathcal{T}_8 = \{\varnothing, \widehat{D_4}, \{1,4\}\} \\ \mathcal{T}_9 = \{\varnothing, \widehat{D_4}, \{1,5\}\} & \mathcal{T}_{10} = \{\varnothing, \widehat{D_4}, \{2,3\}\} \\ \mathcal{T}_{11} = \{\varnothing, \widehat{D_4}, \{2,4\}\} & \mathcal{T}_{12} = \{\varnothing, \widehat{D_4}, \{2,5\}\} \\ \mathcal{T}_{13} = \{\varnothing, \widehat{D_4}, \{3,4\}\} & \mathcal{T}_{14} = \{\varnothing, \widehat{D_4}, \{3,5\}\} \\ \mathcal{T}_{15} = \{\varnothing, \widehat{D_4}, \{4,5\}\} & \mathcal{T}_{16} = \{\varnothing, \widehat{D_4}, \{1,2,3\}\} \\ \mathcal{T}_{17} = \{\varnothing, \widehat{D_4}, \{1,2,4\}\} & \mathcal{T}_{18} = \{\varnothing, \widehat{D_4}, \{1,2,5\}\} \\ \mathcal{T}_{19} = \{\varnothing, \widehat{D_4}, \{2,3,4\}\} & \mathcal{T}_{20} = \{\varnothing, \widehat{D_4}, \{2,3,5\}\} \\ \mathcal{T}_{21} = \{\varnothing, \widehat{D_4}, \{2,4,5\}\} & \mathcal{T}_{22} = \{\varnothing, \widehat{D_4}, \{3,4,5\}\} \\ \mathcal{T}_{23} = \{\varnothing, \widehat{D_4}, \{1,2,3,4\}\} & \mathcal{T}_{24} = \{\varnothing, \widehat{D_4}, \{1,2,3,5\}\} \\ \mathcal{T}_{25} = \{\varnothing, \widehat{D_4}, \{2,3,4,5\}\}. & \end{array}$$

This means that $(\widehat{D_4}, \mathcal{T}_{dis})$ and $(\widehat{D_4}, \mathcal{T}_{ndis})$ are topological polygroups. Subpolygroups of $\widehat{D_4}$ are $\varnothing$, $\widehat{D_4}, \{1\}, \{1,2\}, \{1,3\}, \{1,4\}, \{1,2,3,4\}$. Let A be a arbitrary set and $a_1, a_2, a_3, a_4, a_5 \in A$. Then we define a soft set $\mathbb{F}$ by

$$\mathbb{F}(x) = \begin{cases} \{1\} & \text{if } x = a_1 \\ \{1,2\} & \text{if } x = a_2 \\ \{1,3\} & \text{if } x = a_3 \\ \{1,4\} & \text{if } x = a_4 \\ \{1,2,3,4\} & \text{if } x = a_5 \\ \varnothing & \text{otherwise.} \end{cases}$$

Since restriction of topology $\mathcal{T}_5 = \{\varnothing, \widehat{D_4}, \{5\}\}$ to subspaces $\mathbb{F}(x)$ are discrete or anti-discrete topologies $(\mathbb{F}, A, \mathcal{T}_5)$ is a soft topological polygroup. With this method we can make many examples of soft topological polygroups.

Definition 2. Let $(\mathbb{F}, A, \mathcal{T})$ be a soft topological polygroup over P. Then the closure of $(\mathbb{F}, A, \mathcal{T})$ denoted by $(\overline{\mathbb{F}}, A, \mathcal{T})$ and defined as $\overline{\mathbb{F}}(a) = \overline{\mathbb{F}(a)}$ where $\overline{\mathbb{F}(a)}$ is the closure of $\mathbb{F}(a)$ in topology defined on P.

Theorem 1. [2] Let A and B be subsets of a topological polygroup P with the property that every open subset of P is a complete part. Then

(1) $\overline{A} \circ \overline{B} \subseteq \overline{A \circ B}$,

(2) $(\overline{A})^{-1} = \overline{(A^{-1})}$.

Theorem 2. [2] Let P be a topological polygroup with the property that every open subset of P is a complete part. Then

(1) If K is a subsemihypergroup of P, then as well as $\overline{K}$,

(2) If K is a subpolygroup of P, then as well as $\overline{K}$.

Theorem 3. [8] Let $(\mathbb{F}, A, \mathcal{T})$ be a soft topological polygroup over a topological polygroup $(P, \mathcal{T})$ and every open subset of P is a complete part. Then the following are true.

(1) $(\overline{\mathbb{F}}, A, \mathcal{T})$ is also a soft topological polygroup over $(P, \mathcal{T})$,

(2) $(\mathbb{F}, A, \mathcal{T}) \widehat{\subset} (\overline{\mathbb{F}}, A, \mathcal{T})$.

• Now instead of the usual topology we use the soft topology to define the soft topological polygroup based on [10].

• A family θ of soft sets over U is called a soft topology on U if the following axioms hold:

(1) $\widehat{\varnothing}$ and $\widehat{U}$ are in θ,

(2) θ is closed under finite soft intersection,

(3) θ is closed under (arbitrary) soft union.

• We will use the symbol (U, θ, E) to denote a soft topological space and soft set $\mathbb{F}$ is called a soft close set if $\mathbb{F}^{\widehat{c}}$ is soft open set, where each member of θ said to be a soft open set.

Example 4. Let U be $\mathbb{Z}_2$ and θ be $\{\widehat{\varnothing}, \{e_2\} \times \mathbb{Z}_2, \widehat{\mathbb{Z}_2}\}$, where $E = \{e_1, e_2\}$ and $\{e_2\} \times \mathbb{Z}_2$ be soft set $\mathbb{F} : E \longmapsto P(\mathbb{Z}_2)$ with the property that $\mathbb{F}(e_1) = \varnothing; \mathbb{F}(e_2) = \mathbb{Z}_2$. Then $(\mathbb{Z}_2, \theta, E)$ is soft topological space.

Example 5. Let P be $\{1, 2\}$ and hyperoperation $\divideontimes$ be as follows:

$\divideontimes$	1	2
1	1	2
2	2	$\{1,2\}$

polygroup P with topology $\theta = \{\widehat{\varnothing}, \{e_2\} \times P, \widehat{P}\}$ is a soft topological space.

Example 6. Let P be $\{e, a, b, c\}$ and hyperoperation $\circ$ be as follows:

$\circ$	e	a	b	c
e	e	a	b	c
a	a	$\{e,a\}$	c	$\{b,c\}$
b	b	c	e	a
c	c	$\{b,c\}$	a	$\{e,a\}$

polygroup P with topologies $\theta_1 = \{\widehat{\varnothing}, \{e_1\} \times P, \widehat{P}\}$, $\theta_2 = \{\widehat{\varnothing}, \{e_2\} \times P, \widehat{P}\}$ are soft topological spaces.

• The closure of $\mathbb{F}$ is denoted by $\overline{\mathbb{F}}$ and is defined by soft intersection of all soft closed supersets of $\mathbb{F}$, where $\mathbb{F}$ is soft set over U.

• A soft set $\mathbb{F}$ said to be a soft neighborhood of x if there exists a soft open set $\mathbb{G}$ with the property that $x\widehat{\in}\mathbb{G}\widehat{\subseteq}\mathbb{F}$, where x be an element of the universe U. The soft neighborhood system of x we will consider the collection of all soft neighborhoods of x.

• Let V be a subset of the universe U. A soft set $\mathbb{F}$ said to be a soft neighborhood of V if there exists a soft open set $\mathbb{G}$ with the property that $V\widehat{\subseteq}\mathbb{G}\widehat{\subseteq}\mathbb{F}$ (i.e. $\forall e\in E: V\subseteq\mathbb{G}(e)\subseteq\mathbb{F}(e)$).

In this section, we will define soft continuity and express its equivalent theorems, then define soft topological polygroups with the idea of Hida [10] and study the properties of soft topological polygroups.

Definition 3. Let P_1, P_2 be polygroups and $(P_1,\theta_1,E),(P_2,\theta_2,E)$ be soft topological spaces. The function $\varphi:(P_1,\theta_1,E)\longmapsto(P_2,\theta_2,E)$ said to be a soft continuous function if for all $x\in P_1$ and for all soft neighborhood $\mathbb{F}_{\varphi(x)}$ of $\varphi(x)$, there exists a soft neighborhood $\mathbb{F}_x$ of x with the property that $\varphi(\mathbb{F}_x)\widehat{\subseteq}\mathbb{F}_{\varphi(x)}$.

Theorem 4. Let $\varphi:(P_1,\theta_1,E)\longmapsto(P_2,\theta_2,E)$ be a function such that for all soft open set $\mathbb{F}'\in\theta_2$ the inverse image $\varphi^{-1}(\mathbb{F}')$ is soft open set if and only if for every soft closed set $\mathbb{F}'$ the inverse image $\varphi^{-1}(\mathbb{F}')$ is soft closed set.

Proof. This is easily seen to be an equivalence relation.

Theorem 5. Let $\varphi:(P_1,\theta_1,E)\longmapsto(P_2,\theta_2,E)$ be function. For every soft closed set $\mathbb{F}'$, the inverse image $\varphi^{-1}(\mathbb{F}')$ is also soft closed if and only if for all soft set $\mathbb{F}$, $\varphi(\overline{\mathbb{F}})\widehat{\subseteq}\overline{\varphi(\mathbb{F})}$.

Proof.

(i) $\Longleftarrow$ Let $\mathbb{F}'$ be a soft closed set. Then we have $\varphi(\varphi^{-1}(\mathbb{F}'))\widehat{\subseteq}\mathbb{F}'$. The soft closeness of $\mathbb{F}'$, together with the assumption (for all soft set $\mathbb{F}$, we have $\varphi(\overline{\mathbb{F}})\widehat{\subseteq}\overline{\varphi(\mathbb{F})}$, proves that

$$\varphi(\overline{(\varphi^{-1}(\mathbb{F}'))})\widehat{\subseteq}\overline{\varphi(\varphi^{-1}(\mathbb{F}'))}\widehat{\subseteq}\mathbb{F}'.$$

Therefore, it holds that $\overline{\varphi^{-1}(\mathbb{F}')}\widehat{\subseteq}\varphi^{-1}(\mathbb{F}')\widehat{\subseteq}\overline{\varphi^{-1}(\mathbb{F}')}$, which shows that $\varphi^{-1}(\mathbb{F}')$ is soft closed.

(ii) $\Longrightarrow$ We have $\mathbb{F}\widehat{\subseteq}\varphi^{-1}(\overline{\varphi(\mathbb{F})})$ for any soft set $\mathbb{F}$. Since (for every soft closed set $\mathbb{F}'$, the inverse image $\varphi^{-1}(\mathbb{F}')$ is also soft closed, we have $\overline{\mathbb{F}}\widehat{\subseteq}\varphi^{-1}(\overline{\varphi(\mathbb{F})})$. Thus, we have

$$\varphi(\overline{\mathbb{F}})\widehat{\subseteq}\varphi(\varphi^{-1}(\overline{\varphi(\mathbb{F})}))\widehat{=}\overline{\varphi(\mathbb{F})}.$$

Theorem 6. Let $\varphi:(P_1,\theta_1,E)\longmapsto(P_2,\theta_2,E)$ be a function. If for all soft open set $\mathbb{F}'\in\theta_2$, the inverse image $\varphi^{-1}(\mathbb{F}')$ is also soft open set then φ is a soft continuous function.

Proof. For all $x\in P_1$ and a soft open neighborhood $\mathbb{F}'$ of $\varphi(x)$, $\varphi^{-1}(\mathbb{F}')$ is a soft open set having x as a soft element. Since $\varphi(\varphi^{-1}(\mathbb{F}'))\widehat{\subseteq}\mathbb{F}'$, give $F=\varphi^{-1}(\mathbb{F}')$ in this case $\varphi(\mathbb{F})\widehat{\subseteq}\mathbb{F}'$.

Example 7. The opposite Theorem 6 is not true.

Let P_1 be $<\{e\},\theta_1,\{a_1,a_2\}>$ and P_2 be $<\{e\},\theta_2,\{a_1,a_2\}>$, where

$$\theta_1=\{\widehat{\varnothing},\{(a_1,e),(a_2,e)\}\},$$
$$\theta_2=\{\widehat{\varnothing},\{(a_2,e)\},\{(a_1,e),(a_2,e)\}\}.$$

The unique soft neighborhood of the point e is $\{a_1,a_2\}\times\{e\}$ in θ_2. The inverse image of $\{a_1,a_2\}\times\{e\}$ under $id:P_1\longmapsto P_2$ is $\{a_1,a_2\}\times\{e\}$. Thus $id:P_1\longmapsto P_2$ satisisfies in the second part of Theorem 6, but $id^{-1}(\{(a_2,e)\})$ is not soft open in P_1.

Definition 4. A bijection $\varphi:P_1\longmapsto P_2$ said to be a soft homeomorphism between (P_1,θ_1,E) and (P_2,θ_2,E) if φ and φ^{-1} are soft continuous.

Theorem 7. Let $\varphi : (P_1, \theta_1, E) \longmapsto (P_2, \theta_2, E)$ be a soft continuous function. Then for all soft open set $\mathbb{F}_2 \in \theta_2$, there exists a soft open set $\mathbb{F}_1 \in \theta_1$ with the property that for all $x \in P_1$; $x \widehat{\in} \mathbb{F}_1$ if and only if $x \widehat{\in} \varphi^{-1}(\mathbb{F}_2)$.

Proof. For every $x \in P_1$ with $\varphi(x) \widehat{\in} \mathbb{F}_2$, choose a soft open $\mathbb{F}_x \in \theta_1$ with the property that $x \widehat{\in} \mathbb{F}_x$ and $\varphi(\mathbb{F}_x) \widehat{\subseteq} \mathbb{F}_2$. Then define $\mathbb{F}_1 = \widehat{\bigcup}\{\mathbb{F}_x | x \in P_1, \varphi(x) \widehat{\in} \mathbb{F}_2\}$, $\mathbb{F}_1$; has the required properties.

Definition 5. Let $(P, \circ, e, ^{-1})$ be a polygroup and θ be a soft topology on P with a parameter set E. Then (P, θ, E) is a soft topological polygroup if the following items are true:

(i) For each soft neighborhood $\mathbb{F}_{p \circ q}$ of $p \circ q$, where $(p, q) \in P \times P$ there exist soft neighborhoods $\mathbb{F}_p$ and $\mathbb{F}_q$ of p and q with the property that $\mathbb{F}_p \circ \mathbb{F}_q \widehat{\subseteq} \mathbb{F}_{p \circ q}$,

(ii) The inversion function $^{-1} : P \longmapsto P$ is soft continuous.

Every soft topological group is a soft topological polygroup.

Example 8. Let E be $\{e_1, e_2\}$ and θ be $\{\widehat{\varnothing}, \{(e_1, \overline{1})\}, \widehat{\mathbb{Z}_2}\}$. In conclusion, $(\mathbb{Z}_2, \theta, E)$ is a soft topological polygroup.

Example 9. Show that $(\mathbb{R}, \theta, E)$ is a soft topological group, where $E = \{e_1, e_2\}$ and θ is the soft topology generated by the following subbase:

$$\{\widehat{\varnothing}, \widehat{\mathbb{R}}\} \cup \{\{(e_1, r), (e_2, x) | r - \epsilon < x < r + \epsilon\} | r \in \mathbb{R}, \epsilon > 0\}.$$

Example 10. Every polygroup with discrete or anti-discrete topology is soft topological polygroup.

Example 11. Let P be $\{e, a, b, c\}$ and hyperoparation $\circ$ be as follow:

$\circ$	e	a	b	c
e	e	a	b	c
a	a	$\{e, a\}$	c	$\{b, c\}$
b	b	c	e	a
c	c	$\{b, c\}$	a	$\{e, a\}$

Let E be $\{e_1, e_2, e_3\}$. Then polygroup P with topologies

$$\begin{aligned}
&\theta_1 = \{\widehat{\varnothing}, \{e_1\} \times P, \widehat{P}\} \\
&\theta_2 = \{\widehat{\varnothing}, \{e_2\} \times P, \widehat{P}\} \\
&\theta_3 = \{\widehat{\varnothing}, \{e_3\} \times \{a, b\}, \widehat{P}\} \\
&\theta_3 = \{\widehat{\varnothing}, \{e_3\} \times \{a, b\}, \{e_1\} \times \{e, b\}, \widehat{P}\} \\
&\theta_4 = \{\widehat{\varnothing}, \{e_3\} \times \{a, b\}, \{e_1\} \times \{e, b\}, \{e_2\} \times \{e, b, c\}, \widehat{P}\}
\end{aligned}$$

are soft topological polygroups.

Theorem 8. (P, θ, E) is a soft topological polygroup if and only if for all $x, y \in P$ and for each soft open set $\mathbb{F}$ with $x \circ y^{-1} \widehat{\subseteq} \mathbb{F}$, there exist soft open sets $\mathbb{F}_x, \mathbb{F}_y$ with the property that $x \widehat{\in} \mathbb{F}_x, y \widehat{\in} \mathbb{F}_y$ and $\mathbb{F}_x \circ \mathbb{F}_y^{-1} \widehat{\subseteq} \mathbb{F}$.

Proof. $\Longrightarrow$ From item (i) of Definition 5 we know that there exist soft open sets $\mathbb{F}_x, \mathbb{F}_{y^{-1}}$ with the property that $x \widehat{\in} \mathbb{F}_x, y^{-1} \widehat{\in} \mathbb{F}_{y^{-1}}$ and $\mathbb{F}_x \circ \mathbb{F}_{y^{-1}} \widehat{\subseteq} \mathbb{F}$. From item (ii) Definition 5 we know that there exists a soft open set $\mathbb{F}_y$ satisfying $y \widehat{\in} \mathbb{F}_y$ and $(\mathbb{F}_y)^{-1} \widehat{\subseteq} \mathbb{F}_{y^{-1}}$. In conclusion, $x \widehat{\in} \mathbb{F}_x, y \widehat{\in} \mathbb{F}_y$ and $\mathbb{F}_x \circ (\mathbb{F}_y)^{-1} \widehat{\subseteq} \mathbb{F}_x \circ \mathbb{F}_{y^{-1}} \widehat{\subseteq} \mathbb{F}$. $\Longleftarrow$ Let $\mathbb{F}$ be a soft open set with the property that $x^{-1} \widehat{\in} \mathbb{F}$. Since $x^{-1} = e \circ x^{-1}$, there exist soft open sets $\mathbb{F}_e, \mathbb{F}_x$ with the property that $e \widehat{\in} \mathbb{F}_e, x \widehat{\in} \mathbb{F}_x$ and $\mathbb{F}_e \circ \mathbb{F}_x^{-1} \widehat{\subseteq} \mathbb{F}$. In conclusion, $\mathbb{F}_x^{-1} \widehat{\subseteq} \mathbb{F}_e \circ \mathbb{F}_x^{-1} \widehat{\subseteq} \mathbb{F}$, which that the item (ii) of Definition 5 is proved. Let $\mathbb{F}$ be a soft open set satisfying $x \circ y \widehat{\subseteq} \mathbb{F}$. Since $x \circ y = x \circ (y^{-1})^{-1}$, we can find soft open sets $\mathbb{F}_x, \mathbb{F}_{y^{-1}}$ with the property that $x \widehat{\in} \mathbb{F}_x, y^{-1} \widehat{\in} \mathbb{F}_{y^{-1}}$ and $\mathbb{F}_x \circ (\mathbb{F}_{y^{-1}})^{-1} \widehat{\subseteq} \mathbb{F}$. Since $^{-1} : P \longmapsto P$ is soft continuous, we can find a soft open set $\mathbb{F}_y$ with the property that $y \widehat{\in} \mathbb{F}_y$ and $\mathbb{F}_y^{-1} \widehat{\subseteq} \mathbb{F}_{y^{-1}}$. In conclusion, $\mathbb{F}_x \circ \mathbb{F}_y \widehat{=} \mathbb{F}_x \circ ((\mathbb{F}_y)^{-1})^{-1} \widehat{\subseteq} \mathbb{F}_x \circ (\mathbb{F}_{y^{-1}})^{-1} \widehat{\subseteq} \mathbb{F}$.

Definition 6. Let (P, θ, E) be a soft topological polygroup and for all $x, y \in P$ with $x \neq y$, there exists a soft open set $\mathbb{F}$ with the property that either $x \widehat{\in} \mathbb{F} \bigwedge \forall e \in E(y \notin \mathbb{F}(e))$ or $y \widehat{\in} \mathbb{F} \bigwedge \forall e \in E(x \notin \mathbb{F}(e))$ holds, then (P, θ, E) is a soft $\mathcal{T}_0 space$.

Theorem 9. Let $\varphi : (P_1, \theta_1, E) \longmapsto (P_2, \theta_2, E)$ be a soft continuous injection and (P_2, θ_2, E) be a soft $\mathcal{T}_0 space$. Then (P_1, θ_1, E).

Proof. Let (P_2, θ_2, E) be a soft $\mathcal{T}_0 space$ and φ be a soft continuous injection, if $x, y \in P_1$ and $x \neq y$ then $\varphi(x), \varphi(y) \in P_2$ and $\varphi(x) \neq \varphi(y)$, on the other hand, (P_2, θ_2, E) is a soft $\mathcal{T}_0 space$ hence there exists a soft open set $\mathbb{F}$ with the property that $\varphi(x) \widehat{\in} \mathbb{F}$ or $\varphi(y) \widehat{\in} \mathbb{F}$. Without loss of generality let $\varphi(x) \widehat{\in} \mathbb{F}$ then since φ is continuous, there exists $\mathbb{F}_x \in \theta_1$ with the property that $x \widehat{\in} \mathbb{F}_x$ and $\varphi(\mathbb{F}_x) \widehat{\subseteq} \mathbb{F}$ hence $x \widehat{\in} \mathbb{F} \bigwedge \forall e \in E(y \notin \mathbb{F}(e))$.

Definition 7. Let (P, θ, E) be a soft topological polygroup and for every distinct points $x_1, x_2 \in P$, there exist soft open sets $\mathbb{F}_1, \mathbb{F}_2$ with the property that both $x_1 \widehat{\in} \mathbb{F}_1 \bigwedge \forall e \in E(x_2 \notin \mathbb{F}_1(e))$ and $x_2 \widehat{\in} \mathbb{F}_2 \bigwedge \forall e \in E(x_1 \notin \mathbb{F}_2(e))$ hold, then (P, θ, E) is a soft $\mathcal{T}_1 space$.

Theorem 10. Let $\varphi : (P_1, \theta_1, E) \longmapsto (P_2, \theta_2, E)$ be a soft continuous injection and (P_2, θ_2, E) be a soft $\mathcal{T}_1 space$. Then (P_1, θ_1, E).

Proof. It is simillar to the proof of Theorem 9.

Definition 8. Let $\varphi : (P_1, \theta_1, E) \longmapsto (P_2, \theta_2, E)$ be a soft continuous injection and for every distinct elements $x_1, x_2 \in P$, there exist soft open sets $\mathbb{F}_1, \mathbb{F}_2 \in \theta$ with $x_1 \widehat{\in} \mathbb{F}_1, x_2 \widehat{\in} \mathbb{F}_2$ and $\mathbb{F}_1 \widehat{\cap} \mathbb{F}_2 = \widehat{\varnothing}$, then (P, θ, E) is a soft Hausdorff (or soft $\mathcal{T}_2 space$ [14]).

Example 12. Let $\mathbb{R}$ be real number, E be $\{e_1, e_2\}$, and θ be the soft topology generated by the following subbase:

$$\{\widehat{\varnothing}, \widehat{\mathbb{R}}\} \cup \{\{(e_1, r), (e_2, x) | r - \epsilon < x < r + \epsilon\} | r \in \mathbb{R}, \epsilon > 0\}.$$

It can be shown that $(\mathbb{R}, \theta, E)$ is a soft Hausdorff space.

Theorem 11. Let $\varphi : (P_1, \theta_1, E) \longmapsto (P_2, \theta_2, E)$ be a soft continuous injection and (P_2, θ_2, E) be a soft Hausdorff space. Then (P_1, θ_1, E).

Proof. Take distinct elements x and y from P_1. We can separate $\varphi(x)$ from $\varphi(y)$ by soft open sets, $\mathbb{F}_{\varphi(x)}, \mathbb{F}_{\varphi(y)}$. Since φ is soft continuous, we have soft open neighborhoods $\mathbb{F}_x, \mathbb{F}_y$ of x, y, respectively, satisfying that $\varphi(\mathbb{F}_x) \widehat{\subseteq} \mathbb{F}_{\varphi(x)}$ and $\varphi(\mathbb{F}_y) \widehat{\subseteq} \mathbb{F}_{\varphi(y)}$. Clearly $\mathbb{F}_x$ and $\mathbb{F}_y$ separate x from y.

Definition 9. Let (P, θ, E) be a soft topological polygroup and for every $x \in P$, every soft neighborhood of x contains a soft closed neighborhood of x, then (P, θ, E) is a soft regular space [14].

Example 13. Let P be $\mathbb{Z}_2$ and E be $\{e_1, e_2, e_3\}$. Define soft topology θ on P by $\theta = \{\widehat{\varnothing}, \{(e_2, \overline{1})\}, \widehat{P}\}$. Since in soft topological polygroup, every point $x \in P$ has only one soft clopen neighborhood $\widehat{P}$, it is obvious that soft space is soft regular.

Definition 10. A family C of soft open sets over P is said to be a soft open covering of P if for all $x \in P$ there exists an $\mathbb{F} \in C$ with the property that $x \widehat{\in} \mathbb{F}$.

Definition 11. A soft space (P, θ, E) is soft compact if for any soft open covering C of (P, θ, E), there exist $\mathbb{F}_1, ..., \mathbb{F}_n \in C$ with the property that $\{\mathbb{F}_1, ..., \mathbb{F}_n\}$ is a soft open covering.

Theorem 12. If V is soft compact with respect to θ_1, then so is $\varphi(V)$ with respect to θ_2, where $\varphi : (P_1, \theta_1, E) \longmapsto (P_2, \theta_2, E)$ be a soft continuous function and $V \subseteq P_1$ be a subset.

Proof. Let C' be a soft open covering of $\varphi(V)$. For every $v \in V$, there exists an $\mathbb{G}'_{\varphi(v)} \in C'$ with the property that $\varphi(v) \widehat{\in} \mathbb{G}'_{\varphi(v)}$. Since φ is soft continuous, there exists a soft open neighborhood $\mathbb{G}_v$ of v with the property that $\varphi(\mathbb{G}_v) \widehat{\subseteq} \mathbb{G}'_{\varphi(v)}$. Then the family $\{\mathbb{G}_v | v \in V\}$ is a soft covering of V. V is soft compact with respect to θ_1, there exist $v_1, ..., v_n \in V$ with the property that $\{\mathbb{G}_{v_i}\}_{i=1}^n$ is a soft covering of V. Thus we have $\varphi(\mathbb{G}_{v_i}) \widehat{\subseteq} \mathbb{G}'_{\varphi(v_i)}$, in conclusion $\{\mathbb{G}'_{\varphi(v_i)}\}_{i=1}^n$ is a soft covering of $\varphi(V)$.

Definition 12. If for any soft open covering $\mathbb{F}_1, \mathbb{F}_2$ of X subject to the condition that $\nexists x \in X(x \widehat{\in} \mathbb{F}_1 \bigwedge x \widehat{\in} \mathbb{F}_2)$, either $\forall x \in X(x \widehat{\notin} \mathbb{F}_1)$ or $\forall x \in X(x \widehat{\notin} \mathbb{F}_2)$ holds, then subset X of P said to be soft connected.

Example 14. Let E be $\{e_1, e_2\}$ and θ be $\{\widehat{\varnothing}, \{(e_1, \overline{1}), (e_2, \overline{0})\}, \{(e_1, \overline{0}), (e_2, \overline{1})\}, \widehat{\mathbb{Z}_2}\}$ on $\mathbb{Z}_2$. In this case, since both $\overline{0}$ and $\overline{1}$ have only one soft neighborhood $\widehat{\mathbb{Z}_2}$ the soft topological polygroup $(\mathbb{Z}_2, \theta, E)$ is soft connected.

Example 15. Let $\mathbb{Z}$ be integer numbers and consider soft topological polygroup $(\mathbb{Z}, \theta, E)$ where $E = \{e_1, e_2\}$ and $\theta = \{\widehat{\varnothing}, \widehat{\mathbb{Z}}\} \cup \{\{e_1\} \times S | S \subseteq \mathbb{Z}\}$. Since $\widehat{\mathbb{Z}}$ is the unique soft neighborhood of $i \in \mathbb{Z}$ thus $(\mathbb{Z}, \theta, E)$ is soft compact and also soft connected.

Theorem 13. If both X_1 and X_2 are soft connected then $X_1 \cup X_2$ is soft connected, where X_1 and X_2 are subsets of P having non-empty intersection.

Proof. Suppose that $\{\mathbb{F}_1, \mathbb{F}_2\}$ be soft open covering of $X_1 \cup X_2$ Since $\{\mathbb{F}_1, \mathbb{F}_2\}$ is a soft open covering also of X_1. Without loss of generality assume $\forall z \in X_1(z \widehat{\notin} \mathbb{F}_1)$. In particular, $x \widehat{\notin} \mathbb{F}_1$ holds for every $x \in X_1 \cap X_2$. Suppose that for the contradiction that there were a $z \in X_2$ with the property that $z \widehat{\in} \mathbb{F}_1$. Since $\{\mathbb{F}_1, \mathbb{F}_2\}$ is a soft open covering of X_2, it would hold that $\forall z \in X_2(z \widehat{\notin} \mathbb{F}_2)$. In particular, $x \widehat{\notin} \mathbb{F}_2$ holds for every $x \in X_1 \cap X_2$, which gives a contradiction. Therefore, $\forall z \in X_2(z \widehat{\notin} \mathbb{F}_1)$ holds; so has obtained $\forall z \in X_1 \cup X_2(z \widehat{\notin} \mathbb{F}_1)$.

Theorem 14. If $\varphi : (P_1, \theta_1, E) \longmapsto (P_2, \theta_2, E)$ is a soft continuous and X subset of P is soft connected, then $\varphi(X)$ is soft connected.

Proof. Let X be a soft connected subset of P_1. Select an arbitrary soft open covering $\{\mathbb{F}'_1, \mathbb{F}'_2\}$ of $\varphi(x)$. By Theorem 7, there exist soft open sets $\mathbb{F}_i \in \theta_1$ with the property that $\varphi(\mathbb{F}_i) \widehat{\subseteq} \mathbb{F}'_i$ and $\forall y \in P_2(y \widehat{\in} \mathbb{F}'_i$ if and only if $y \widehat{\in} \varphi(\mathbb{F}_i))(i = 1, 2)$. Thus for every $x \in X$, exactly one of $x \widehat{\in} \mathbb{F}_1$ and $x \widehat{\in} \mathbb{F}_2$ holds. Since X is soft connected, there is no loss of generality in assuming $\forall x \in X(x \widehat{\notin} \mathbb{F}_1)$. Therefore have $\forall y \in \varphi(X)(y \widehat{\notin} \varphi(\mathbb{F}_1))$, and $\forall y \in \varphi(X)(y \widehat{\notin} \mathbb{F}'_1)$.

Theorem 15. If $\varphi : (P_1, \theta_1, E) \longmapsto (P_2, \theta_2, E)$ is a soft homeomorphism, then $X \subseteq P$ is soft connected if and only if $\varphi(X)$ is soft connected.

Proof. We have showed a left-to-right direction in the previous theorem. The converse direction follows from the same argument applied to φ^{-1}.

Definition 13. We say that Property $\mathcal{P}$ of soft topological polygroups is a soft topological property if the following condition holds for any soft space (P, θ, E).

A soft space (P, θ, E) has the property $\mathcal{P}$ if and only if every soft space which is soft homeomorphic to (P, θ, E) has the property $\mathcal{P}$.

Theorem 16. Soft compactness and soft connectedness are soft topological properties.

Definition 14. For every soft topological polygroups (P_1, θ_1, E) and (P_2, θ_2, E), the set $\{\mathbb{F}_1 \times \mathbb{F}_2 | \mathbb{F}_1 \in \theta_1, \mathbb{F}_2 \in \theta_2\}$ generates a soft topology $\theta^{\times}$ over $P_1 \times P_2$. The soft space $(P_1 \times P_2, \theta^{\times}, E)$ said to be the soft product of (P_1, θ_1, E) and (P_2, θ_2, E), where $\mathbb{F}_1 \times \mathbb{F}_2$ is the soft set on $P_1 \times P_2$ defined by $(\mathbb{F}_1 \times \mathbb{F}_2)(e) := \mathbb{F}_1(e) \times \mathbb{F}_2(e)$ for every $e \in E$.

Theorem 17. The soft product of every two soft $\mathcal{T}_0$*spaces* is a soft $\mathcal{T}_0$*space*.

Proof. Suppose that (P, θ, E) and (P', θ', E) are soft $\mathcal{T}_0$*spaces*. Take distinct points $(x, x'), (y, y') \in P \times P'$. Without loss of generality, suppose that $x \neq y$. Since (P, θ, E) is a soft $\mathcal{T}_0$*space*, there exists a soft open set $\mathbb{F}$ with the property that either $x \widehat{\in} \mathbb{F} \wedge \forall e \in E(y \widehat{\notin} \mathbb{F}(e))$ or $y \widehat{\in} \mathbb{F} \wedge \forall e \in E(x \widehat{\notin} \mathbb{F}(e))$ holds. Thus we have:

$$(x, x') \widehat{\in} \mathbb{F} \times \widehat{P} \wedge \forall e \in E((y, y') \notin \mathbb{F}(e) \times P)$$

or

$$(y, y')\widehat{\in}\mathbb{F} \times \widehat{P} \wedge \forall e \in E((x, x') \notin \mathbb{F}(e) \times P).$$

Theorem 18. The soft product of every two soft $\mathcal{T}_1 spaces$ is a soft $\mathcal{T}_1 space$.

Proof. It is clear.

Theorem 19. The soft product of every two soft Hausdorff spaces is a soft Hausdorff space.

Proof. It is clear.

Theorem 20. Let h be an element of a polygroup P and (P, θ, E) be soft topological polygroup. Then:

(i) $\varphi_L(h) : P \longmapsto \{h \circ x\}_{x \in P}; x \longmapsto h \circ x(\varphi_R(h) : P \longmapsto \{h \circ x\}_{x \in P}; x \longmapsto x \circ h)$ is a soft homeomorphism.

(ii) $\varphi(h) : P \longmapsto \{h \circ x \circ h^{-1}\}_{x \in P}; x \longmapsto h \circ x \circ h^{-1}$ is a soft homeomorphism.

Proof. For every $h, x \in P$ and a soft neighborhood $\mathbb{F}$ of $h \circ x$, by the definition of soft topological polygroup, there exist soft neighborhood $\mathbb{F}_h$ and $\mathbb{F}_x$ of h and x with the property that $\mathbb{F}_h \circ \mathbb{F}_x \widehat{\subseteq} \mathbb{F}$. Thus, we have $\varphi_L(h)(\mathbb{F}_x) = (h \circ \mathbb{F}_x)\widehat{\subseteq}\mathbb{F}_h \circ \mathbb{F}_x \widehat{\subseteq}\mathbb{F}$, in conclusion $\varphi_L(h)$ is soft continuous. Since $\varphi_L(h)$ is soft continuous for each $h \in P$, for both h and h^{-1}, the first case follows at once by $(\varphi_L(h))^{-1} = \varphi_L(h^{-1})$. The second case can be proved similarly.

Theorem 21. For every soft topological polygroup (P, θ, E), the following items are equivalent:

(i) (P, θ, E) is a soft $\mathcal{T}_0 space$,

(ii) (P, θ, E) is a soft $\mathcal{T}_1 space$,

(iii) (P, θ, E) is a soft Hausdorff space.

Proof. $(i) \Longrightarrow (ii)$ We prove that $\widehat{\{e\}}$ is soft closed. For this, note that every $x \neq e$ can be separated from e by a soft open set. Take an $x \in p \setminus \{e\}$ arbitrarily. By the item (i), there exists a soft open set $\mathbb{F}$ with the property that either $x\widehat{\in}\mathbb{F} \wedge \forall e \in E(e \notin \mathbb{F}(e))$ or $e\widehat{\in}\mathbb{F} \wedge \forall e \in E(x \notin \mathbb{F}(e))$ holds. If the first happens, it is done. In the second property, the soft continuity of $\varphi_L(x)$ and the inversion $^{-1} : p \longmapsto P$ guarantees the existence of a soft set $\mathbb{F}'$ satisfying that $x\widehat{\in}\mathbb{F}'$ and $x \circ (\mathbb{F}')^{-1}\widehat{\subseteq}\mathbb{F}$. Thus, we have $\mathbb{F}'\widehat{\subseteq}\mathbb{F}^{-1} \circ x$. If e were in $\mathbb{F}'(e)$ for some $e \in E$, then we would have $e \in u^{-1} \circ x$ for some $u \in \mathbb{F}(e)$. Thus x is equal to $u(\in \mathbb{F}(e))$, contradicting the assumption that $\forall e \in E(x \notin \mathbb{F}(e))$. Therefore, e is not in $\mathbb{F}'(e)$ for any $e \in E$, in conclusion $\widehat{\{e\}}\widehat{\cap}\mathbb{F}' = \widehat{\varnothing}$ holds for this soft neighborhood $\mathbb{F}'$ of x. Take every distinct x, y from P. Since $x^{-1} \circ y$ is a soft subset of a soft open set $\{e\}^{\widehat{c}}$, the soft continuity of $\varphi_L(x^{-1})$ implies the existence of a soft open set $\mathbb{F}$ with the property that $y\widehat{\in}\mathbb{F}$ and $x^{-1} \circ \mathbb{F}\widehat{\subseteq}\{e\}^{\widehat{c}}$. In conclusion, this soft open set $\mathbb{F}$ satisfies $\forall e \in E(x \notin \mathbb{F}(e))$.

$(ii) \Longrightarrow (i)$ it is clear.

$(ii) \Longrightarrow (iii)$ Take $x \neq y$ from P. Since $e \neq x^{-1} \circ y$, item (ii) implies that $\widehat{\{x^{-1} \circ y\}}^{\widehat{c}}$ is soft open. Take a soft neighborhood $\mathbb{F}$ of e with the property that $\mathbb{F} \circ \mathbb{F}^{-1}\widehat{\subseteq}\widehat{\{x^{-1} \circ y\}}^{\widehat{c}}$. Suppose that for the contradiction, for some $e \in E$, the soft sets $x \circ \mathbb{F}(e)$ and $y \circ \mathbb{F}(e)$ had a common element, say g. Take $g \in x \circ h, g \in y \circ k$ for $h, k \in \mathbb{F}(e)$. However, then we would have

$g^{-1} \in h^{-1} \circ x^{-1}, g \in y \circ k$ Thus $e \in h^{-1} \circ x^{-1} \circ y \circ k$ then $h \in x^{-1} \circ y \circ k$ hence $h \circ k^{-1} \subseteq x^{-1} \circ y$ and $h \circ k^{-1} \subseteq \mathbb{F}(e) \circ \mathbb{F}(e)^{-1}$ and $\mathbb{F}(e) \circ \mathbb{F}(e)^{-1} \subseteq \{x^{-1} \circ y\}^{\widehat{c}}$ then $(h \circ k^{-1}) \subseteq (x^{-1} \circ y) \cap \{x^{-1} \circ y\}^{\widehat{c}}$. This is a contradiction. In conclusion $x \circ \mathbb{F}\widehat{\cap} y \circ \mathbb{F}\widehat{=}\widehat{\varphi}$. The soft continuity of $\varphi_L(x^{-1})(resp.\varphi_L(y^{-1}))$, presents a soft open $\mathbb{F}_x(resp.\mathbb{F}_y)$ with the property that $x\widehat{\in}\mathbb{F}_x\widehat{\subseteq}x \circ \mathbb{F}(resp. y\widehat{\in}\mathbb{F}_y\widehat{\subseteq}y \circ \mathbb{F}$. Obviously, $\mathbb{F}_x$ and $\mathbb{F}_y$ are soft disjoint, as

$$\mathbb{F}_x\widehat{\cap}\mathbb{F}_y\widehat{\subseteq}x \circ \mathbb{F}\widehat{\cap}y \circ \mathbb{F}\widehat{=}\widehat{\varphi}.$$

$(iii) \Longrightarrow (ii)$ it is clear.

Definition 15. The soft connected component of x, is the largest soft connected subset of P containing x, for every $x \in P$.

Definition 16. The soft connected component of P is the soft connected component of $e \in P$.

Theorem 22. Let N_e be the soft connected component of P. Then, for each $q \in P$ the connected component of q is $q \circ N_e$.

Proof. If N_q be The soft connected component of q by Theorems 14 and 20 $q \circ N_e \subseteq N_q$ since $q \circ N_e$ is a soft connection containing of q. Notably $N_e \circ q \subseteq N_q$ Nonetheless, $N_e \subseteq N_q \circ q^{-1} \subseteq N_e$ then $N_e = N_q \circ q^{-1}$ in conclusion $N_e \circ q = N_q$.

Theorem 23. If N is a soft connected component of P, then N is normal subpolygroup of P.

Proof. Suppose that $a, b \in N$ Since $^{-1} : P \longmapsto P$ is soft homeomorphism and $\varphi_L(h) : P \longmapsto \{h \circ x\}_{x \in P}; x \longmapsto h \circ x (resp. \varphi_R(h) : P \longmapsto \{x \circ h\}_{x \in P}; x \longmapsto x \circ h)$ are soft continuous. By Theorems 14 and 15 $a \circ N^{-1}$ is soft connected. Since $a \circ N^{-1}$ contains $e \in a \circ a^{-1}$, we have $a \circ N^{-1} \subseteq N$. Obviously, $a \circ b^{-1} \subseteq a \circ N^{-1}$, thus have $a \circ b^{-1} \subseteq a \circ N^{-1} \subseteq N$. This proves that N is a subpolygroup of P. Note that both $a^{-1} \circ N \circ a$ and $a \circ N \circ a^{-1}$ are soft connected, and contain e. Above all N is the largest soft connected subset containing e, we have $a^{-1} \circ N \circ a$ and $a \circ N \circ a^{-1}$ are a subset of N, in conclusion, N is normal subpolygroup of P.

Theorem 24. Let H and K be soft connected subsets of a soft topological polygroup P. Then $H \circ K$ subset of P is soft connected.

Proof. Suppose that $\{\mathbb{F}_1, \mathbb{F}_2\}$ is a soft open covering of $H \circ K$ with the property that $\nexists g \in H \circ K$ satisfies both $g \widehat{\in} \mathbb{F}_1$ and $g \widehat{\in} \mathbb{F}_2$. Due to the Theorem 14, $h \circ K = (\varphi_L(h))(K)$ is soft connected for each $h \in H$. Note that $\{\mathbb{F}_1, \mathbb{F}_2\}$ is a soft covering of $h \circ K$ for all $h \in H$. Take an $h \in H$ arbitrarily. We suppose that $\forall g \in h \circ K (g \widehat{\notin} \mathbb{F}_1)$ without loss of generality. Assume for the contradiction that $\exists g' \in h' \circ K (g' \widehat{\in} \mathbb{F}_1)$ holds for some $h' \in H$. Select a g' from $h' \circ K$, and deposit $g' \in h' \circ k' (k' \in K)$. In conclusion, both $(\forall t \in h \circ k') t \widehat{\notin} \mathbb{F}_1$ and $h' \circ k' \ni g' \widehat{\in} \mathbb{F}_1$ are true, contradicting the soft connectedness of $H \circ k'$. Thus, $\forall g \in h \circ K (g \widehat{\notin} \mathbb{F}_1)$ holds for each $h \in H$. Notably, $\forall g \in H \circ K (g \widehat{\notin} \mathbb{F}_1)$. Therefore, $H \circ K$ is soft connected.

Theorem 25. If H is a subpolygroup of P with the property that $\widehat{H}$ is soft open, then $\widehat{H}$ is soft closed.

Proof. Suppose that $P = H \bigcup (\bigcup_{\alpha \in \Omega} H \circ g_\alpha)$ is a right coset decomposition. First, prove that $\widehat{H \circ g_\alpha}$ is soft open for all $\alpha \in \Omega$. For all $h \in H$, from the soft continuity of $\varphi_R(g_\alpha^{-1})$, it can be select a soft neighborhood $\mathbb{F}_{h \circ g_\alpha}$ of $h \circ g_\alpha$ with the property that $\mathbb{F}_{h \circ g_\alpha} \circ g_\alpha^{-1} \widehat{\subseteq} \widehat{H}$. Above all for every $h \in H$, we have $h \widehat{\in} \widehat{\bigcup}_{h \in H} \mathbb{F}_{h \circ g_\alpha} \circ g_\alpha^{-1} \widehat{\subseteq} \widehat{H}$. Therefore $\widehat{H} \widehat{=} \widehat{\bigcup}_{h \in H} \mathbb{F}_{h \circ g_\alpha} \circ g_\alpha^{-1}$, and $\widehat{H \circ g_\alpha}$ is soft equal to $\widehat{\bigcup}_{h \in H} \mathbb{F}_{h \circ g_\alpha}$. As a soft union of soft open sets $\mathbb{F}_{h \circ g_\alpha}$, $\widehat{H \circ g_\alpha}$ is also soft open.

In summary, $\widehat{\bigcup}_{\alpha \in \Omega} \widehat{H \circ g_\alpha}$ is soft open as it is the soft union of soft open sets. Therefore $\widehat{H} = \widehat{G} \setminus \widehat{\bigcup}_{\alpha \in \Omega} \widehat{H \circ g_\alpha}$ is soft closed.

Theorem 26. Let H be a subpolygroup of G. $\widehat{H}$ is soft open if and only if there exist an $h \in H$ and a soft neighborhood $\mathbb{F}$ of h with the property that $\mathbb{F} \widehat{\subseteq} \widehat{H}$.

Proof. $\Longrightarrow$: Select h and $\mathbb{F}$ as above. For every $h' \in H$, there exists a soft neighborhood $\mathbb{F}'_{h'}$ of h' with the property that $h \circ (h')^{-1} \circ \mathbb{F}'_{h'} \widehat{\subseteq} \mathbb{F}$ as $\varphi_L(h \circ (h')^{-1}) : P \longmapsto \{h \circ (h')^{-1} \circ x\}_{x \in P}$ is soft continuous. Since $\mathbb{F}'_{h'} \widehat{\subseteq} h' \circ h^{-1} \circ \mathbb{F}$ and H is a subpolygroup we have $\mathbb{F}'_{h'} \widehat{\subseteq} h' \circ h^{-1} \circ \mathbb{F} \widehat{\subseteq} \widehat{H}$. Thus $\widehat{H} \widehat{=} \widehat{\bigcup}_{h \in H} \mathbb{F}'_h$ is soft open.

$\Longleftarrow$: It is clear.

Conclusion and Future Work

This study presented two different definitions of the soft topological polygroup. The authors provided attributes for each definition along with examples. The connection between the complete parts and the concepts such as soft continuous function, soft Hausdorff space, soft $\mathcal{T}_0$ space, soft $\mathcal{T}_1$ space, soft open covering, soft compact, soft connected in soft topological polygroups was examined. Lastly, necessary arrangements were made.

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Жұмсақ топологиялық полигруппаларға тағы бір көзқарас

Жұмсақ топологиялық полигруппалар екі түрлі жолмен анықталады. Бірінші анықтамада тұрақты топология, ал екінші анықтамада жұмсақ топология бар. Екінші анықтамада жұмсақ маңай, жұмсақ үзіліссіздік, жұмсақ компакт, жұмсақ байланыс, жұмсақ хаусдорф кеңістігі сияқты ұғымдар пайда болады және олардың жұмсақ топологиялық полигруппалардағы жұмсақ үздіксіз функциялармен байланысы зерттеледі. Кәдімгі топологияда үздіксіздіктің бес баламалы анықтамасы бар, бірақ олардың барлығы міндетті түрде жұмсақ үзіліссіздікте анықталмаған.

Кілт сөздер: жұмсақ жиын, жұмсақ үзіліссіздік, жұмсақ топологиялық полигруппалар, жұмсақ хаусдорф кеңістігі, жұмсақ ашық жабын, жұмсақ компакт, жұмсақ байланыс.

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Другой взгляд на мягкие топологические полигруппы

Мягкие топологические полигруппы определяются двумя разными способами. Первое определение имеет обычную топологию, а второе — мягкую топологию. Во втором определении появляются такие понятия, как мягкая окрестность, мягкая непрерывность, мягкий компакт, мягкая связность, мягкое хаусдорфово пространство, и изучается их связи с мягкими непрерывными функциями в мягких топологических полигруппах. В обычной топологии есть пять эквивалентных определений непрерывности, но не все они обязательно установлены в мягкой непрерывности.

Ключевые слова: мягкое множество, мягкая непрерывность, мягкие топологические полигруппы, мягкое хаусдорфово пространство, мягкое открытое покрытие, мягкий компакт, мягкая связность.