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A.S. Berdyshev[1,2], A.R. Ryskan[1*]

[1] *Abai Kazakh National Pedagogical University, Almaty, Kazakhstan;*
[2] *Institute of Information and Computational Technologies, Almaty, Kazakhstan*
(E-mail: berdyshev@mail.ru, ryskan.a727@gmail.com)

Boundary value problem for the four-dimensional Gellerstedt equation

In this work, the solvability of the problem with Neumann and Dirichlet boundary conditions for the Gellerstedt equation in four variables was investigated. The energy integral method was used to prove the uniqueness of the solution to the problem. In addition to it, formulas for differentiation, autotransformation, and decomposition of hypergeometric functions were applied. The solution was obtained explicitly and expressed by Lauricella's hypergeometric function.

Keywords: Gellerstedt equation, boundary value problem with mixed conditions, fundamental solution,

Introduction

The study of boundary value problems for degenerate equations is one of the important directions of modern theory of partial differential equations. The solution of many boundary value problems for partial differential equations has an applied nature [1–2]. The boundary value problems for degenerate elliptic equations were well studied in works [3–6].

In the formulation of problems and questions of the solvability of local and nonlocal boundary value problems for degenerate elliptic equations, fundamental solutions of these equations are essentially used [7]. The explicit form of the fundamental solutions makes it possible to correctly formulate the problem statement and study in detail the various properties of the considered equation solutions.

Fundamental solutions of degenerate elliptic equations are expressed in terms of the Lauricella's multidimensional hypergeometric functions and the Gauss hypergeometric function of one variable. Many problems of natural science, such as problems of dynamics and heat conduction, the theory of electromagnetic oscillations, aerodynamics, quantum mechanics, quantum chemistry, potential theory, etc. lead to the study of various properties of multidimensional hypergeometric functions [8–14].

For two-dimensional and three-dimensional elliptic equations with singular coefficients, fundamental solutions were constructed, which were applied in the study of the various problems solvability in many works [15–19].

In [20] fundamental solutions for the generalized Gellerstedt equation of four variables were constructed

$$y^m z^k t^l u_{xx} + x^n z^k t^l u_{yy} + x^n y^m t^l u_{zz} + x^n y^m z^k u_{tt} = 0, \;\; m,n,k,l > 0, \;\; m,n,k,l \equiv const.$$

Since the generalized Gellerstedt equation has four hypersurfaces of degeneration of the equation type, accordingly sixteen fundamental solutions were obtained. It was proved that the fundamental solutions have a singularity of the order $\frac{1}{r^2}$, at $r \to 0$, where $r = \sqrt{x^2+y^2+z^2+t^2}$.

These fundamental solutions are expressed in terms of Lauricella's hypergeometric functions, each of the fundamental solutions is applied in solving the corresponding boundary value problems [21–23].

1. Preliminary information

By definition, the Gauss hypergeometric function has the form

*Corresponding author.
E-mail: ryskan.a727@gmail.com

$$F(a,b;c;x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}x^n,\quad c\neq 0,\ -1,\ -2,\ \ldots,$$

where

$$(a)_n=a(a+1)(a+2)\ldots(a+n-1)\ =\frac{\Gamma(a+n)}{\Gamma(a)}\quad (n=0,1,2,3,\ldots)$$

is Pochhammer symbol. Here $\Gamma(a)$ is Euler's gamma function, for it the formula of the doubled argument is valid [24; 19, (15)]

$$\Gamma(2z)=2^{2z-1}\pi^{-\frac{1}{2}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right). \tag{1}$$

We present the main properties of the Euler's gamma function, the Gauss hypergeometric function, and Lauricella's hypergeometric function of many variables, which will be used in what follows.

The Gauss hypergeometric function has the following property [25; 3, (5)]:

$$F(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\quad \mathrm{Re}(c-a-b)>0. \tag{2}$$

The Gauss hypergeometric function satisfies the Bolz autotransformation formula [26; 64, (22)]:

$$F(a,b;c;x)=(1-x)^{-b}F\left(c-a,b;c;\frac{x}{x-1}\right). \tag{3}$$

Lauricella's hypergeometric function of n variables [25; 114]

$$F_D^{(n)}(a;b_1,b_2,b_3,...,b_n;c;x_1,x_2,...,x_n)=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(a)_{m_1+\ldots+m_n}(b_1)_{m_1}\ldots(b_n)_{m_n}}{(c)_{m_1+\ldots+m_n}m_1!\ldots m_n!}x_1^{m_1}...x_n^{m_n}, \tag{4}$$
$$(|x_1|<1,\ |x_2|<1,\ \ldots\ ,|x_n|<1),$$

the form [25; 117]

$$F_D^{(n)}(a;b_1,b_2,b_3,...,b_n;c;1,1,...,1)=\frac{\Gamma(c)\Gamma(c-a-b_1-b_2-...-b_n)}{\Gamma(c-a)\Gamma(c-b_1-b_2-...-b_n)},\ \ n=1,2,\ldots \tag{5}$$
$$\mathrm{Re}(c-a-b_1-b_2-...-b_n)>0,\ \ c\neq 0,-1,-2,\ldots.$$

in the case when all variables in (4) take the value 1.

Lauricella's hypergeometric function in the case of four variables has the form [25; 114, (1)]:

$$F_A^{(4)}(a;b_1,b_2,b_3,b_4;c_1,c_2,c_3,c_4;x,y,z,t)=\sum_{m,n,p,q}^{\infty}\frac{(a)_{m+n+p+q}(b_1)_m(b_2)_n(b_3)_p(b_4)_q}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q m!n!p!q!}x^m y^n z^p t^q, \tag{6}$$
$$(|x|+|y|+|z|+|t|<1).$$

The validity of the decomposition formula for a hypergeometric function of three variables, was proved in [27]:

$$F_A^{(3)}(a,b_1,b_2,b_3;c_1,c_2,c_3;x,y,z)=\sum_{n_1,n_2,n_3=0}^{\infty}\frac{(a)_{n_1+n_2+n_3}(b_1)_{n_1+n_2}(b_2)_{n_1+n_3}(b_3)_{n_2+n_3}}{(c_1)_{n_1+n_2}(c_2)_{n_1+n_3}(c_3)_{n_2+n_3}n_1!n_2!n_3!}\times$$
$$\times x^{n_1+n_2}y^{n_1+n_3}z^{n_2+n_3}F(a+n_1+n_2,b_1+n_1+n_2;c_1+n_1+n_2;x) \tag{7}$$
$$\times F(a+n_1+n_2+n_3,b_2+n_1+n_3;c_2+n_1+n_3;y)$$
$$\times F(a+n_1+n_2+n_3,b_3+n_2+n_3;c_3+n_2+n_3;z).$$

We also use the formula for the differentiation of hypergeometric functions of three variables [25]

$$\frac{\partial^{i+j+k}}{\partial x^i \partial y^j \partial z^k} F_A^{(3)}\left(\alpha;\beta_1,\beta_2,\beta_3;\gamma_1,\gamma_2,\gamma_3;x,y,z\right) = \frac{(\alpha)_{i+j+k}(\beta_1)_i(\beta_2)_j(\beta_3)_k}{(\gamma_1)_i(\gamma_2)_j(\gamma_3)_k}\times$$
$$\times F_A^{(3)}\left(\alpha+i+j+k;\beta_1+i,\beta_2+j,\beta_3+k;\gamma_1+i,\gamma_2+j,\gamma_3+k;x,y,z\right),$$
$$i,j,k\in\mathbb{N}_0=\{0,1,2,...\}. \tag{8}$$

For Lauricella's function $F_A^{(n)}$ the following adjacent relations are valid

$$\frac{b_1}{c_1}x_1F_A\left(a+1;b_1+1,b_2,\ldots,b_n;c_1+1,c_2,\ldots,c_n;x_1,\ldots,x_n\right)$$
$$+\frac{b_2}{c_2}x_2F_A\left(a+1;b_1,b_2+1,\ldots,b_n;c_1,c_2+1,\ldots,c_n;x_1,\ldots,x_n\right)$$
$$+\ldots+\frac{b_n}{c_n}x_nF_A\left(a+1;b_1,b_2,\ldots,b_n+1;c_1,c_2,\ldots,c_n+1;x_1,\ldots,x_n\right)=$$
$$=F_A\left(a+1;b_1,\ldots,b_n;c_1,\ldots,c_n;x_1,\ldots,x_n\right)-F_A\left(a;b_1,\ldots,b_n;c_1,\ldots,c_n;x_1,\ldots,x_n\right). \tag{9}$$

To calculate the value of a multiple integral, we use the formula [28; 637, (3)]

$$\int\limits_0^\infty\int\limits_0^\infty\cdots\int\limits_0^\infty \frac{x_1^{p_1-1}x_2^{p_2-1}\ldots x_n^{p_n-1}}{\left[1+(r_1x_1)^{q_1}+(r_2x_2)^{q_2}+\ldots+(r_nx_n)^{q_n}\right]^s}dx_1dx_2\ldots dx_n=$$
$$=\frac{\Gamma\left(\frac{p_1}{q_1}\right)\Gamma\left(\frac{p_2}{q_2}\right)\ldots\Gamma\left(\frac{p_n}{q_n}\right)}{q_1q_2\ldots q_n{r_1}^{p_1q_1}{r_2}^{p_2q_2}\ldots{r_n}^{p_nq_n}}\frac{\Gamma\left(s-\frac{p_1}{q_1}-\frac{p_2}{q_2}-\ldots-\frac{p_n}{q_n}\right)}{\Gamma(s)},\ (p_i>0,q_i>0,r_i>0,s>0), \tag{10}$$

and for integrals expressed in terms of the beta function, the formulas [24; 25, (16), (19)]

$$\int\limits_0^\infty(1+bt^z)^{-y}t^xdt=z^{-1}b^{-\frac{x+1}{z}}\beta\left(\frac{x+1}{z},y-\frac{x+1}{z}\right),\ \left(z>0,\ b>0,\ 0<\mathrm{Re}\,\frac{x+1}{z}<\mathrm{Re}\,y\right), \tag{11}$$

$$\int\limits_0^{\frac{\pi}{2}}(\sin t)^{2x-1}(\cos t)^{2y-1}dt=\frac{1}{2}\beta\left(x,y\right),\ (\mathrm{Re}\,x>0,\mathrm{Re}\,y>0). \tag{12}$$

2. Statement of the problem

Considering the generalized Gellerstedt equation:

$$H(u)=y^mz^kt^lu_{xx}+x^nz^kt^lu_{yy}+x^ny^mt^lu_{zz}+x^ny^mz^ku_{tt}=0, m,n,k,l>0,\ \ m,n,k,l\equiv const, \tag{13}$$

we introduce the following notations:

$$D=\{(x,y,z,t):\ x>0,y>0,\ z>0,t>0\},$$

$$S_1=\{(0,y,z,t):\ x=0,y>0,\ z>0,t>0\},$$
$$S_2=\{(x,0,z,t):\ x>0,y=0,\ z>0,t>0\},$$
$$S_3=\{(x,y,0,t):\ x>0,y>0,\ z=0,t>0\},$$
$$S_4=\{(x,y,z,0):\ x>0,y>0,\ z>0,t=0\},$$
$$R^2=\frac{4}{(n+2)^2}x^{n+2}+\frac{4}{(m+2)^2}y^{m+2}+\frac{4}{(k+2)^2}z^{k+2}+\frac{4}{(l+2)^2}t^{l+2}.$$

Problem ND_2. Find a regular solution $u(x,y,z,t)$ of the equation (13) from the class $C\left(\overline{D}\right)\cap C^1\left(D\cup\overline{S_3}\cup\overline{S_4}\right)\cap C^2(D)$ satisfying the condition:

$$u(x,y,z,t)|_{x=0} = \tau_1(y,z,t), \quad (y,z,t) \in \overline{S_1}, \tag{14}$$

$$u(x,y,z,t)|_{y=0} = \tau_2(x,z,t), \quad (x,z,t) \in \overline{S_2}, \tag{15}$$

$$\left.\frac{\partial}{\partial z} u(x,y,z,t)\right|_{z=0} = \nu_3(x,y,t), \quad (x,y,t) \in S_3, \tag{16}$$

$$\left.\frac{\partial}{\partial t} u(x,y,z,t)\right|_{t=0} = \nu_4(x,y,z), \quad (x,y,z) \in S_4, \tag{17}$$

$$\lim_{R\to\infty} u(x,y,z,t) = 0, \tag{18}$$

where $\tau_1(y,z,t), \tau_2(x,z,t), \nu_3(x,y,t), \nu_4(x,y,z) \in \mathbb{C}$ are given continuous functions, moreover the function $\nu_3(x,y,t), \nu_4(x,y,z)$ at the origin of coordinates can go to integrable order infinity. Also, for the large enough values R, the following inequalities hold:

$$|\tau_1(y,z,t)| \le \frac{c_1}{\left[1 + \frac{4}{(n+2)^2} y^{m+2} + \frac{4}{(k+2)^2} z^{k+2} + \frac{4}{(l+2)^2} t^{l+2}\right]^{\varepsilon_1}}, \tag{19}$$

$$|\tau_2(x,z,t)| \le \frac{c_2}{\left[1 + \frac{4}{(n+2)^2} x^{n+2} + \frac{4}{(k+2)^2} z^{k+2} + \frac{4}{(l+2)^2} t^{l+2}\right]^{\varepsilon_2}}, \tag{20}$$

$$|\nu_3(x,y,t)| \le \frac{c_3}{\left[1 + \frac{4}{(n+2)^2} x^{n+2} + \frac{4}{(m+2)^2} y^{m+2} + \frac{4}{(l+2)^2} t^{l+2}\right]^{\frac{1-2\gamma+\varepsilon_3}{2}}}, \tag{21}$$

$$|\nu_4(x,y,z)| \le \frac{c_4}{\left[1 + \frac{4}{(n+2)^2} x^{n+2} + \frac{4}{(m+2)^2} y^{m+2} + \frac{4}{(k+2)^2} z^{k+2}\right]^{\frac{1-2\delta+\varepsilon_4}{2}}}, \tag{22}$$

here c_1, c_2, c_3, $c_4 > 0$ and ε_1, ε_2, ε_3, ε_4 are small enough positive numbers.

Theorem 1. The boundary value problem ND_2 has at most one solution.

Proof. Let $u(x,y,z,t)$ be the solution of a homogeneous problem ND_2, i.e. $u(x,y,z,t)$ is the solution of the equation (13) satisfying the conditions (14)–(18).

By D_R we denote the bounded domain with the boundary $\partial D_R = S_{1R} \cup S_{2R} \cup S_{3R} \cup S_{4R} \cup \sigma_R$, where $S_{1R} = S_1 \cap \{x = 0,\, 0 < y < R,\, 0 < z < R,\, 0 < t < R\}$, $S_{2R} = S_2 \cap \{0 < x < R,\, y = 0,\, 0 < z < R,\, 0 < t < R\}$, $S_{3R} = S_3 \cap \{0 < x < R,\, 0 < y < R,\, z = 0,\, 0 < t < R\}$,$S_{4R} = S_4 \cap \{0 < x < R,\, 0 < y < R,\, 0 < z < R,\, t = 0\}$, $\sigma_R = \left\{(x,\,y,\,z,\,t) : \frac{4}{(n+2)^2} x^{n+2} + \frac{4}{(m+2)^2} y^{m+2} + \frac{4}{(k+2)^2} z^{k+2} + \frac{4}{(l+2)^2} t^{l+2} = R^2,\, x \ge 0,\, y \ge 0,\, z \ge 0,\, t \ge 0\right\}$.

Choosing large enough R, we integrate equation (13) over the domain D_R, previously multiplied it by a function $u(x,y,z,t)$, we obtain

$$\iiiint_{D_R} \left[y^m z^k t^l u u_{xx} + x^n z^k t^l u u_{yy} + x^n y^m t^l u u_{zz} + x^n y^m z^k u u_{tt}\right] dxdydzdt = 0. \tag{23}$$

Taking into account (23) we obtain the following equalities:

$$y^m z^k t^l u u_{xx} = \frac{\partial}{\partial x}\left(y^m z^k t^l u u_x\right) - y^m z^k t^l u_x^2, \quad x^n z^k t^l u u_{yy} = \frac{\partial}{\partial y}\left(x^n z^k t^l u u_y\right) - x^n z^k t^l u_y^2,$$

$$x^n y^m t^l u u_{zz} = \frac{\partial}{\partial z}\left(x^n y^m t^l u u_z\right) - x^n y^m t^l u_z^2, \quad x^n y^m z^k u u_{tt} = \frac{\partial}{\partial t}\left(x^n y^m z^k u u_t\right) - x^n y^m z^k u_t^2,$$

after applying the Gauss-Ostrogradsky formula, we have

$$\begin{aligned} &\iiiint_{D_R} \left[y^m z^k t^l u_x^2 + x^n z^k t^l u_y^2 + x^n y^m t^l u_z^2 + x^n y^m z^k u_t^2\right] dxdydzdt = \\ &= \iiint_{S_{1R}} y^m z^k t^l \tau_1 u_x dydzdt + \iiint_{S_{2R}} x^n z^k t^l \tau_2 u_y dxdzdt + \iiint_{S_{3R}} x^n y^m t^l u \nu_3 dxdydt \\ &+ \iiint_{S_{4R}} x^n y^m z^k u \nu_4 dxdydz + \iiint_{\sigma_R} x^n y^m z^k t^l u \frac{\partial u}{\partial n} dS, \end{aligned} \tag{24}$$

where,

$$\frac{\partial u}{\partial n} = u_x \cos(n,x) + u_y \cos(n,y) + u_z \cos(n,z) + u_t \cos(n,t),$$

$\cos(n,x)\,dS = dydzdt$, $\cos(n,y)\,dS = dxdzdt$, $\cos(n,z)\,dS = dxdydt$, $\cos(n,t)\,dS = dxdydz$, n is outer normal to ∂D_R.

Since for the function u $\tau_1 = \tau_2 = \nu_3 = \nu_4 = 0$, then from (24) we have

$$\iiiint\limits_{D_R} \left[y^m z^k t^l u_x^2 + x^n z^k t^l u_y^2 + x^n y^m t^l u_z^2 + x^n y^m z^k u_t^2 \right] dxdydzdt = \iiint\limits_{\sigma_R} x^n y^m z^k t^l u \frac{\partial u}{\partial n} dS. \tag{25}$$

By virtue of condition (18) for $R \to \infty$ $\lim\limits_{R\to\infty} \iiint\limits_{\sigma_R} x^n y^m z^k t^l u \frac{\partial u}{\partial n} dS = 0$, then from (25) we have

$$\iiiint\limits_{D} \left[y^m z^k t^l u_x^2 + x^n z^k t^l u_y^2 + x^n y^m t^l u_z^2 + x^n y^m z^k u_t^2 \right] dxdydzdt \equiv 0. \tag{26}$$

From (26), we get $u_x = u_y = u_z = u_t = 0$, which means $u = const$, and from the conditions $u|_{x=0} = u|_{y=0} = u_z|_{z=0} = u_t|_{t=0} = 0$ follows that $u \equiv 0$. So, we have proved the uniqueness of the problem ND_2 .

3. Existence of a problem solution

The solution to the ND_2 problem has the form

$$\begin{aligned} u(x_0,y_0,z_0,t_0) = &\int\limits_0^\infty\int\limits_0^\infty\int\limits_0^\infty y^m z^k t^l \tau_1(y,z,t) \frac{\partial}{\partial x} g_6(x,y,z,t;x_0,y_0,z_0,t_0)\Big|_{x=0} dydzdt + \\ &+ \int\limits_0^\infty\int\limits_0^\infty\int\limits_0^\infty x^n z^k t^l \tau_2(x,z,t) \frac{\partial}{\partial y} g_6(x,y,z,t;x_0,y_0,z_0,t_0)|_{y=0} dxdzdt - \\ &- \int\limits_0^\infty\int\limits_0^\infty\int\limits_0^\infty x^n y^m t^l \nu_3(x,y,t)\, g_6(x,y,0,t;x_0,y_0,z_0,t_0)\, dxdydt - \\ &- \int\limits_0^\infty\int\limits_0^\infty\int\limits_0^\infty x^n y^m z^k \nu_4(x,y,z)\, g_6(x,y,z,0;x_0,y_0,z_0,t_0)\, dxdydz, \end{aligned} \tag{27}$$

where

$$g_6(x,y,z,t;x_0,y_0,z_0,t_0) = k_6 \left(\frac{4}{n+2}\right)^{\frac{4}{n+2}} \left(\frac{4}{m+2}\right)^{\frac{4}{m+2}} \left(r^2\right)^{\alpha+\beta-\gamma-\delta-3} xyx_0y_0 \times$$
$$\times F_A^{(4)}(3-\alpha-\beta+\gamma+\delta; 1-\alpha, 1-\beta, \gamma, \delta; 2-2\alpha, 2-2\beta, 2\gamma, 2\delta; \xi, \eta, \zeta, \varsigma)$$

is fundamental solution to the equation (13). Here function $F_A^{(4)}$ is Lauricella's function (6),

$$\begin{aligned} k_6 = \frac{1}{4\pi^2} \left(\frac{4}{n+2}\right)^{2\alpha} \left(\frac{4}{m+2}\right)^{2\beta} \left(\frac{4}{k+2}\right)^{2\gamma} \left(\frac{4}{l+2}\right)^{2\delta} \times \\ \times \frac{\Gamma(3-\alpha-\beta+\gamma+\delta)\,\Gamma(1-\alpha)\,\Gamma(1-\beta)\,\Gamma(\gamma)\,\Gamma(\delta)}{\Gamma(2-2\alpha)\,\Gamma(2-2\beta)\,\Gamma(2\gamma)\,\Gamma(2\delta)}, \end{aligned} \tag{28}$$

$$\xi = \frac{r^2 - r_1^2}{r^2},\ \ \eta = \frac{r^2 - r_2^2}{r^2},\ \ \zeta = \frac{r^2 - r_3^2}{r^2},\ \ \varsigma = \frac{r^2 - r_4^2}{r^2},$$

$$\left.\begin{matrix} r^2 \\ r_1^2 \\ r_2^2 \\ r_3^2 \\ r_4^2 \end{matrix}\right\} = \left(\frac{2}{n+2}x^{\frac{n+2}{2}} \begin{matrix} - \\ + \\ - \\ - \\ - \end{matrix} \frac{2}{n+2}x_0^{\frac{n+2}{2}}\right)^2 + \left(\frac{2}{m+2}y^{\frac{m+2}{2}} \begin{matrix} - \\ - \\ + \\ - \\ - \end{matrix} \frac{2}{m+2}y_0^{\frac{m+2}{2}}\right)^2$$

$$+\left(\frac{2}{k+2}z^{\frac{k+2}{2}} \begin{matrix} - \\ - \\ - \\ + \\ - \end{matrix} \frac{2}{k+2}z_0^{\frac{k+2}{2}}\right)^2 + \left(\frac{2}{l+2}t^{\frac{l+2}{2}} \begin{matrix} - \\ - \\ - \\ - \\ + \end{matrix} \frac{2}{l+2}t_0^{\frac{l+2}{2}}\right)^2,$$

$$\alpha = \frac{n}{2\left(n+2\right)},\ \ \beta = \frac{m}{2\left(m+2\right)},\ \ \gamma = \frac{k}{2\left(k+2\right)},\ \ \delta = \frac{l}{2\left(l+2\right)}.$$

Since the function q_6 is a fundamental solution to equation (13), it is obvious that the solution to problem (27) satisfies equation (13).

Let us prove that function (27) satisfies conditions (14) - (17) of problem ND_2. We apply differentiation formulas (8) and decomposition of hypergeometric functions (9) to (27) and represent (27) as the sum:

$$u\left(x_0, y_0, z_0, t_0\right) = I_1\left(x_0, y_0, z_0, t_0\right) + I_2\left(x_0, y_0, z_0, t_0\right) + I_3\left(x_0, y_0, z_0, t_0\right) + I_4\left(x_0, y_0, z_0, t_0\right), \tag{29}$$

where

$$I_1\left(x_0, y_0, z_0, t_0\right) = k_6\left(\frac{4}{n+2}\right)^{\frac{4}{n+2}}\left(\frac{4}{m+2}\right)^{\frac{4}{m+2}} x_0 y_0 \int\limits_0^{\infty}\int\limits_0^{\infty}\int\limits_0^{\infty} y^{m+1}z^k t^l \tau_1\left(y, z, t\right) \times$$
$$\times\left(r^2\right)^{\alpha+\beta-\gamma-\delta-3} F_A^{(3)}\left(3-\alpha-\beta+\gamma+\delta; 1-\beta, \gamma, \delta; 2-2\beta, 2\gamma, 2\delta; \eta, \zeta, \varsigma\right)\Big|_{x=0} dydzdt, \tag{30}$$

$$I_2\left(x_0, y_0, z_0, t_0\right) = k_6\left(\frac{4}{n+2}\right)^{\frac{4}{n+2}}\left(\frac{4}{m+2}\right)^{\frac{4}{m+2}} x_0 y_0 \int\limits_0^{\infty}\int\limits_0^{\infty}\int\limits_0^{\infty} x^{n+1}z^k t^l \tau_2\left(x, z, t\right) \times$$
$$\times\left(r^2\right)^{\alpha+\beta-\gamma-\delta-3} F_A^{(3)}\left(3-\alpha-\beta+\gamma+\delta; 1-\alpha, \gamma, \delta; 2-2\alpha, 2\gamma, 2\delta; \xi, \zeta, \varsigma\right)\Big|_{y=0} dxdzdt, \tag{31}$$

$$I_3\left(x_0, y_0, z_0, t_0\right) = -k_6\left(\frac{4}{n+2}\right)^{\frac{4}{n+2}}\left(\frac{4}{m+2}\right)^{\frac{4}{m+2}} x_0 y_0 \int\limits_0^{\infty}\int\limits_0^{\infty}\int\limits_0^{\infty} x^{n+1}y^{m+1} t^l \nu_3\left(x, y, t\right) \times$$
$$\times\left(r^2\right)^{\alpha+\beta-\gamma-\delta-3} F_A^{(3)}\left(3-\alpha-\beta+\gamma+\delta; 1-\alpha, 1-\beta, \delta; 2-2\alpha, 2-2\beta, 2\delta; \xi, \eta, \varsigma\right)\Big|_{z=0} dxdydt, \tag{32}$$

$$I_4\left(x_0, y_0, z_0, t_0\right) = -k_6\left(\frac{4}{n+2}\right)^{\frac{4}{n+2}}\left(\frac{4}{m+2}\right)^{\frac{4}{m+2}} x_0 y_0 \int\limits_0^{\infty}\int\limits_0^{\infty}\int\limits_0^{\infty} x^{n+1}y^{m+1} z^k \nu_4\left(x, y, z\right) \times$$
$$\left(r^2\right)^{\alpha+\beta-\gamma-\delta-3} F_A^{(3)}\left(3-\alpha-\beta+\gamma+\delta; 1-\alpha, 1-\beta, \gamma; 2-2\alpha, 2-2\beta, 2\gamma; \xi, \eta, \zeta\right)\Big|_{t=0} dxdydz. \tag{33}$$

Let us check condition (14). Consider the first term of the solution, written in the form (29), function (30). We decompose the function $F_A^{(3)}$ in (30) by formula (7), then after performing some transformations in (30) and applying the Bolz autotransformation formula (3), we obtain

$$F_A\left(3-\alpha-\beta+\gamma+\delta; 1-\beta, \gamma, \delta; 2-2\beta, 2\gamma, 2\delta; \eta, \zeta, \varsigma\right) =$$
$$= \left(r^2\right)^{1-\beta+\gamma+\delta}\left(r_2^2\right)^{\beta-1}\left(r_3^2\right)^{-\gamma}\left(r_4^2\right)^{-\delta} P_1\left(0, y, z, t; x_0, y_0, z_0, t_0\right), \tag{34}$$

where

$$
\begin{aligned}
P_1\left(0,y,z,t;x_0,y_0,z_0,t_0\right) = & \sum_{l_1,l_2,l_3=0}^{\infty} \frac{(3-\alpha-\beta+\gamma+\delta)_{l_1+l_2+l_3}(1-\beta)_{l_1+l_2}(\gamma)_{l_1+l_3}(\delta)_{l_2+l_3}}{(2-2\beta)_{l_1+l_2}(2\gamma)_{l_1+l_3}(2\delta)_{l_2+l_3}l_1!l_2!l_3!} \\
& \times \left(\frac{r_2^2-r^2}{r_2^2}\right)^{l_1+l_2}\left(\frac{r_3^2-r^2}{r_3^2}\right)^{l_1+l_3}\left(\frac{r_4^2-r^2}{r_4^2}\right)^{l_2+l_3} \\
& \times F\left(\alpha-\beta-\gamma-\delta-1, 1-\beta+l_1+l_2; 2-2\beta+l_1+l_2; \frac{r_2^2-r^2}{r_2^2}\right) \\
& \times F\left(\alpha+\beta+\gamma-\delta-3-l_2, \gamma+l_1+l_3; 2\gamma+l_1+l_3; \frac{r_3^2-r^2}{r_3^2}\right) \\
& \times F\left(\alpha+\beta-\gamma+\delta-3-l_1, \delta+l_2+l_3; 2\delta+l_2+l_3; \frac{r_4^2-r^2}{r_4^2}\right).
\end{aligned} \tag{35}
$$

Thus, substituting (34) into (30), we have

$$
\begin{aligned}
& I_1\left(x_0,y_0,z_0,t_0\right) = k_6\left(\frac{4}{n+2}\right)^{\frac{4}{n+2}}\left(\frac{4}{m+2}\right)^{\frac{4}{m+2}} x_0 y_0 \times \\
& \times \int\limits_0^\infty\int\limits_0^\infty\int\limits_0^\infty y^{m+1}z^k t^l \tau_1\left(y,z,t\right) \left.\frac{P_1\left(0,y,z,t;x_0,y_0,z_0,t_0\right)}{\left(r^2\right)^{2-\alpha}\left(r_2^2\right)^{1-\beta}\left(r_3^2\right)^{\gamma}\left(r_4^2\right)^{\delta}}\right|_{x=0} dydzdt.
\end{aligned} \tag{36}
$$

In (36), we make the change of variables

$$
\begin{gathered}
\frac{2}{m+2}y^{\frac{m+2}{2}} = \frac{2}{m+2}y_0^{\frac{m+2}{2}} + \frac{2}{n+2}x_0^{\frac{n+2}{2}}s_1, \ \frac{2}{k+2}z^{\frac{k+2}{2}} = \frac{2}{k+2}z_0^{\frac{k+2}{2}} + \frac{2}{n+2}x_0^{\frac{n+2}{2}}s_2, \\
\frac{2}{l+2}t^{\frac{l+2}{2}} = \frac{2}{l+2}t_0^{\frac{l+2}{2}} + \frac{2}{n+2}x_0^{\frac{n+2}{2}}s_3.
\end{gathered} \tag{37}
$$

Then, we obtain the following equality

$$
\begin{aligned}
& I_1\left(x_0,y_0,z_0,t_0\right) = k_6\left(\frac{4}{n+2}\right)^{\frac{4}{n+2}}\left(\frac{4}{m+2}\right)^{\frac{4}{m+2}} x_0 y_0 \left(\frac{2}{n+2}x_0^{\frac{n+2}{2}}\right)^3 \\
& \int\limits_{-a}^\infty\int\limits_{-b}^\infty\int\limits_{-c}^\infty \left[\left[\frac{m+2}{2}\left(\frac{2}{m+2}y_0^{\frac{m+2}{2}} + \frac{2}{n+2}x_0^{\frac{n+2}{2}}s_1\right)\right]^{\frac{2}{m+2}}\right]^{m+1} \left[\left[\frac{k+2}{2}\left(\frac{2}{k+2}z_0^{\frac{k+2}{2}} + \frac{2}{n+2}x_0^{\frac{n+2}{2}}s_2\right)\right]^{\frac{2}{k+2}}\right]^{k} \\
& \times \left[\left[\frac{l+2}{2}\left(\frac{2}{l+2}t_0^{\frac{l+2}{2}} + \frac{2}{n+2}x_0^{\frac{n+2}{2}}s_3\right)\right]^{\frac{2}{l+2}}\right]^{l} \left.\frac{P_1\left(0,y,z,t;x_0,y_0,z_0,t_0\right)}{\left(1+s_1^2+s_2^2+s_3^2\right)^{2-\alpha}\left(r_2^2\right)^{1-\beta}\left(r_3^2\right)^{\gamma}\left(r_4^2\right)^{\delta}}\right|_{x=0} \\
& \times \tau_1\left(\left[\frac{m+2}{2}\left(\frac{2}{m+2}y_0^{\frac{m+2}{2}} + \frac{2}{n+2}x_0^{\frac{n+2}{2}}s_1\right)\right]^{\frac{2}{m+2}}, \left[\frac{k+2}{2}\left(\frac{2}{k+2}z_0^{\frac{k+2}{2}} + \frac{2}{n+2}x_0^{\frac{n+2}{2}}s_2\right)\right]^{\frac{2}{k+2}},\right. \\
& \left.\left[\frac{l+2}{2}\left(\frac{2}{l+2}t_0^{\frac{l+2}{2}} + \frac{2}{n+2}x_0^{\frac{n+2}{2}}s_3\right)\right]^{\frac{2}{l+2}}\right) \left[\frac{m+2}{2}\left(\frac{2}{m+2}y_0^{\frac{m+2}{2}} + \frac{2}{n+2}x_0^{\frac{n+2}{2}}s_1\right)\right]^{-\frac{m}{m+2}} \times \\
& \times \left[\frac{k+2}{2}\left(\frac{2}{k+2}z_0^{\frac{k+2}{2}} + \frac{2}{n+2}x_0^{\frac{n+2}{2}}s_2\right)\right]^{-\frac{k}{k+2}} \left[\frac{l+2}{2}\left(\frac{2}{l+2}t_0^{\frac{l+2}{2}} + \frac{2}{n+2}x_0^{\frac{n+2}{2}}s_3\right)\right]^{-\frac{l}{l+2}} ds_1 ds_2 ds_3,
\end{aligned} \tag{38}
$$

where

$$
a = \frac{\frac{2}{m+2}y_0^{\frac{m+2}{2}}}{\frac{2}{n+2}x_0^{\frac{n+2}{2}}}, b = \frac{\frac{2}{k+2}z_0^{\frac{k+2}{2}}}{\frac{2}{n+2}x_0^{\frac{n+2}{2}}}, \ c = \frac{\frac{2}{l+2}t_0^{\frac{l+2}{2}}}{\frac{2}{n+2}x_0^{\frac{n+2}{2}}}.
$$

At $x_0 \to 0$ from (35) we have

$$\begin{aligned}
&\lim_{x_0 \to 0} P_1(0,y,z,t;x_0,y_0,z_0,t_0) = \sum_{l_1,l_2,l_3=0}^{\infty} \frac{(3-\alpha-\beta+\gamma+\delta)_{l_1+l_2+l_3}(1-\beta)_{l_1+l_2}(\gamma)_{l_1+l_3}(\delta)_{l_2+l_3}}{(2-2\beta)_{l_1+l_2}(2\gamma)_{l_1+l_3}(2\delta)_{l_2+l_3}l_1!l_2!l_3!} \\
&\times F(\alpha-\beta-\gamma-\delta-1, 1-\beta+l_1+l_2; 2-2\beta+l_1+l_2; 1) \\
&\times F(\alpha+\beta+\gamma-\delta-3-l_2, \gamma+l_1+l_3; 2\gamma+l_1+l_3; 1) \\
&\times F(\alpha+\beta-\gamma+\delta-3-l_1, \delta+l_2+l_3; 2\delta+l_2+l_3; 1)
\end{aligned} \tag{39}$$

Applying formulas (2) and (5) to (39), we determine

$$\lim_{x_0 \to 0} P_1(0,y,z,t;x_0,y_0,z_0,t_0) = \frac{\Gamma(2-2\beta)\Gamma(2\gamma)\Gamma(2\delta)\Gamma(2-\alpha)}{\Gamma(3-\alpha-\beta+\gamma+\delta)\Gamma(1-\beta)\Gamma(\gamma)\Gamma(\delta)} \tag{40}$$

By virtue of (40), from (38) at $x_0 \to 0$, we obtain

$$\begin{aligned}
&\lim_{x_0 \to 0} I_1(x_0,y_0,z_0,t_0) = k_6 \left(\frac{8}{n+2}\right)^{\frac{2}{n+2}} \left(\frac{4}{m+2}\right)^{-2\beta} \left(\frac{4}{k+2}\right)^{-2\gamma} \left(\frac{4}{l+2}\right)^{-2\delta} \times \\
&\times \frac{\Gamma(2-2\beta)\Gamma(2\gamma)\Gamma(2\delta)\Gamma(2-\alpha)}{\Gamma(3-\alpha-\beta+\gamma+\delta)\Gamma(1-\beta)\Gamma(\gamma)\Gamma(\delta)} \tau_1(y_0,z_0,t_0) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{ds_1 ds_2 ds_3}{(1+s_1^2+s_2^2+s_3^2)^{2-\alpha}}.
\end{aligned} \tag{41}$$

To calculate the triple integral from (41), using formula (10), we get

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{ds_1 ds_2 ds_3}{(1+s_1^2+s_2^2+s_3^2)^{2-\alpha}} = 8\int_{0}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} \frac{ds_1 ds_2 ds_3}{(1+s_1^2+s_2^2+s_3^2)^{2-\alpha}} = \frac{\pi\sqrt{\pi}\Gamma\left(\frac{1}{2}-\alpha\right)}{\Gamma(2-\alpha)}. \tag{42}$$

Applying formula (1) in (42), as a result, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{ds_1 ds_2 ds_3}{(1+s_1^2+s_2^2+s_3^2)^{2-\alpha}} = \frac{\pi^2\Gamma(2-2\alpha)}{2^{-2\alpha}\Gamma(2-\alpha)(1-2\alpha)\Gamma(1-\alpha)}. \tag{43}$$

Substituting (43) into (41), we finally have

$$\begin{aligned}
&\lim_{x_0 \to 0} I_1(x_0,y_0,z_0,t_0) = 4\pi^2 k_6 \left(\frac{4}{n+2}\right)^{-2\alpha} \left(\frac{4}{m+2}\right)^{-2\beta} \left(\frac{4}{k+2}\right)^{-2\gamma} \left(\frac{4}{l+2}\right)^{-2\delta} \times \\
&\times \frac{\Gamma(2-2\alpha)\Gamma(2-2\beta)\Gamma(2\gamma)\Gamma(2\delta)}{\Gamma(3-\alpha-\beta+\gamma+\delta)\Gamma(1-\alpha)\Gamma(1-\beta)\Gamma(\gamma)\Gamma(\delta)} \tau_1(y_0,z_0,t_0)
\end{aligned} \tag{44}$$

Taking into account (28), from (44), we obtain

$$\lim_{x_0 \to 0} I_1(x_0,y_0,z_0,t_0) = \tau_1(y_0,z_0,t_0).$$

It is easy to show that

$$\lim_{x_0 \to 0} I_2(x_0,y_0,z_0,t_0) = 0, \ \lim_{x_0 \to 0} I_3(x_0,y_0,z_0,t_0) = 0, \ \lim_{x_0 \to 0} I_4(x_0,y_0,z_0,t_0) = 0.$$

Accordingly, $\lim_{x_0 \to 0} u(x_0,y_0,z_0,t_0) = \tau_1(y_0,z_0,t_0)$, hence, function (29) satisfies condition (14) of the problem ND_2. Similarly, can be convinced that function (29) also satisfies conditions (15), (16), and (17) of the problem ND_2.

Let us show that if the given functions satisfy inequalities (19) – (22) for large enough values of the argument, then the solution (29) of the Problem ND_2 also satisfies condition (18). Indeed, let inequalities (19) – (22) are hold, in expressions (30) – (33) we make the following change of variables

$$\xi_1 = \frac{1}{R_0}\frac{2}{n+2}x^{\frac{n+2}{2}}, \ \eta_1 = \frac{1}{R_0}\frac{2}{m+2}y^{\frac{m+2}{2}}, \ \zeta_1 = \frac{1}{R_0}\frac{2}{k+2}z^{\frac{k+2}{2}}, \ \varsigma_1 = \frac{1}{R_0}\frac{2}{l+2}t^{\frac{l+2}{2}},$$
$$\sigma_1 = \frac{1}{R_0}\frac{2}{n+2}x_0^{\frac{n+2}{2}}, \ \sigma_2 = \frac{1}{R_0}\frac{2}{m+2}y_0^{\frac{m+2}{2}}, \ \sigma_3 = \frac{1}{R_0}\frac{2}{k+2}z_0^{\frac{k+2}{2}}, \ \sigma_4 = \frac{1}{R_0}\frac{2}{l+2}t_0^{\frac{l+2}{2}},$$

where

$$R_0{}^2 = \frac{4}{(n+2)^2} x_0{}^{n+2} + \frac{4}{(m+2)^2} y_0{}^{m+2} + \frac{4}{(k+2)^2} z_0{}^{k+2} + \frac{4}{(l+2)^2} t_0{}^{l+2}.$$

Then at $R_0 \to \infty$ from (30)–(33) we obtain the following inequalities:

$$\lim_{R_0\to\infty} |I_1(x_0,y_0,z_0,t_0)| \le \frac{k_6 c_1}{R_0^{2\varepsilon_1}} 4^{\frac{2}{n+2}+\frac{2}{m+2}} (\sigma_1)^{\frac{2}{n+2}} (\sigma_2)^{\frac{2}{m+2}} \left(\frac{2}{n+2}\right)^{1-2\alpha} \left(\frac{2}{m+2}\right)^{-2\beta} \times$$
$$\left(\frac{2}{k+2}\right)^{-2\gamma} \left(\frac{2}{l+2}\right)^{-2\delta} \int_0^\infty\int_0^\infty\int_0^\infty \frac{\eta_1 \zeta_1{}^{2\gamma} \varsigma_1{}^{2\delta}}{\left(1+\eta_1{}^2+\zeta_1{}^2+\varsigma_1{}^2\right)^{3-\alpha-\beta+\gamma+\delta} \left(\eta_1{}^2+\zeta_1{}^2+\varsigma_1{}^2\right)^{\varepsilon_1}} d\eta_1 d\zeta_1 d\varsigma_1, \quad (45)$$

$$\lim_{R_0\to\infty} |I_2(x_0,y_0,z_0,t_0)| \le \frac{k_6 c_2}{R_0^{2\varepsilon_2}} 4^{\frac{2}{n+2}+\frac{2}{m+2}} (\sigma_1)^{\frac{2}{n+2}} (\sigma_2)^{\frac{2}{m+2}} \left(\frac{2}{n+2}\right)^{-2\alpha} \left(\frac{2}{m+2}\right)^{1-2\beta} \times$$
$$\left(\frac{2}{k+2}\right)^{-2\gamma} \left(\frac{2}{l+2}\right)^{-2\delta} \int_0^\infty\int_0^\infty\int_0^\infty \frac{\xi_1 \zeta_1{}^{2\gamma} \varsigma_1{}^{2\delta}}{\left(1+\xi_1{}^2+\zeta_1{}^2+\varsigma_1{}^2\right)^{3-\alpha-\beta+\gamma+\delta} \left(\xi_1{}^2+\zeta_1{}^2+\varsigma_1{}^2\right)^{\varepsilon_2}} d\xi_1 d\zeta_1 d\varsigma_1, \quad (46)$$

$$\lim_{R_0\to\infty} |I_3(x_0,y_0,z_0,t_0)| \le \frac{k_6 c_3}{R_0^{\varepsilon_3}} 4^{\frac{2}{n+2}+\frac{2}{m+2}} (\sigma_1)^{\frac{2}{n+2}} (\sigma_2)^{\frac{2}{m+2}} \left(\frac{2}{n+2}\right)^{-2\alpha} \left(\frac{2}{m+2}\right)^{-2\beta} \times$$
$$\left(\frac{2}{l+2}\right)^{-2\delta} \int_0^\infty\int_0^\infty\int_0^\infty \frac{\xi_1 \eta_1 \varsigma_1{}^{2\delta}}{\left(1+\xi_1{}^2+\eta_1{}^2+\varsigma_1{}^2\right)^{3-\alpha-\beta+\gamma+\delta} \left(\xi_1{}^2+\eta_1{}^2+\varsigma_1{}^2\right)^{\frac{1-2\gamma+\varepsilon_3}{2}}} d\xi_1 d\eta_1 d\varsigma_1, \quad (47)$$

$$\lim_{R_0\to\infty} |I_4(x_0,y_0,z_0,t_0)| \le \frac{k_6 c_4}{R_0^{\varepsilon_4}} 4^{\frac{2}{n+2}+\frac{2}{m+2}} (\sigma_1)^{\frac{2}{n+2}} (\sigma_2)^{\frac{2}{m+2}} \left(\frac{2}{n+2}\right)^{-2\alpha} \left(\frac{2}{m+2}\right)^{-2\beta} \times$$
$$\left(\frac{2}{k+2}\right)^{-2\gamma} \int_0^\infty\int_0^\infty\int_0^\infty \frac{\xi_1 \eta_1 \zeta_1{}^{2\gamma}}{\left(1+\xi_1{}^2+\eta_1{}^2+\zeta_1{}^2\right)^{3-\alpha-\beta+\gamma+\delta} \left(\xi_1{}^2+\eta_1{}^2+\zeta_1{}^2\right)^{\frac{1-2\delta+\varepsilon_4}{2}}} d\xi_1 d\eta_1 d\zeta_1. \quad (48)$$

Let us show that the triple integrals in inequalities (45) – (48) are bounded.
Considering the integrals from inequality (45) – (46), these integrals satisfy the identity.

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{x y^{2b} z^{2c} dx dy dz}{\left(1+x^2+y^2+z^2\right)^{3-a-b+c+d} \left(x^2+y^2+z^2\right)^{\varepsilon}} =$$
$$= \frac{1}{8} \frac{\Gamma\left(\frac{1}{2}+c\right)\Gamma\left(\frac{1}{2}+d\right)\Gamma\left(2+c+d-\varepsilon\right)\Gamma\left(1-a-b+\varepsilon\right)}{\Gamma\left(2+c+d\right)\Gamma\left(3-a-b+c+d\right)}, \quad a+b-1<\varepsilon<2+c+d. \quad (49)$$

Indeed, in integral (49), passing into spherical coordinates, we have

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{x y^{2b} z^{2c} dx dy dz}{\left(1+x^2+y^2+z^2\right)^{3-a-b+c+d} \left(x^2+y^2+z^2\right)^{\varepsilon}} =$$
$$= \int_0^{\frac{\pi}{2}} \sin^{2c}\varphi \cos\varphi d\varphi \int_0^{\frac{\pi}{2}} \sin^{2+2c}\theta \cos^{2d}\theta d\theta \int_0^\infty r^{3+2c+2d-2\varepsilon} \left(1+r^2\right)^{a+b-c-d-3} dr. \quad (50)$$

Using the values of integrals (11) and (12) in expression (50), we obtain the identity (49)

$$\int_0^{\frac{\pi}{2}} \sin^{2c}\varphi \cos\varphi d\varphi \int_0^{\frac{\pi}{2}} \sin^{2+2c}\theta \cos^{2d}\theta d\theta \int_0^\infty r^{3+2c+2d-2\varepsilon} \left(1+r^2\right)^{a+b-c-d-3} dr =$$
$$= \frac{1}{8} \frac{\Gamma\left(\frac{1}{2}+c\right)\Gamma\left(\frac{1}{2}+d\right)\Gamma\left(2+c+d-\varepsilon\right)\Gamma\left(1-a-b+\varepsilon\right)}{\Gamma\left(2+c+d\right)\Gamma\left(3-a-b+c+d\right)}, \quad a+b-1<\varepsilon<2+c+d.$$

Thus, inequalities (45) - (46) by virtue of the value of integral (49) the inequalities follow

$$\lim_{R_0\to\infty} |I_1(x_0,y_0,z_0,t_0)| \le \frac{k_6\overline{c_1}}{R_0{}^{2\varepsilon_1}}, \ \lim_{R_0\to\infty} |I_2(x_0,y_0,z_0,t_0)| \le \frac{k_6\overline{c_2}}{R_0{}^{2\varepsilon_2}}, \tag{51}$$

where $\overline{c_1}$, $\overline{c_2}$ are constants.

Let us show that the integrals in (47) – (48) are bounded. For inequalities (47) – (48), the identity is true

$$\begin{gathered}\int_0^\infty\int_0^\infty\int_0^\infty \frac{xyt^{2d}}{(1+x^2+y^2+t^2)^{3-a-b+c+d}(x^2+y^2+t^2)^{\frac{1-2c+\varepsilon}{2}}}dxdydt = \\ = \frac{1}{8}\frac{\Gamma\left(\frac{1}{2}+d\right)}{\Gamma\left(\frac{5}{2}+d\right)}\frac{\Gamma\left(2+c+d-\frac{\varepsilon}{2}\right)\Gamma\left(1-a-b+\frac{\varepsilon}{2}\right)}{\Gamma(3-a-b+c+d)}, \quad 2a+2b-2<\varepsilon<4+2c+2d.\end{gathered} \tag{52}$$

Passing into spherical coordinates in (52), we obtain

$$\begin{gathered}\int_0^\infty\int_0^\infty\int_0^\infty \frac{xyt^{2d}}{(1+x^2+y^2+t^2)^{3-a-b+c+d}(x^2+y^2+t^2)^{\frac{1-2c+\varepsilon}{2}}}dxdydt = \\ = \int_0^{\frac{\pi}{2}} \sin\varphi\cos\varphi d\varphi \int_0^{\frac{\pi}{2}} \sin^3\theta r \cos^{2d}\theta d\theta \int_0^\infty \left(1+r^2\right)^{a+b-c-d-3} r^{3+2c+2d-\varepsilon} dr.\end{gathered} \tag{53}$$

Using formulas (11) and (12) to the right-hand side of (53), we define

$$\begin{gathered}\int_0^{\frac{\pi}{2}} \sin\varphi\cos\varphi d\varphi \int_0^{\frac{\pi}{2}} \sin^3\theta r \cos^{2d}\theta d\theta \int_0^\infty \left(1+r^2\right)^{a+b-c-d-3} r^{3+2c+2d-\varepsilon} d = \\ = \frac{1}{8}\frac{\Gamma\left(\frac{1}{2}+d\right)}{\Gamma\left(\frac{5}{2}+d\right)}\frac{\Gamma\left(2+c+d-\frac{\varepsilon}{2}\right)\Gamma\left(1-a-b+\frac{\varepsilon}{2}\right)}{\Gamma(3-a-b+c+d)}, \ 2a+2b-2<\varepsilon<4+2c+2d.\end{gathered}$$

Thus, we have shown that the integrals in inequalities (47) – (48) are bounded; the integrals satisfy the inequalities

$$\lim_{R_0\to\infty} |I_3(x_0,y_0,z_0,t_0)| \le \frac{k_6\overline{c_3}}{R_0{}^{\varepsilon_3}}, \lim_{R_0\to\infty} |I_4(x_0,y_0,z_0,t_0)| \le \frac{k_6\overline{c_4}}{R_0{}^{\varepsilon_4}}, \tag{54}$$

where $\overline{c_3}$, $\overline{c_4}$ are constants. Inequalities (51) and (54) show that solution (27) at $R_0 \to \infty$ tends to zero. Thereby, condition (18) of Problem ND_2 is satisfied. In this connection, solution (27) of Problem ND_2 satisfies all conditions of Problem ND_2.

Conclusions

We have proved the following theorem.

Theorem 2. Let conditions (19) – (22) be satisfied, then a regular solution to problem ND_2 (13), (14) – (18) exists and is expressed by formula (27).

In four-dimensional space in an infinite domain for the degenerate elliptic Gellerstedt equation, the problem ND_2 with two Neumann boundary conditions and with two Dirichlet conditions is solved. The solution is written explicitly. The uniqueness and existence of a solution to the equation are proved.

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А.С. Бердышев[1,2], А.Р. Рысқан[1,2]

[1] *Абай атындағы Қазақ ұлттық педагогикалық университеті, Алматы, Қазақстан;*
[2] *Ақпараттық және есептеуіш технологиялар институты, Алматы, Қазақстан*

Төрт өлшемді Геллерстедт теңдеуі үшін шеттік есеп

Мақалада төрт айнымалы Геллерстедт теңдеу үшін Нейман және Дирихле шарттары бар шеттік есептің шешілуі зерттелген. Есеп шешімінің жалғыздығын дәлелдеу үшін энергия интегралы әдісі қолданылған. Сонымен қатар, шешімінің бар болуына гипергеометриялық функцияларды дифференциациялау, автотрансформациялау және жіктеу формулалары пайдаланылған. Шешім айқын түрде алынған және Лауричелла гипергеометриялық функцияларымен өрнектелген.

Кілт сөздер: Геллерстедт теңдеуі, аралас шарттары бар шеттік есеп, фундаментальді шешім, Лауричелла гипергеометриялық функциясы.

А.С. Бердышев[1,2], А.Р. Рыскан[1,2]

[1] *Казахский национальный педагогический университет имени Абая, Алматы, Казахстан;*
[2] *Институт информационных и вычислительных технологий МОН РК, Алматы, Казахстан*

Краевая задача для четырехмерного уравнения Геллерстедта

В статье исследована разрешимость задачи с краевыми условиями Неймана и Дирихле для уравнения Геллерстедта от четырех переменных. В ходе доказательства единственности решения задачи применен метод интеграла энергии, кроме того, существовании решения в задачи использованы формулы дифференцирования, автотрансформации, разложения гипергеометрических функций. Решение получено в явном виде и выражено гипергеометрическими функциями Лауричеллы.

Ключевые слова: уравнение Геллерстедта, краевая задача со смешанными условиями, фундаментальное решение, гипергеометрическая функция Лауричеллы.

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