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Analyzing Restrained Pitchfork Domination Across Path-Related Graph Structures

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Let $G = (V, E)$ be a finite, simple, and undirected graph without an isolated vertex. A dominating subset $D \subseteq V(G)$ is a restrained pitchfork dominating set if $1 \leq |N(u) \cap V - D| \leq 2$ for every $u \in D$ and every vertex not in D is adjacent to at least one vertex in the same set. The cardinality of a minimum restrained pitchfork dominating set is the restrained pitchfork domination number $\gamma_{rpf}(G)$. In the course of this investigation, we undertake an examination of the restrained pitchfork domination number within various path-related graphs. This analysis encompasses a range of graph structures, including the coconut tree, double star, banana tree, binomial tree, thorn path, thorn graph, and the square of the path denoted as P_n.

Keywords: Domination, restrained domination, pitchfork domination, restrained pitchfork domination, path graph.

2020 Mathematics Subject Classification: 05C69, 05C38.

Introduction

Graph theory provides a fundamental framework for understanding and analyzing various systems and networks, ranging from social networks to biological pathways to communication networks. A graph $G = (V, E)$ comprises a set V of vertices (or nodes) and a set E of edges (or connections) that link pairs of vertices. The order of a graph, denoted as n, represents the number of vertices in the graph, while the size, denoted as m, indicates the number of edges. For basic and detailed concepts, we refer [1, 2].

Dominating sets play a crucial role in graph theory, offering insights into the structure and connectivity of graphs. A dominating set $D \subseteq V(G)$ within a graph G ensures that every vertex not in D is adjacent to at least one vertex in D. A dominating set D is considered minimal if no proper subset of D retains the dominating property. The cardinality of the smallest dominating set in a graph G is known as the domination number $\gamma(G)$, representing a fundamental parameter of the graph's structure.

In certain contexts, such as when studying path-related graphs or tree structures [3, 4], additional constraints on dominating sets may be considered. A dominating subset D is deemed restrained if each vertex outside of D is adjacent to at least one vertex within D. Furthermore, a specialized form of dominating set, known as a pitchfork dominating set, imposes stricter conditions: each vertex within the dominating set must dominate at least one vertex and at most two vertices outside of the set [5, 6].

The concept of restrained domination has garnered attention [7, 8], particularly in the study of path-related graphs, as explored by Vaidya [9, 10]. Additionally, research on restrained domination in tree structures has been well-documented. These investigations highlight the significance of understanding and characterizing various types of dominating sets in different graph structures, shedding light on their properties and implications in diverse applications.

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1 Restrained pitchfork domination in paths

Definition 1. Let $G = (V, E)$ be a finite, simple, undirected graph without isolated vertices. A dominating subset D of $V(G)$ is a restrained pitchfork dominating set if $1 \leq |N(u) \cap (V - D)| \leq 2$ for every $u \in D$, and every vertex in $V - D$ is adjacent to at least one vertex in $V - D$. D is minimal if it has no proper restrained pitchfork dominating subset. The restrained pitchfork domination number is denoted by $\gamma_{rpf}(G)$, which is the minimum cardinality of a minimal restrained pitchfork dominating set.

Observation 1. Let G be a graph with restrained pitchfork domination number $\gamma_{rpf}(G)$ and a restrained pitchfork dominating set D. Then:

(i) $\gamma_{rpf} \geq 1$.

(ii) The degree of each vertex is greater than or equal to 1 for every $u \in D$.

(iii) Each restrained pitchfork dominating set has a vertex of degree one that belongs to it.

Definition 2. [11] (see Fig. 1) For any positive integers n and m greater than 2, the coconut tree graph $CT(m, n)$ is constructed by appending n additional pendant edges at the final vertex of the path P_m.

Definition 3. [3] The double star graph $ST(m, n)$ is formed by connecting the centres of two stars, $ST(m)$ and $ST(n)$, thereby creating an edge between them.

Definition 4. [3] (see Fig. 2) A banana tree, denoted as $B(m, n)$, is obtained by linking one leaf from each of m copies of an n-star network to a new single root vertex, represented by v.

Definition 5. [3](see Fig. 3) The binomial tree B_n of order zero consists of a single node R if $n = 0$. For $n > 0$, B_n includes the root R and n subtrees $B_0, B_1, \ldots, B_{n-1}$.

Definition 6. [3](see Fig. 4) A thorn path $P_{n,p,k}$ is created by adding p neighbors to each non-terminal vertex of the path P_n, and k neighbors to each terminal vertex.

Definition 7. [3](see Fig. 5) A thorn rod $P_{n,m}$ consists of terminal vertices of degree m at both ends and a linear network with n vertices in between.

Definition 8. [3](see Fig. 6) The square of a graph G, denoted as G^2, shares the same vertex set as G and includes an edge between any two vertices u and v if the distance between them in G is less than 3.

2 Main Results

Theorem 1. Let $CT(m, n)$ be a coconut tree where $m \geqslant 2$, $n \geqslant 3$ then

$$\gamma_{rpf} CT(m,n) = \begin{cases} m + [\frac{n}{3}] & \text{for} \quad n \equiv 0 \pmod 3, \\ m + \lfloor\frac{n-2}{3}\rfloor + 1 & \text{for} \quad n \equiv 1 \pmod 3, \\ m + \lfloor\frac{n-3}{3}\rfloor + 3 & \text{for} \quad n \equiv 2 \pmod 3. \end{cases}$$

Proof. Let $CT(m, n)$ be a coconut tree with the dominating set D. As it is a restrained dominating set all the pendent vertices are in D. Since it has m pendent edges that are adjacent to the n^{th} vertex of p_n. It has m pendent vertices and all of m are in D. Now we consider only P_n. There are three cases in D. Hence D is of any one of the forms. Let $V(P_n) = \{v_1, v_2, ..., v_n\}$ be the set of vertices. Hence

$$D = \begin{cases} v_{3i+1} & i = 0, 1, \ldots, \frac{n}{3} - 1 \quad \text{for} \quad n \equiv 0 \pmod 3, \\ v_{3i+1} & i = \left\{0, 1, \ldots, \lceil \frac{n}{3} \rceil - 2\right\} \cup \{v_{n-2}\} \quad \text{for} \quad n \equiv 1 \pmod 3, \\ v_{3i+1} & i = \left\{0, 1, \ldots, \lceil \frac{n}{3} \rceil - 3\right\} \cup \{v_{n-2}, v_{n-3}, v_{n-6}\} \quad \text{for} \quad n \equiv 2 \pmod 3. \end{cases}$$

Case (i): If $n \equiv 0 \pmod 3$.

Let us divide the vertex sets into $\frac{n}{3}$ subsets. It contains $\frac{n}{3}$ subsets, each containing 3 elements. From that, we take the first element. Hence, we get $m + [\frac{n}{3}]$.

Case (ii): If $n \equiv 1 \pmod 3$.

Let us divide the vertex sets into $\frac{n}{3}$ subsets. It contains $\lfloor \frac{n-2}{3} \rfloor$ subsets, each contains 3 elements. From that, we take the first vertex. Since v_n cannot be in D, from the remaining subsets, we take v_{n-2}. Hence, we get $m + \lfloor \frac{n-2}{3} \rfloor + 1$.

Case (iii): If $n \equiv 2 \pmod 3$.

Let us divide the vertex sets into $\frac{n}{3}$ subsets. It contains $\lfloor \frac{n-3}{3} \rfloor$ subsets, each contains 3 elements. From that, we take the first vertex. Since v_n cannot be in D, from the remaining subsets we take v_{n-2}, v_{n-3}, v_{n-6}. Hence, we get $m + \lfloor \frac{n-3}{3} \rfloor + 3$.

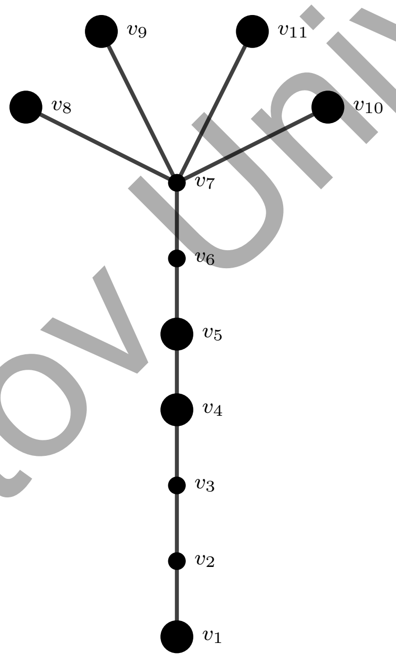

Figure 1. $\gamma_{rpf} CT(7, 4)$

Theorem 2. Let $ST(m, n)$ be a double star graph with $m, n \geqslant 1$, then $\gamma_{rpf} ST(m, n) = m + n$.

Proof. Since this graph contains $m + n$ pendent vertices, hence the result.

Theorem 3. Let $B(m, n)$ be a banana graph, then $\gamma_{rpf} B(m, n) = m(n-2)+1$ if and only if $m = 2$.

Proof. Let the banana tree comprise star graphs and its root vertex v_0. Let the vertex set of each (m) star graph be $\{v_1, v_2, \ldots, v_n\}$. Suppose that one of the vertices, say v_3, has degree n, and all other vertices have degree one. Among them, one pendent vertex, say (v_1), is adjacent to the root vertex v_0 whose degree is two. Hence each star graph has $(n-2)$ pendent vertices and these belong to D. Moreover, v_0 is in D. Hence $m(n-2)+1$.

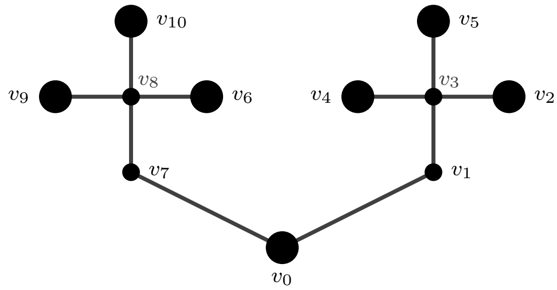

Figure 2. $\gamma_{rpf}B(2,5)$

Theorem 4. Let B_n be a binomial tree with $n \geqslant 2$, then $\gamma_{rpf}Bn = 2(n-1)$.

Proof. Let $\{v_1, v_2, \ldots, v_n\}$ be the set of vertices in B_n. Since B_n can be formed from two copies of B_{n-1}, each with $(n-1)$ children at the root, all of which have degree one. Obviously, these two $(n-1)$ vertices belong to D. Hence, $2(n-1)$.

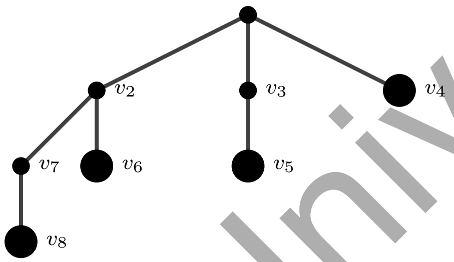

Figure 3. $\gamma_{rpf}B_3$

Theorem 5. Let $P_{n,p,k}$ be a thorn path graph then

$$\gamma_{rpf}(P_{n,p,k}) = \begin{cases} 2k & \text{for} \quad n = 2, \\ 2k + (n-2)p & \text{for} \quad n \geqslant 3. \end{cases}$$

Proof. Case (i): If $n = 2$.

Let $P_{n,p.k}$ be a thorn path graph. Now we are adding k vertices to the terminal vertices. Hence, we get $2k$ pendent vertices which are all in D.

Case (ii): If $n \geqslant 3$.

In this case, there are $(n-2)$ non terminal vertices and 2 terminal vertices. Then each of $(n-2)$ non terminal vertices has p pendent vertices, and each of two terminal vertices is attached to k pendent vertices. Thus it contains $2k + (n-2)p$ vertices in D.

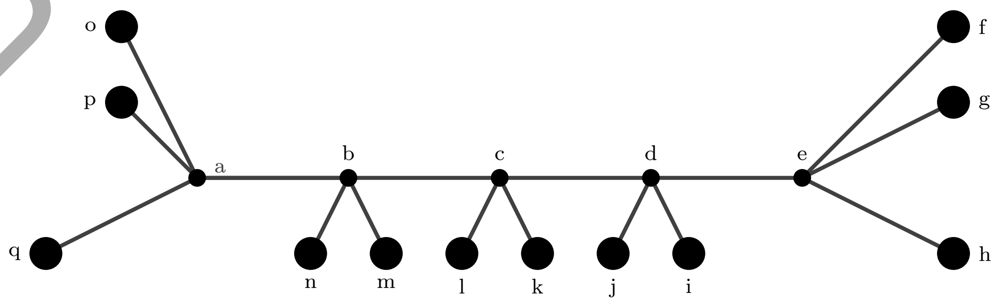

Figure 4. $\gamma_{rpf}P_{5,3,2}$

Theorem 6. Let $(P_{n,m})$ be Thorn rod graph then

$$\gamma_{rpf}(P_{n,m}) = \begin{cases} 2m & \text{for} \quad n = 2, \\ 2m + \lfloor \frac{n-1}{3} \rfloor + 1 & \text{for} \quad n \equiv 0 \pmod 3, \\ 2m + \lfloor \frac{n-5}{3} \rfloor + 3 & \text{for} \quad n \equiv 1 \pmod 3, \\ 2m + \lfloor \frac{n}{3} \rfloor & \text{for} \quad n \equiv 2 \pmod 3. \end{cases}$$

Proof. Case (i): If $n = 2$.

It is obvious, that all the pendent vertices belong to D.

Case (ii): If $n \equiv 0 \pmod 3$.

Let $\{v_1, v_2, v_3, \ldots, v_n\}$ be the set of vertices in P_n. Moreover, v_1 and v_n are adjacent to m pendent vertices. Then it has $2m$ pendent vertices. Hence, v_1 and v_n are not in D. Therefore v_{n-2} is in D. Thus, D is in any one of the form

$$D = \begin{cases} v_{3i+3} & i = \left\{0, 1, \ldots, \frac{n}{3} - 1\right\} \cup \{v_{n-2}\} \quad \text{for} \quad n \equiv 0 \pmod 3, \\ v_{3i+3} & i = \left\{0, 1, \ldots, \lfloor \frac{n-5}{3} \rfloor - 1\right\} \cup \{v_{n-2}, v_{n-3}, v_{n-6}\} \quad \text{for} \quad n \equiv 1 \pmod 3, \\ v_{3i+3} & i = \left\{0, 1, \ldots, \lfloor \frac{n}{3} \rfloor\right\} \quad \text{for} \quad n \equiv 2 \pmod 3. \end{cases}$$

Let us divide the vertex set into $\frac{n}{3}$ subsets. Since v_{n-2} is in D, from $\lfloor \frac{n-1}{3} \rfloor$ subsets we take the last vertex. Hence D becomes $2m + \lfloor \frac{n-1}{3} \rfloor + 1$.

Case (iii): If $n \equiv 1 \pmod 3$.

Let us divide the vertex set into $\frac{n}{3}$ subsets. Since v_{n-2} is always included in D, we can consider only $\lfloor \frac{n-5}{3} \rfloor$ subsets. From that set, we take one (the last) vertex. Still, it does not satisfy our condition, so we also take v_{n-3}, v_{n-6}. Then we get $D = 2m + \lfloor \frac{n-5}{3} \rfloor + 3$.

Case (iv): If $n \equiv 2 \pmod 3$.

Let us divide $P(V_n)$ into $\frac{n}{3}$ subsets. It has $\lfloor \frac{n}{3} \rfloor$ subsets. From each subset, we take one (the last) vertex. Hence, D is $2m + \lfloor \frac{n}{3} \rfloor$.

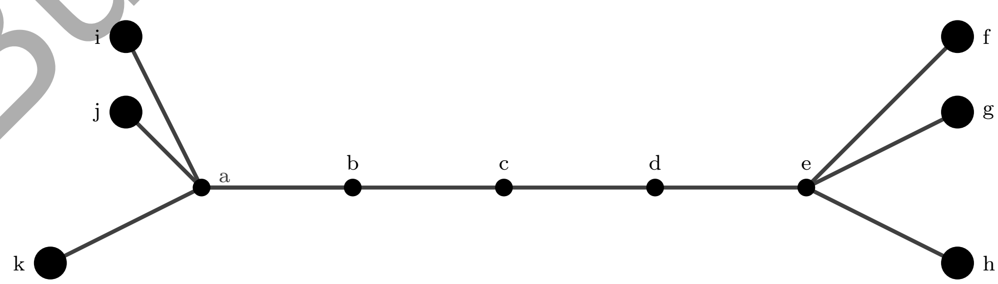

Figure 5. $\gamma_{rpf} P_{5,3}$

Theorem 7. Let the square of the path graph be P_n^2 with $n \geqslant 3$, then

$$\gamma_{rpf}(P_n^2) = \begin{cases} 1 & \text{for} \quad n = 3, \\ 2 & \text{for} \quad n = 4, 5, 6, \\ 3 & \text{for} \quad n = 7, \\ 4 & \text{for} \quad n = 8, \\ 4[\frac{n}{8}] & \text{for} \quad n \equiv 0, 1 \pmod 8, \\ 4\lfloor\frac{n}{8}\rfloor & \text{for} \quad n \equiv 2 \pmod 8, \\ 4\lfloor\frac{n}{8}\rfloor + 1 & \text{for} \quad n \equiv 1, 3, 4, 5 \pmod 8, \\ 4\lfloor\frac{n}{8}\rfloor + 2 & \text{for} \quad n \equiv 1, 6 \pmod 8, \\ 4\lfloor\frac{n}{8}\rfloor + 3 & \text{for} \quad n \equiv 1, 7 \pmod 8. \end{cases}$$

Proof. Case (i): If $n = 3$. It is obvious.

Case (ii): If $n = 4, 5, 6$. Then v_1 and v_n are in the dominating set.

Case (iii): If $n = 7$. Here v_1, v_{n-1}, and v_n are vertices, which satisfy our conditions and hence, they belong to D.

Case (iv): If $n = 7$. Here v_1, v_2, v_{n-1}, and v_n are vertices, which satisfy our conditions and hence, they belong to D.

Case (v): If $n \equiv 1 \pmod 8$.

Let $\{v_1, v_2, v_3, \ldots, v_n\}$ be the set of vertices. Let us divide the vertex set into $\frac{n}{8}$ subsets. Here $\lfloor\frac{n}{8}\rfloor$ subsets contain 4 dominating vertices and the remaining subsets may be P_1, P_2, P_3, P_4, P_5, P_6 or P_7. Now we consider the following cases:

Case (a): If $n \equiv 1 \pmod 8$ and the remaining subset is either P_1 or P_2, then each $\lfloor\frac{n}{8}\rfloor$ has 4 vertices. Therefore D is of the form $v_{8i+1}, v_{8i+5}, v_{8i+6}, v_{8i+7}$, $i = 0, 1, \ldots, \lfloor\frac{n-1}{8}\rfloor - 1$. Hence, D is $4\lfloor\frac{n}{8}\rfloor$.

Case (b): If $n \equiv 1 \pmod 8$ and the remaining subset is P_3, then each $\lfloor\frac{n}{8}\rfloor$ has 4 vertices. Therefore D is of the form $v_{8i+1}, v_{8i+5}, v_{8i+6}, v_{8i+7} \cup \{v_n\}$, $i = 0, 1, \ldots, \lfloor\frac{n-1}{8}\rfloor - 1$. Hence, D is $4\lfloor\frac{n}{8}\rfloor + 1$.

Case (c): If $n \equiv 1 \pmod 8$ and the remaining subset is either P_4 or P_5, then each $\lfloor\frac{n}{8}\rfloor$ has 4 vertices. Therefore D is of the form $v_{8i+1}, v_{8i+5}, v_{8i+6}, v_{8i+7} \cup \{v_n, v_{n-3}\}$, $i = 0, 1, \ldots, \lfloor\frac{n-1}{8}\rfloor - 1$. Hence, D is $4\lfloor\frac{n}{8}\rfloor + 2$.

Case (d): If $n \equiv 1 \pmod 8$ and the remaining subset is either P_6 or P_7, then each $\lfloor\frac{n}{8}\rfloor$ has 4 vertices. Therefore D is of the form $v_{8i+1}, v_{8i+5}, v_{8i+6}, v_{8i+7} \cup \{v_n, v_{n-1}, v_{n-5}\}$, $i = 0, 1, \ldots, \lfloor\frac{n-1}{8}\rfloor - 1$. Hence, D is $4\lfloor\frac{n}{8}\rfloor + 3$.

Case (vi): If $n \equiv 0 \pmod 8$, let us divide $V(P_n)$ into $\frac{n}{8}$ subsets, each containing 8 vertices. From each subset, we take 4 vertices of the form $v_{8i+1}, v_{8i+2}, v_{8i+7}, v_{8i+8}$, $i = 0, 1, \ldots, \frac{n}{8} - 1$. Hence, D is $4\lfloor\frac{n}{8}\rfloor$.

Case (vii): If $n \equiv 2 \pmod 8$, let us divide $V(P_n)$ into $\frac{n}{8}$ subsets, each containing 8 vertices. From each subset, we take 4 vertices of the form $v_{8i+1}, v_{8i+6}, v_{8i+7}, v_{8i+8}$, $i = 0, 1, \ldots, \frac{n-2}{8} - 1$. Hence, D is $4\lfloor\frac{n}{8}\rfloor$.

Case (viii): If $n \equiv 3 \pmod 8$, let us divide $V(P_n)$ into $\frac{n}{8}$ subsets each containing 8 vertices. From each subset, we take 4 vertices of the form $v_{8i+1}, v_{8i+6}, v_{8i+7}, v_{8i+8} \cup \{v_n\}$, $i = 0, 1, \ldots, \frac{n-3}{8} - 1$. Hence, D is $4\lfloor\frac{n}{8}\rfloor + 1$.

Case (ix): If $n \equiv 4 \pmod 8$, let us divide $V(P_n)$ into $\frac{n}{8}$ subsets each containing 8 vertices. From each subset, we take 4 vertices of the form $v_{8i+1}, v_{8i+6}, v_{8i+7}, v_{8i+8} \cup \{v_n\}$, $i = 0, 1, \ldots, \frac{n-4}{8} - 1$. Hence, D is $4\lfloor \frac{n}{8} \rfloor + 1$.

Case (ix): If $n \equiv 5 \pmod 8$, let us divide $V(P_n)$ into $\frac{n}{8}$ and each containing 8 vertices. From each subset, we take 4 vertices of the form $v_{8i+1}, v_{8i+6}, v_{8i+7}, v_{8i+8} \cup \{v_n\}$, $i = 0, 1, \ldots, \frac{n-5}{8} - 1$. Hence, D is $4\lfloor \frac{n}{8} \rfloor + 1$.

Case (x): If $n \equiv 6 \pmod 8$, let us divide $V(P_n)$ into $\frac{n}{8}$ subsets each containing 8 vertices. From each subset, we take 4 vertices of the form $v_{8i+1}, v_{8i+6}, v_{8i+7}, v_{8i+8} \cup \{v_n, v_{n-1}\}$, $i = 0, 1, \ldots, \frac{n-6}{8} - 1$. Hence, D is $4\lfloor \frac{n}{8} \rfloor + 2$.

Case (xi): If $n \equiv 7 \pmod 8$, let us divide $V(P_n)$ into $\frac{n}{8}$ subsets each containing 8 vertices. From each subset, take 4 vertices of the form $v_{8i+1}, v_{8i+6}, v_{8i+7}, v_{8i+8} \cup \{v_n, v_{n-1}, v_{n-5}\}$, $i = 0, 1, \ldots, \frac{n-6}{8} - 1$. Hence, D is $4\lfloor \frac{n}{8} \rfloor + 3$.

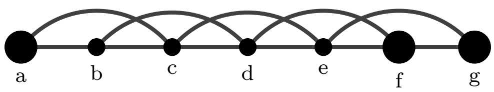

Figure 6. $\gamma_{rpf} {P_7}^2$

Conclusion

In this study, we examined the concept of restrained pitchfork domination across various path-related graph structures, establishing key results for their domination numbers. Through rigorous mathematical analysis, we derived explicit formulations for the restrained pitchfork domination number in structures such as the coconut tree, double star, banana tree, binomial tree, thorn path, thorn rod, and the square of a path. The results obtained contribute to the broader understanding of domination in graph theory, particularly in specialized graph classes.

The findings presented in this paper not only provide theoretical insights but also hold potential for applications in network optimization, communication systems, and combinatorial optimization problems where controlled domination constraints are relevant. The scientific novelty of this work lies in the extension of existing domination parameters by incorporating restrained pitchfork constraints, thereby refining structural characterizations of these graphs.

Future research in this area can explore variations of restrained pitchfork domination in more complex graph families, including weighted graphs and directed graphs. Additionally, investigating algorithmic approaches to efficiently compute restrained pitchfork domination numbers in large-scale graphs remains an open direction for further study.

Author Contributions

P. Vijayalakshmi collected and analyzed data, and led manuscript preparation. K. Karuppasamy served as the principal investigator of the research and supervised the research process. All authors participated in the revision of the manuscript and approved the final submission.

Conflict of Interest

The authors declare no conflict of interest.

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