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On the time-optimal control problem for a fourth order parabolic equation in a two-dimensional domain

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Previously, boundary control problems for the second order parabolic type equation in the bounded domain were studied. In this paper, a boundary control problem associated with a fourth-order parabolic equation in a bounded two-dimensional domain was considered. On the part of the considered domain's boundary, the value of the solution with control function is given. Restrictions on the control are given in such a way that the average value of the solution in the considered domain gets a given value. By the method of separation of variables the given problem is reduced to a Volterra integral equation of the first kind. The existence of the control function was proved by the Laplace transform method and an estimate was found for the minimal time at which the given average temperature in the domain is reached.

Keywords: initial-boundary problem, fourth-order parabolic equation, minimal time, admissible control, Volterra integral equation, Laplace transform method.

2020 Mathematics Subject Classification: 35K25, 35K35.

Introduction

In this paper, we consider the fourth order parabolic equation in the domain $\Omega = \{(x, y) : 0 < x < \pi, \quad 0 < y < \pi\}$

$$u_t(x, y, t) + \Delta^2 u(x, y, t) = 0, \quad (x, y, t) \in \Omega_T := \Omega \times (0, \infty), \tag{1}$$

with boundary value conditions

$$u(0, y, t) \ = \ \psi(y)\,\nu(t), \quad u_x(\pi, y, t) \ = \ 0, \quad u_{xx}(0, y, t) \ = \ 0, \quad u_{xxx}(\pi, y, t) \ = \ 0, \tag{2}$$

$$u(x, 0, t) \ = \ 0, \quad u_y(x, \pi, t) \ = \ 0, \quad u_{yy}(x, 0, t) \ = \ 0, \quad u_{yyy}(x, \pi, t) \ = \ 0, \tag{3}$$

and initial value condition

$$u(x, y, 0) \ = \ 0, \quad 0 \le x, y \le \pi, \tag{4}$$

where $\Delta^2 u(x, y, t) = u_{xxxx}(x, y, t) + u_{yyyy}(x, y, t)$, $\psi(y)$ is a given function and $\nu(t)$ is the control function.

Suppose $M > 0$ is a given constant. If the control function $\nu(t) \in W_2^1(\mathbb{R}_+)$ satisfies the conditions $\nu(0) = 0$ and $|\nu(t)| \le M$ on the half-line $t \ge 0$, we call it *admissible control*. We will prove later in Section 2 that the function ν belongs to the class $W_2^1(\mathbb{R}_+)$.

Now we present the following minimum time problem.

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Time-Optimal Problem. Assume that $\theta > 0$ is given constant. Then, find the minimal value of $T > 0$ such that for $t > 0$ the solution $u(x, y, t)$ of the problem (1)–(4) with a control function $\nu(t)$ exists and for some $T_1 > T$ satisfies the equation

$$\int_0^\pi \int_0^\pi u(x, y, t)\, dy\, dx = \theta, \quad T \le t \le T_1. \tag{5}$$

It is known that fourth-order parabolic equations were introduced to describe the epitaxial growth of nanoscale thin films [1]. Therefore, interest in materials science has been increasing in recent years.

Control problems related to second-order parabolic type equations were first studied by Fattorini and Friedman [2,3]. Control problems for the infinite-dimensional case were studied by Egorov [4], who generalized Pontryagin's maximum principle to a class of equations in Banach space, and the proof of a bang-bang principle was shown in the particular conditions.

The optimal time problem related to the second-order parabolic type equation in the bounded $n-$dimensional domain was studied in a new method by Albeverio and Alimov [5] and the optimal time's estimate for achieving a given average temperature was found. In [6,7], mathematical models of thermocontrol processes for the second order parabolic equation are considered. The control problem for the second-order parabolic equation associated with the Neumann boundary condition in a bounded three-dimensional domain is studied in [8]. In this work, an estimate of the optimal time was found when the average temperature is close to the critical value.

In [9, 10], the control problems of the second-order parabolic type equation associated with the Dirichlet boundary condition in the two-dimensional domain are studied. In these articles, an estimate of the minimum time for achieving a given average temperature was found, and the existence of a control function is proved by the Laplace transform method. The boundary control problem related to the fast heating of the thin rod for the inhomogeneous heat conduction equation was studied in works [11, 12] and the existence of the admissible control function was proved.

The optimal time problem for the heat equation with the Neumann boundary condition in a one-dimensional domain is studied in [13]. The difference of this work from the previous works is that the required estimate for the minimum time is found with a non-negative definite weight function under the integral condition. In [14], the control problem for a second-order parabolic type equation with two control functions was studied and the existence of admissible control functions was proved by the Laplace transform method.

A lot of information on the optimal control problems was given in detail in the monographs of Lions and Fursikov [15, 16]. Practical approaches to general numerical optimization and optimal control for equations of the second order parabolic type are studied in works such as [17, 18].

Boundary control problems related to the second-order pseudo-parabolic equation in a bounded domain are studied in detail in works [19–21]. In these works, the existence of the control function is proved using the method of Laplace transform.

In [22], Guo considered the null boundary control problem for a fourth order parabolic equation in one-dimensional bounded domain by the method reducing the control problem to the well-posed problems, proposed by Guo and Littman [23]. In [24], the null interior controllability for a fourth order parabolic equation was studied. The method that they used is based on Lebeau-Rabbiano inequality. The initial boundary value problem for equations from a class of fourth order semilinear parabolic equations was studied by Xu, et al. [25], and the global existence and nonexistence of solutions with initial data in the potential well are derived. Further research results on the global dynamic behavior of solutions associated with fourth-order parabolic equations for the epitaxial thin film model were studied by Chen [26].

In this work, the boundary control problem for the fourth-order parabolic equation is considered. The difference between this work and the previous works is that in this problem, the control problem associated with the fourth order parabolic type equation is studied. In Section 1, the boundary control problem studied is reduced to the Volterra integral equation of the first kind by the Fourier method. In Section 2, the existence of a solution to the Volterra integral equation is proved using the Laplace transform method. Section 3 gives an estimate of the minimum time required to reach a given average temperature of the plate.

We now consider the eigenvalue problem

$$\Delta^2 X(x,y) = \lambda\, X(x,y), \quad (x,y) \in \Omega,$$

with the boundary value conditions

$$X(0,y) = X_{xx}(0,y) = 0, \quad X_x(\pi,y) = X_{xxx}(\pi,y) = 0,$$

and

$$X(x,0) = X_{yy}(x,0) = 0, \quad X_y(x,\pi) = X_{yyy}(x,\pi) = 0, \quad (x,y) \in \partial\Omega.$$

Then we have the eigenvalue and eigenfunctions defined as follows

$$\lambda_{mn} = \left(\frac{2\,m+1}{2}\right)^4 + \left(\frac{2\,n+1}{2}\right)^4, \quad X_{mn}(x,y) = \sin\frac{2\,m+1}{2}x\,\sin\frac{2\,n+1}{2}y, \quad m,n = 0,1,\ldots$$

Suppose that the function $\psi \in H^4(\Omega)$ satisfies the following conditions

$$\psi(0) = \psi^{(1)}(\pi) = \psi^{(2)}(0) = \psi^{(3)}(\pi) = 0, \quad \psi_n \geq 0,$$

where ψ_n is the Fourier coefficient of the function $\psi(y)$ and as follows

$$\psi_n = \frac{2}{\pi}\int_0^\pi \psi(y)\,\sin\frac{2n+1}{2}y\,dy, \quad n = 0,1,\ldots \tag{6}$$

We set

$$\beta_{mn} = \frac{1}{\pi}\frac{(2m+1)^2\,\psi_n}{2n+1}, \quad m,n = 0,1,\ldots, \tag{7}$$

where ψ_n is defined by (6).

Theorem 1. Let be

$$0 < \theta < \frac{\beta_0\, M}{\lambda_0}.$$

Set

$$T_0 \;=\; -\frac{1}{\lambda_0}\ln\left(1 - \frac{\theta\,\lambda_0}{\beta_0\, M}\right).$$

Then a solution T_{min} of the time-optimal problem exists and the estimate $T_{min} \leq T_0$ is valid.

1 *Main integral equation*

In this section, we consider how the given control problem can be reduced to a Volterra integral equation of the first kind.

By the solution of the initial-boundary problem (1)–(4), we mean the function $u(x,y,t)$, which is expressed in the following form

$$u(x,y,t) \;=\; \psi(y)\,\nu(t) - w(x,y,t), \tag{8}$$

where the function $w(x,y,t)$ with the regularity $w(x,y,t) \in C^{4,4,1}_{x,y,t}(\Omega_T) \cap C(\bar{\Omega}_T)$ and $w_{xx}, w_{yy} \in C(\bar{\Omega})$ is the solution to the initial-boundary problem

$$w_t(x,y,t) + \Delta^2 w(x,y,t) \; = \; \psi(y)\,\nu'(t) + \psi^{(4)}(y)\,\nu(t),$$

with the boundary value conditions

$$w(0,y,t) \; = \; w_{xx}(0,y,t) \; = \; 0, \quad w_x(\pi,y,t) \; = \; w_{xxx}(\pi,y,t) \; = \; 0,$$

$$w(x,0,t) \; = \; w_{yy}(x,0,t) \; = \; 0, \quad w_y(x,\pi,t) \; = \; w_{yyy}(x,\pi,t) \; = \; 0,$$

and the initial condition

$$w(x,y,0) = 0.$$

As a result, we get the following solution

$$w(x,y,t) \; = \; \frac{4}{\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\psi_n}{2m+1}\Big(\int_0^t e^{-\lambda_{mn}(t-s)}\,\nu'(s)\,ds\Big)\,\sin\frac{2\,m+1}{2}x\,\sin\frac{2\,n+1}{2}y+$$

$$+\frac{1}{4\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(2n+1)^4\,\psi_n}{2m+1}\Big(\int_0^t e^{-\lambda_{mn}(t-s)}\,\nu(s)\,ds\Big)\,\sin\frac{2\,m+1}{2}x\,\sin\frac{2\,n+1}{2}y. \tag{9}$$

By (8) and (9), we have the solution of the initial-boundary problem (1)–(4) (see, [27]):

$$u(x,y,t) = \psi(y)\,\nu(t)-$$

$$-\frac{4}{\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\psi_n}{2m+1}\Big(\int_0^t e^{-\lambda_{mn}(t-s)}\,\nu'(s)\,ds\Big)\,\sin\frac{2\,m+1}{2}x\,\sin\frac{2\,n+1}{2}y-$$

$$-\frac{1}{4\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(2n+1)^4\,\psi_n}{2m+1}\Big(\int_0^t e^{-\lambda_{mn}(t-s)}\,\nu(s)\,ds\Big)\,\sin\frac{2\,m+1}{2}x\,\sin\frac{2\,n+1}{2}y.$$

Using condition (5) and the solution to problem (1)–(4), we can write

$$h(t) \; = \; \int_0^{\pi}\int_0^{\pi} u(x,y,t)\,dx\,dy = \nu(t)\int_0^{\pi}\int_0^{\pi}\psi(y)\,dx\,dy-$$

$$-\frac{16}{\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\psi_n}{(2m+1)^2\,(2n+1)}\int_0^t e^{-\lambda_{mn}(t-s)}\,\nu'(s)\,ds-$$

$$-\frac{1}{\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(2n+1)^3\,\psi_n}{(2m+1)^2}\int_0^t e^{-\lambda_{mn}(t-s)}\,\nu(s)\,ds. \tag{10}$$

From the definition of the function $\nu(t)$ and from (10), we may write

$$h(t) \; = \; \nu(t)\int_0^{\pi}\int_0^{\pi}\psi(y)\,dx\,dy - \nu(t)\,\frac{16}{\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\psi_n}{(2m+1)^2\,(2n+1)}+$$

$$+\,\frac{1}{\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(2m+1)^2\,\psi_n}{2n+1}\int\limits_0^t e^{-\lambda_{mn}(t-s)}\,\nu(s)\,ds. \qquad (11)$$

Note that

$$\int\limits_0^{\pi}\int\limits_0^{\pi}\psi(y)\,dx\,dy \;=\; \frac{16}{\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\psi_n}{(2m+1)^2\,(2n+1)}. \qquad (12)$$

Then, from (11) and (12), we obtain

$$h(t) \;=\; \frac{1}{\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(2m+1)^2\,\psi_n}{2n+1}\int\limits_0^t e^{-\lambda_{mn}(t-s)}\,\nu(s)\,ds.$$

We set

$$B(t) \;=\; \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\beta_{mn}\,e^{-\lambda_{mn}t}, \quad t>0, \qquad (13)$$

where β_{mn} is defined by (7).

Let there exist $M_0>0$ constant. Denote by $W(M_0)$ the set of function $h\in W_2^2(-\infty,+\infty)$, which satisfies the condition

$$\|h\|_{W_2^2(R_+)}\le M_0, \quad h(t)=0 \quad \text{for all} \quad t\le 0.$$

Thus, we have the following Volterra integral equation

$$\int\limits_0^t B(t-s)\,\nu(s)\,ds \;=\; h(t), \quad t>0, \qquad (14)$$

where $h(t)=\theta$ for $T\le t\le T_1$.

Theorem 2. Assume that $M_0>0$ exists. Then, for any function $h\in W(M_0)$ the solution $\nu(t)$ of integral equation (14) exists and satisfies the condition

$$|\nu(t)| \;\le\; M.$$

2 Proof of Theorem 2

Proposition 1. Suppose that $\alpha\in(\frac{3}{4},1)$. Then for the function $B(t)$ defined by (13) the following estimate

$$0<B(t)\le\; C_\alpha\,t^{-\alpha}, \quad 0<t\le 1, \qquad (15)$$

is valid.

Proof. Using the definition (13) and $\lambda_{mn}=\left(\frac{2m+1}{2}\right)^4+\left(\frac{2n+1}{2}\right)^4$, we may write

$$B(t)=\frac{1}{\pi}\sum_{m=0}^{\infty}(2m+1)^2\,e^{-(\frac{2m+1}{2})^4 t}\sum_{n=0}^{\infty}\frac{\psi_n}{2n+1}\,e^{-(\frac{2n+1}{2})^4 t}.$$

We set

$$A(t) \;=\; \sum_{n=0}^{\infty}\frac{\psi_n}{2n+1}\,e^{-(\frac{2n+1}{2})^4 t}, \quad t>0.$$

Clearly, for any $0 < t \le T$, this function satisfies the following inequality

$$0 \ < \ A(T) \ \le \ A(t) \ < \ A(0). \tag{16}$$

Let $\delta > 0$ be constant. We know the maximum value of the function $g(t,\delta) = t^\alpha e^{-\delta t}$ is reached at the point $t = \frac{\alpha}{\delta}$ and this value is equal to $\frac{\alpha^\alpha}{\delta^\alpha} e^{-\alpha}$.

As a result, for any $\alpha \in (\frac{3}{4}, 1)$, we have the following estimate

$$\sum_{m=0}^{\infty} (2m+1)^2 \, e^{-(\frac{2m+1}{2})^4 \, t} \ = \ t^{-\alpha} \sum_{m=0}^{\infty} (2m+1)^2 \, t^\alpha \, e^{-(\frac{2m+1}{2})^4 \, t} \le$$

$$\le \frac{16^\alpha \, \alpha^\alpha \, e^{-\alpha}}{t^\alpha} \sum_{m=0}^{\infty} \frac{(2m+1)^2}{(2m+1)^{4\,\alpha}} \le C_\alpha \, t^{-\alpha}, \tag{17}$$

where

$$\sum_{m=0}^{\infty} \frac{(2m+1)^2}{(2m+1)^{4\,\alpha}} = \sum_{m=0}^{\infty} \frac{1}{(2m+1)^{4\alpha-2}} < +\infty.$$

Then the required estimate (15) follows from (16) and (17).

Proposition 1 is proved.

As we know, the Laplace transform of the function $\nu(t)$ is defined as follows

$$\widetilde{\nu}(p) = \int_0^\infty e^{-pt} \, \nu(t) \, dt, \quad \text{where} \quad p = \sigma + i\,\tau, \quad \sigma > 0, \quad \tau \in \mathbb{R}.$$

We rewrite integral equation (14) as follows

$$\int_0^t B(t-s) \, \nu(s) ds \ = \ h(t), \quad t > 0.$$

Then we use Laplace transform and obtain the following equation

$$\widetilde{h}(p) = \int_0^\infty e^{-pt} \, dt \int_0^t B(t-s) \, \nu(s) ds = \widetilde{B}(p) \, \widetilde{\nu}(p).$$

Thus, we have

$$\widetilde{\nu}(p) = \frac{\widetilde{h}(p)}{\widetilde{B}(p)},$$

and

$$\nu(t) = \frac{1}{2\pi i} \int_{\sigma - i\,\infty}^{\sigma + i\,\infty} \frac{\widetilde{h}(p)}{\widetilde{B}(p)} \, e^{pt} dp = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\widetilde{h}(\sigma + i\,\tau)}{\widetilde{B}(\sigma + i\,\tau)} \, e^{(\sigma + i\,\tau) t} \, d\tau. \tag{18}$$

Then we can write

$$\widetilde{B}(p) = \int_0^\infty B(t) \, e^{-pt} \, dt = \sum_{m,n=0}^{\infty} \beta_{mn} \int_0^\infty e^{-(p + \lambda_{mn}) t} \, dt = \sum_{m,n=0}^{\infty} \frac{\beta_{mn}}{p + \lambda_{mn}},$$

where $B(t)$ is defined by (13) and

$$\widetilde{B}(\sigma+i\,\tau)=\sum_{m,n=0}^{\infty}\frac{\beta_{mn}}{\sigma+\lambda_{mn}+i\,\tau}=\sum_{m,n=0}^{\infty}\frac{\beta_{mn}\,(\sigma+\lambda_{mn})}{(\sigma+\lambda_{mn})^2+\tau^2}-i\,\tau\sum_{m,n=0}^{\infty}\frac{\beta_{mn}}{(\sigma+\lambda_{mn})^2+\tau^2}$$

$$=\mathrm{Re}\widetilde{B}(\sigma+i\,\tau)+i\,\mathrm{Im}\widetilde{B}(\sigma+i\,\tau),$$

where

$$\mathrm{Re}\widetilde{B}(\sigma+i\,\tau)=\sum_{m,n=0}^{\infty}\frac{\beta_{mn}\,(\sigma+\lambda_{mn})}{(\sigma+\lambda_{mn})^2+\tau^2},\quad \mathrm{Im}\widetilde{B}(\sigma+i\,\tau)=-\tau\sum_{m,n=0}^{\infty}\frac{\beta_{mn}}{(\sigma+\lambda_{mn})^2+\tau^2}.$$

Obviously, the following inequality holds

$$(\sigma+\lambda_{mn})^2+\tau^2\le[(\sigma+\lambda_{mn})^2+1](1+\tau^2),$$

and we further have

$$\frac{1}{(\sigma+\lambda_{mn})^2+\tau^2}\ge\frac{1}{1+\tau^2}\,\frac{1}{(\sigma+\lambda_{mn})^2+1}. \tag{19}$$

Thus, due to (19), we can obtain the following estimates

$$|\mathrm{Re}\widetilde{B}(\sigma+i\,\tau)|=\sum_{m,n=0}^{\infty}\frac{\beta_{mn}\,(\sigma+\lambda_{mn})}{(\sigma+\lambda_{mn})^2+\tau^2}\ge$$

$$\ge\frac{1}{1+\tau^2}\sum_{m,n=0}^{\infty}\frac{\beta_{mn}\,(\sigma+\lambda_{mn})}{(\sigma+\lambda_{mn})^2+1}=\frac{C_{1,\sigma}}{1+\tau^2}, \tag{20}$$

and

$$|\mathrm{Im}\widetilde{B}(\sigma+i\,\tau)|=|\tau|\sum_{m,n=0}^{\infty}\frac{\beta_{mn}}{(\sigma+\lambda_{mn})^2+\tau^2}\ge$$

$$\ge\frac{|\tau|}{1+\tau^2}\sum_{m,n=0}^{\infty}\frac{\beta_{mn}}{(\sigma+\lambda_{mn})^2+1}=\frac{C_{2,\sigma}\,|\tau|}{1+\tau^2}, \tag{21}$$

where $C_{1,\sigma}$, $C_{2,\sigma}$ are defined as follows

$$C_{1,\sigma}=\sum_{m,n=0}^{\infty}\frac{\beta_{mn}\,(\sigma+\lambda_{mn})}{(\sigma+\lambda_{mn})^2+1},\quad C_{2,\sigma}=\sum_{m,n=0}^{\infty}\frac{\beta_{mn}}{(\sigma+\lambda_{mn})^2+1}.$$

From (20) and (21), we have the following estimate

$$|\widetilde{B}(\sigma+i\,\tau)|^2=|\mathrm{Re}\widetilde{B}(\sigma+i\,\tau)|^2+|\mathrm{Im}\widetilde{B}(\sigma+i\,\tau)|^2\ge\frac{\min(C_{1,\sigma}^2,C_{2,\sigma}^2)}{1+\tau^2},$$

and

$$|\widetilde{B}(\sigma+i\,\tau)|\ge\frac{C_\sigma}{\sqrt{1+\tau^2}},\quad\text{where}\quad C_\sigma=\min(C_{1,\sigma},C_{2,\sigma}). \tag{22}$$

Proceeding to the limit as $\sigma\to0$ from (18), we have

$$\nu(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\widetilde{h}(i\,\tau)}{\widetilde{B}(i\,\tau)}\,e^{i\,\tau t}\,d\tau. \tag{23}$$

Proposition 2. [20] Assume that $h(t) \in W(M_0)$. Then for the imaginary part of the Laplace transform of function $h(t)$ the inequality

$$\int_{-\infty}^{+\infty} |\widetilde{h}(i\,\tau)| \sqrt{1+\tau^2}\, d\tau \leq C_1 \, \|h\|_{W_2^2(R_+)}$$

is valid, where $C_1 > 0$ is a constant.

Proof of the Theorem 2. Now we prove that $\nu \in W_2^1(\mathbb{R}_+)$. By (22) and (23), we obtain

$$\int_{-\infty}^{+\infty} |\widetilde{\nu}(\tau)|^2 (1+|\tau|^2)\, d\tau \;=\; \int_{-\infty}^{+\infty} \left| \frac{\widetilde{h}(i\,\tau)}{\widetilde{B}(i\,\tau)} \right|^2 (1+|\tau|^2)\, d\tau \;\leq C_0 \int_{-\infty}^{+\infty} |\widetilde{h}(i\,\tau)|^2 (1+|\tau|^2)^2\, d\tau \;=\; C_0 \|h\|^2_{W_2^2(\mathbb{R})},$$

$C_0 = \min(C_{1,0}, C_{2,0})$ which is defined by (22). Further,

$$|\nu(t) - \nu(s)| \;=\; \left| \int_s^t \nu'(\xi)\, d\xi \right| \;\leq\; \|\nu'\|_{L_2} (t-s)^{1/2}.$$

From (22), (23) and Proposition 2, we have the estimate

$$|\nu(t)| \leq \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{|\widetilde{h}(i\,\tau)|}{|\widetilde{B}(i\,\tau)|} d\tau \leq \frac{1}{2\pi C_0} \int_{-\infty}^{+\infty} |\widetilde{h}(i\,\tau)| \sqrt{1+\tau^2} d\tau \leq$$

$$\leq \frac{C_1}{2\pi C_0} \|h\|_{W_2^2(R_+)} \leq \frac{C_1\, M_0}{2\pi C_0} = M,$$

where

$$M_0 = \frac{2\pi C_0}{C_1}\, M.$$

Theorem 2 is proved.

3 *Estimate for the Minimal Time*

Now we introduce the following integral equation

$$\int_0^t B(t-s)\, \nu(s)\, ds \;=\; \theta, \quad T \leq t \leq T_1,$$

where $B(t)$ is defined by (13).

We set

$$\beta_0 = \beta_{00}, \quad \lambda_0 = \lambda_{00},$$

where β_{mn} defined by (7).

Proposition 3. The following estimate is valid:

$$B(t) \geq \beta_0 \, e^{-\lambda_0 t},$$

where the function $B(t)$ is defined by Eq. (13).

The proof of this proposition follows from the fact that the functional series defined by (13) is positive for all $t \geq 0$.

We introduce the following function

$$H(t) = \int_0^t B(t-s)\,ds = \int_0^t B(s)\,ds.$$

It is known that the physical meaning of this function is the average temperature in a bounded domain Ω (see, [5]). It is known $H(0) = 0$ and $H'(t) = B(t) > 0$.

We set

$$H^* = \lim_{t\to\infty} H(t) = \int_0^\infty B(s)ds.$$

The average temperature in the bounded domain does not exceed H^*. Clearly, H^* is finite. Indeed,

$$H^* = \int_0^\infty B(s)\,ds = \sum_{m,n=0}^{\infty} \frac{\beta_{mn}}{\lambda_{mn}} < +\infty,$$

where β_{mn} is defined by (7) and $\lambda_{mn} = \left(\frac{2m+1}{2}\right)^4 + \left(\frac{2n+1}{2}\right)^4$.

Proposition 4. Assume that $0 < \theta < MH^*$. Then there exist $T > 0$ and a control function $\nu(t)$ and the following equality

$$\int_0^T B(T-s)\,\nu(s)ds = \theta \tag{24}$$

is valid.

Proof. The proof of this follows directly from the properties of the function H. Indeed, if we set $\nu(t) = M$ then

$$\int_0^t B(t-s)\nu(s)ds = M\int_0^t B(t-s)ds = M\,H(t),$$

and because of (24) there exists $T > 0$ so that $M\,H(T) = \theta$.

Proposition 4 is proved.

Remark 1. It is clear that the value T, which was found in Proposition 4, gives a solution to the problem. That is T is the root of the following equation

$$H(T) = \frac{\theta}{M}. \tag{25}$$

Lemma 1. Let

$$0 < \theta < \frac{\beta_0\,M}{\lambda_0}.$$

Then there exists $T > 0$ so that

$$T < -\frac{1}{\lambda_0}\ln\left(1 - \frac{\theta\,\lambda_0}{\beta_0\,M}\right),$$

and Eq. (25) is fulfilled.

Proof. Using Proposition 3, we can write the following inequality

$$H(t) = \int\limits_0^t B(s)\,ds \ge \beta_0 \int\limits_0^t e^{-\lambda_0 s} ds =$$

$$= \frac{\beta_0}{\lambda_0}\Big(1 - e^{-\lambda_0 t}\Big). \tag{26}$$

To determine T_0, we consider the following equation:

$$\frac{\beta_0}{\lambda_0}\Big(1 - e^{-\lambda_0 T_0}\Big) = \frac{\theta}{M}. \tag{27}$$

Then we get

$$T_0 = -\frac{1}{\lambda_0}\ln\Big(1 - \frac{\theta\,\lambda_0}{\beta_0\,M}\Big).$$

In accordance with (26) and (27) we have

$$0 < \frac{\theta}{M} \le H(T_0).$$

Then obviously there exists T, $0 < T < T_0$, which is a solution to equation (25).

Lemma 1 is proved.

The proof of Theorem 1 follows from Lemma 1.

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Conflict of Interest

The author declare no conflict of interest.

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Екіөлшемді облыстағы төртінші ретті параболалық теңдеу үшін оңтайлы уақытты басқару мәселесі туралы

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Бұрын шектелген облыстағы екінші ретті параболалық типті теңдеу үшін шекаралық бақылау есептері зерттелді. Бұл жұмыста шектелген екіөлшемді облыстағы төртінші ретті параболалық теңдеумен байланысты шекаралық бақылау есебі қарастырылған. Қарастырылатын облыс шекарасының бөлігінде басқару функциясы бар шешімнің мәні берілген. Бақылаудағы шектеулер қарастырылатын облыстағы шешімнің орташа мәні нақты мәнді алатындай етіп берілді. Айнымалыларды бөлу әдісімен берілген есеп бірінші текті Вольтерра интегралдық теңдеуіне келтіріледі. Басқару функциясының бар болуы Лаплас түрлендіру әдісімен дәлелденді және облыста берілген орташа температураға жетудің ең аз уақытының бағасы табылды.

Кілт сөздер: бастапқы-шекаралық есеп, төртінші ретті параболалық теңдеу, ең аз уақыт, рұқсат етілген бақылау, Вольтерра интегралдық теңдеуі, Лапластың түрлендіру әдісі.

О задаче быстродействия параболического уравнения четвертого порядка в двумерной области

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Ранее были исследованы задачи граничного управления для уравнения параболического типа второго порядка в ограниченной области. В данной работе рассмотрена задача граничного управления, связанная с параболическим уравнением четвертого порядка в ограниченной двумерной области. На части границы рассматриваемой области дано значение решения с функцией управления. Ограничения на управление задаются таким образом, чтобы среднее значение решения в рассматриваемой области получало заданное значение. Задача, заданная методом разделения переменных, сводится к интегральному уравнению Вольтерра первого рода. Методом преобразования Лапласа доказано существование функции управления и найдена оценка минимального времени достижения заданной средней температуры в области.

Ключевые слова: начально-краевая задача, параболическое уравнение четвертого порядка, минимальное время, допустимое управление, интегральное уравнение Вольтерра, метод преобразования Лапласа.

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