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On solvability of boundary value problem for a nonlinear Fredholm integro-differential equation

The paper proposes a constructive method to solve a nonlinear boundary value problem for a Fredholm integro-differential equation. Using D.S. Dzhumabaev parametrization method, the problem under consideration is transformed into an equivalent boundary value problem for a system of nonlinear integro-differential equations with parameters on the subintervals. When applying the parametrization method to a nonlinear Fredholm integro-differential equation, the intermediate problem is a special Cauchy problem for a system of nonlinear integro-differential equations with parameters. By substitution the solution to the special Cauchy problem with parameters into the boundary condition and the continuity conditions of the solution to the original problem at the interior partition points, we construct a system of nonlinear algebraic equations in parameters. It is proved that the solvability of this system provides the existence of a solution to the original boundary value problem. The iterative methods are used to solve both the constructed system of algebraic equations in parameters and the special Cauchy problem. An algorithm for solving boundary value problem under consideration is provided.

Keywords: nonlinear Fredholm integro-differential equation, boundary value problem, special Cauchy problem, iterative process, isolated solution, algorithm, Dzhumabaev parametrization method.

Introduction

The research of initial and boundary value problems (BVPs) for integro-differential equations (IDEs) is devoted to the works of many authors [1–15]. Fredholm IDEs have a number of features that should be taken into account in setting problems for these equations and developing methods for solving them. By D.S. Dzhumabaev parametrization method [16] the new Δ_N general solution to linear Fredholm IDE is proposed in [17], the concept of the general solution is extended to Fredholm IDEs with nonlinear differential parts [18]. In [19–21], the criteria for solvability, unique solvability and conditions of well-posedness of linear boundary value problems for Fredholm IDEs are established.

On $[0,T]$ the boundary value problem for nonlinear Fredholm integro-differential equation (IDE) is ыconsidered:

$$\frac{dx}{dt} = A(t)x + \sum_{k=1}^{m} \varphi_k(t) \int_0^T \psi_k(\tau) f_k(\tau, x(\tau)) d\tau, \quad t \in [0,T], \quad x \in R^n, \tag{1}$$

$$g[x(0), x(T)] = 0, \tag{2}$$

where $n \times n$ matrices $A(t)$, $\varphi_k(t)$, $\psi_k(\tau)$ are continuous on $[0,T]$, $f_k : [0,T] \times R^n \to R^n$, $k = \overline{1,m}$ $\|x\| = \max\limits_{i=\overline{1,n}} |x_i|$.

Denote by $C([0,T], R^n)$ the space of continuous functions $x : [0,T] \to R^n$ with the norm $\|x\|_1 = \max\limits_{t\in[0,T]} \|x(t)\|$.

A solution to problem (1), (2) is a continuously differentiable on $[0,T]$ (at the points $t = 0$, $t = T$, equation (1) is satisfied by one-sided derivatives) function $x(t) \in C([0,T], R^n)$, which satisfies equation (1) and boundary condition (2).

The aim of the paper is to propose a constructive method for finding isolated solution to problem (1), (2).

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1 Scheme of the parametrization method

Let Δ_N be a partition of the interval $[0,T]$ into N parts by points $t_0 = 0 < t_1 < \ldots < t_N = T$.

The restriction of the function $x(t)$ on the rth interval $[t_{r-1}, t_r)$ denote by $x_r(t) : x_r(t) = x(t)$, $t \in [t_{r-1}, t_r)$, $r = \overline{1,N}$, and we reduce the problem (1), (2) to equivalent multi-point boundary value problem

$$\frac{dx_r}{dt} = A(t)x_r + \sum_{k=1}^{m} \varphi_k(t) \sum_{j=1}^{N} \int_{t_{r-1}}^{t_r} \psi_k(\tau) f_k(\tau, x_j(\tau)) d\tau, \quad t \in [t_{r-1}, t_r), \quad x_r \in R^n, \quad r = \overline{1,N}, \tag{3}$$

$$g[x_1(0), \lim_{t \to T-0} x_N(t)] = 0, \tag{4}$$

$$\lim_{t \to t_p - 0} x_p(t)] = x_{p+1}(t_p), \quad p = \overline{1, N-1}, \tag{5}$$

where equations (5) are the continuity conditions for solutions to problem (1), (2) at the interior points of partition Δ_N.

Denote by $C\big([0,T], \Delta_N, R^{nN}\big)$ the space consisting of all function systems $x[t] = \big(x_1(t), x_2(t), \ldots, x_N(t)\big)$, where functions $x_r : [t_{r-1}, t_r) \to R^n$, $r = \overline{1,N}$, are continuous and have finite left-sided limits $\lim\limits_{t \to t_r - 0} x_r(t)$, with the norm $\big\|x[\cdot]\big\|_2 = \max\limits_{r=\overline{1,N}} \sup\limits_{t \in [t_{r-1}, t_r)} \|x_r(t)\|$.

A solution to problem (3)–(5) is a function system $x^*[t] = (x_1^*(t), x_2^*(t), \ldots, x_N^*(t)) \in C([0,T], \Delta_N, R^{nN})$, where the function $x^*r(t)$ continuously differentiable on $[t_{r-1}, t_r)$, satisfies equation (3) for all $t \in [t_{r-1}, t_r)$, $r = \overline{1,N}$, and for $x_1^*(0)$, $\lim\limits_{t \to T-0} x_N^*(t)$, $\lim\limits_{t \to t_p - 0} x_p^*(t)$, $x_{p+1}^*(t_p)$, $p = \overline{1, N-1}$, there are equalities (4), (5).

We introduce additional parameters $\lambda_r = x_r(t_{r-1})$ and make substitutions $u_r(t) = x_r(t) - \lambda_r$, $r = \overline{1,N}$, then we obtain the multi-point boundary value problem with parameters

$$\frac{du_r}{dt} = A(t)[u_r + \lambda_r] + \sum_{k=1}^{m} \varphi_k(t) \sum_{j=1}^{N} \int_{t_{r-1}}^{t_r} \psi_k(\tau) f_k(\tau, u_j(\tau) + \lambda_j) d\tau, \quad t \in [t_{r-1}, t_r), \quad r = \overline{1,N}, \tag{6}$$

$$u_r(t_{r-1}) = 0, \quad r = \overline{1,N}, \tag{7}$$

$$g[\lambda_1, \lambda_N + \lim_{t \to T-0} u_N(t)] = 0, \tag{8}$$

$$\lambda_p + \lim_{t \to t_p - 0} u_p(t)] = \lambda_{p+1}, \quad p = \overline{1, N-1}. \tag{9}$$

A solution to problem (6)–(9) is a pair $(\lambda^*, u^*[t])$ with elements $\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_N^*) \in R^{nN}$, $u^*[t] = (u_1^*(t), u_2^*(t), \ldots, u_N^*(t)) \in C([0,T], \Delta_N, R^{nN})$, where the function $u_r^*(t)$ continuously differentiable on $[t_{r-1}, t_r)$ satisfies differential equation (6) for all $t \in [t_{r-1}, t_r)$ (for $t = t_{r-1}$ equation (6) satisfies the right-hand derivative of the function $u_r(t)$,) condition (7), and for λ_1^*, $\lambda_N^* + \lim\limits_{t \to T-0} u_N^*(t)$, $\lambda_p^* + \lim\limits_{t \to t_p - 0} u_p^*(t) = \lambda_{p+1}^*$, $p = \overline{1, N-1}$, equalities (8), (9) hold.

If $(\lambda^*, u^*[t])$ is a solution to problem (6)–(9), then the function $x^*(t)$ defined by the equalities $x^*(t) = \lambda_r^* + u_r^*(t)$, $t \in [t_{r-1}, t_r)$, $r = \overline{1,N}$, $x^*(t) = \lambda_N^* + \lim\limits_{t \to T-0} u_N^*(t)$, is the solution to problem (1), (2). And vice versa, if $\widetilde{x}(t)$ is a solution to problem (1), (2), then the pair $(\widetilde{\lambda}, \widetilde{u}[t])$ with elements $\widetilde{\lambda} = (\widetilde{\lambda}_1, \widetilde{\lambda}_2, \ldots, \widetilde{\lambda}_N) \in R^{nN}$, $\widetilde{u}_1(t), \widetilde{u}_2(t), \ldots, \widetilde{u}_1(t)_N$, where $\widetilde{\lambda}_r = \widetilde{x}(t_{r-1})$, $\widetilde{u}_r(t) = \widetilde{x}(t) - \widetilde{\lambda}_r$, $t \in [t_{r-1}, t_r)$, $r = \overline{1,N}$, is the solution to problem (6)–(9).

Problem (6), (7) is the special Cauchy problem for the system of nonlinear Fredholm IDEs.

2 The solvability of problem (1), (2)

We will use the limit values of the solution to problem (6), (7) later on, when we turn to problem (1), (2). Therefore, it is reasonable to consider the special Cauchy problem on the closed subintervals:

$$\frac{dv_r}{dt} = A(t)[v_r + \lambda_r] + \sum_{k=1}^{m} \varphi_k(t) \sum_{j=1}^{N} \int_{t_{r-1}}^{t_r} \psi_k(\tau) f_k(\tau, v_j(\tau) + \lambda_j) d\tau, \quad t \in [t_{r-1}, t_r], \quad r = \overline{1,N}, \tag{10}$$

$$v_r(t_{r-1}) = 0, \quad r = \overline{1,N}. \tag{11}$$

Denote by $\widetilde{C}\big([0,T],\Delta_N,R^{nN}\big)$ the space consisting of all function systems $v[t] = \big(v_1(t), v_2(t), \ldots, v_N(t)\big)$, where functions $v_r : [t_{r-1}, t_r] \to R^n$, $r = \overline{1,N}$, are continuous, with the norm $\big\|v[\cdot]\big\|_3 = \max\limits_{r=\overline{1,N}} \max\limits_{t\in[t_{r-1},t_r]} \|v_r(t)\|$.

It is obvious that if for fixed value of parameter $\lambda = \widehat{\lambda}$ the function systems $u[t,\widehat{\lambda}]$ and $v[t,\widehat{\lambda}]$ are the solutions to problems (6), (7) and (10), (11) respectively, then the following equalities are valid:

$$u_r(t,\widehat{\lambda}) = v_r(t,\widehat{\lambda}), \quad t \in [t_{r-1}, t_r), \quad r = \overline{1,N}, \tag{12}$$

$$\lim_{t\to t_r-0} u_r(t,\widehat{\lambda}) = v_r(t_r,\widehat{\lambda}), \quad r = \overline{1,N}. \tag{13}$$

The sufficient conditions of the solvability problem (10), (11) are established in [22].

Employing (8), (9) and considering (12), (13), we get the system of nonlinear algebraic equations in parameters

$$g[\lambda_1, \lambda_N + v_N(t,\lambda_1,\lambda_2,\ldots,\lambda_N)] = 0, \tag{14}$$

$$\lambda_p + v_p(t,\lambda_1,\lambda_2,\ldots,\lambda_N)] = \lambda_{p+1}, \quad p = \overline{1,N-1}. \tag{15}$$

We rewrite system (14), (15) in the following form:

$$Q_*(\Delta_N, \lambda, v[t]) = 0. \tag{16}$$

Condition A. There exists $h > 0 : Nh = T$, $N \in \mathbb{N}$, such that the system of nonlinear equations $Q_*(\Delta_N,\lambda,0) = 0$ has the solution $\lambda^{(0)} = (\lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_N^{(0)}) \in R^{nN}$, and for $\lambda = \lambda^{(0)}$ the special Cauchy problem (10), (11) with the initial guess solution $v^{(0,0)}[t] = (0,0,\ldots,0)$, has the solution $v[t,\lambda^{(0)}] \in \widetilde{C}([0,T],\Delta_N,R^{nN})$.

Denote by $PC([0,T],\Delta_N,R^n)$ the space of piecewise continuous functions $x : [0,T] \to R^n$ with the possible discontinuity points t_s, $s = \overline{1,N-1}$, with the norm $\|x\|_4 = \sup\limits_{t\in[0,T]} \|x(t)\|$.

By the equalities $x^{(0)}(t) = \lambda_r^{(0)} + v_r^{(0)}(t)$, $t \in [t_{r-1}, t_r]$, $r = \overline{1,N}$, we define the piecewise continuous function $x^{(0)}(t)$ on $[0,T]$.

Choose $\rho_\lambda > 0$, $\rho_v > 0$, $\rho_x > 0$ we construct sets:

$$S(\lambda^{(0)},\rho_\lambda) = \{\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_N) \in R^{nN} : \|\lambda - \lambda^{(0)}\| = \max_{r=\overline{1,N}} \|\lambda_r - \lambda_r^{(0)}\| < \rho_\lambda\},$$

$$S(v[t,\lambda^{(0)}],\rho_v) = \{v[t] \in C([0,T],\Delta_N,R^{nN}) : \|v[\cdot] - v[\cdot,\lambda^{(0)}]\|_2 < \rho_v\},$$

$$S(x^{(0)}(t),\rho_x) = \{x(t) \in PC([0,T],\Delta_N,R^n) : \|x - x^{(0)}\|_4 < \rho_x\}.$$

Theorem 1. Let $\lambda^* \in S(\lambda^{(0)},\rho_\lambda)$ be a solution to equation (16) and $v[t,\lambda^*] \in S(v[t,\lambda^{(0)}],\rho_v)$ be a solution to the special Cauchy problem (10), (11) for $\lambda = \lambda^*$. Then the function $x^*(t)$, defined by the equalities $x^*(t) = \lambda_r^* + v_r(t,\lambda^*)$, $t \in [t_{r-1}, t_r]$, $r = \overline{1,N}$, is a solution to problem (1), (2) and $x^*(t) \in S(x^{(0)}(t),\rho_x)$.

Using Theorem 2 [23; 45] to equation (16) we get the following assertion.

Theorem 2. Let the following conditions be fulfilled:

(i) the Jacobi matrix $\dfrac{\partial Q_*(\Delta_N;\lambda,v[t])}{\partial\lambda}$ is uniformly continuous in $S(\lambda^{(0)},\rho_\lambda)$;

(ii) $\dfrac{\partial Q_*(\Delta_N;\lambda,v[t])}{\partial\lambda}$ is invertible and $\Big\|\Big[\dfrac{\partial Q_*(\Delta_N;\lambda,v[t])}{\partial\lambda}\Big]^{-1}\Big\| \le \gamma^*$ for all $\lambda \in S(\lambda^{(0)},\rho_\lambda)$, γ^* is constant;

(iii) $\gamma^*\big\|Q_*(\Delta_N;\lambda^{(0)},v^{(0)}[t])\big\| < \rho_\lambda$.

Then there exists $\alpha_0 \ge 1$ such that for any $\alpha \ge \alpha_0$ the sequence $\lambda^{(k)}$, generated by the iterative process

$$\lambda^{(k+1)} = \lambda^{(k)} - \frac{1}{\alpha}\left[\frac{\partial Q_*(\Delta_N;\lambda^{(k)},v[t,\lambda^{(k)}])}{\partial\lambda}\right]^{-1} Q_*(\Delta_N;\lambda^{(k)},v[t,\lambda^{(k)}]), \quad k = 0,1,2,\ldots,$$

converges to λ^*, an isolated solution to equation (16) in $S(\lambda^{(0)},\rho_\lambda)$, and

$$\|\lambda^* - \lambda^{(0)}\| \le \big\|Q_*(\Delta_N;\lambda^{(0)},v[t,\lambda^{(0)}])\big\|.$$

Condition B. The functions $f_k(t,x)$, $k=\overline{1,m}$, $g(v,w)$ have uniformly continuous partial derivatives $\dfrac{\partial f_k(t,x)}{\partial x}$, $\dfrac{\partial g(v,w)}{\partial v}$, $\dfrac{\partial g(v,w)}{\partial w}$, in $G^{0,1}(\rho_x)=\{(t,x): t\in[0,T], \|x-x^{(0)}(t)\|<\rho_x\}$, $G^{0,2}(\rho_x)=\{(v,w)\in R^{2n}: \|v-x^{(0)}(0)\|<\rho_x, \|w-x^{(0)}(T)\|<\rho_x\}$ respectively, and

$$\Big\|\frac{\partial f_k(t,x)}{\partial x}\Big\|\le L_{k,0},\quad \Big\|\frac{\partial g(v,w)}{\partial v}\Big\|\le L_1,\quad \Big\|\frac{\partial g(v,w)}{\partial w}\Big\|\le L_2,$$

where $L_{k,0}$, $k=\overline{1,m}$, L_1, L_2 are const.

Let the function system $v[t,\lambda]$ be the solution to the special Cauchy problem (10), (11), i.e. the following equalities are valid:

$$\frac{dv_r(t,\lambda)}{dt}=A(t)[v_r(t,\lambda)+\lambda_r]+\sum_{k=1}^m\varphi_k(t)\sum_{j=1}^N\int_{t_{r-1}}^{t_r}\psi_k(\tau)f_k(\tau,v_j(\tau,\lambda)+\lambda_j)d\tau,\quad t\in[t_{r-1},t_r],\quad r=\overline{1,N},\tag{17}$$

$$v_r(t_{r-1},\lambda)=0,\quad r=\overline{1,N}.\tag{18}$$

Condition B by Peano's theorem provides the existence of partial derivatives $\dfrac{\partial v_r(t,\lambda)}{\partial\lambda_i}$, $r,i=\overline{1,N}$, for all $\lambda\in S(\lambda^{(0)},\rho_\lambda)$. Differentiating (17), (18) with respect to λ_i, $i=\overline{1,N}$, yields

$$\frac{d}{dt}\Big(\frac{\partial v_r(t,\lambda)}{\partial\lambda_i}\Big)=A(t)\Big[\frac{\partial v_r(t,\lambda)}{\partial\lambda_i}+\sigma_{ri}\Big]+\sum_{k=1}^m\varphi_k(t)\sum_{j=1}^N\int_{t_{r-1}}^{t_r}\psi_k(\tau)\frac{\partial f_k(\tau,v_j(\tau,\lambda)+\lambda_j)}{\partial x}\frac{\partial v_j(\tau,\lambda)}{\partial\lambda_i}d\tau+$$
$$+\sum_{k=1}^m\varphi_k(t)\int_{t_{i-1}}^{t_i}\psi_k(\tau)\frac{\partial f_k(\tau,v_i(\tau,\lambda)+\lambda_i)}{\partial x}d\tau,\quad t\in[t_{r-1},t_r],\quad r=\overline{1,N},$$

$$\frac{\partial v_r(t_{r-1},\lambda)}{\partial\lambda_i}=0,\quad r,i=\overline{1,N},$$

where

$$\sigma_{ri}=\begin{cases}I, & r=i,\quad I \text{ is the identity matrix of dimension } n,\\ O, & r\ne i,\quad O \text{ is the } n\times n \text{ zero matrix.}\end{cases}$$

If we denote by $z_{ri}(t,\lambda)$ the partial derivative $\dfrac{\partial v_r(t,\lambda)}{\partial\lambda_i}=0$, $r,i=\overline{1,N}$, then for each $i=\overline{1,N}$ the function system $z_i[t,\lambda]=(z_1(t,\lambda),\ldots,(z_N(t,\lambda))$ is a solution to the linear special matrix Cauchy problem

$$\frac{dz_{ri}}{dt}=A(t)\big[z_{ri}+\sigma_{ri}\big]+\sum_{k=1}^m\varphi_k(t)\sum_{j=1}^N\int_{t_{r-1}}^{t_r}\psi_k(\tau)\frac{\partial f_k(\tau,v_j(\tau,\lambda)+\lambda_j)}{\partial x}z_{ji}(\tau,\lambda)d\tau+$$
$$+\sum_{k=1}^m\varphi_k(t)\int_{t_{i-1}}^{t_i}\psi_k(\tau)\frac{\partial f_k(\tau,v_i(\tau,\lambda)+\lambda_i)}{\partial x}d\tau,\quad t\in[t_{r-1},t_r],\quad r=\overline{1,N},$$

$$z_{ri}(t_{r-1},\lambda)=0,\quad r,i=\overline{1,N}.$$

Condition B and conditions of Theorem 2 [22] provide the existence of the Jacobi matrix

$$\frac{\partial Q_*(\Delta_N;\widehat{\lambda},\widehat{v}[t])}{\partial\lambda}=\begin{pmatrix}q_{1,1}(\widehat{\lambda}) & \ldots & q_{1,N-1}(\widehat{\lambda}) & q_{1,N}(\widehat{\lambda})\\ q_{2,1}(\widehat{\lambda}) & \ldots & q_{2,N-1}(\widehat{\lambda}) & q_{2,N}(\widehat{\lambda})\\ \ldots & \ldots & \ldots & \ldots\\ q_{N,1}(\widehat{\lambda}) & \ldots & q_{N,N-1}(\widehat{\lambda}) & q_{N,N}(\widehat{\lambda})\end{pmatrix}\tag{19}$$

for all $\widehat{\lambda}\in S(\lambda^{(0)},\rho_\lambda)$ and its uniform continuity in $S(\lambda^{(0)},\rho_\lambda)$. Here the components of $\dfrac{\partial Q_*(\Delta_N;\widehat{\lambda},v[t,\widehat{\lambda}])}{\partial\lambda}$ are the $n\times n$ matrices

$$q_{1,1}(\widehat{\lambda})=g'_v\big[\widehat{\lambda}_1,\widehat{\lambda}_N+v_N(T,\widehat{\lambda})\big]+g'_w\big[\widehat{\lambda}_1,\widehat{\lambda}_N+v_N(T,\widehat{\lambda})\big]z_{N,1}(T,\widehat{\lambda}),$$

$$q_{1,s}\big(\widehat{\lambda}\big) = g'_w\big[\widehat{\lambda}_1, \widehat{\lambda}_N + v_N(T,\widehat{\lambda})\big] z_{N,s}\big(T,\widehat{\lambda}\big), \quad s = \overline{2, N-1},$$

$$q_{1,N}\big(\widehat{\lambda}\big) = g'_w\big[\widehat{\lambda}_1, \widehat{\lambda}_N + v_N(T,\widehat{\lambda})\big]\big[I + z_{N,N}(T,\widehat{\lambda})\big],$$

$$q_{p,r}\big(\widehat{\lambda}\big) = z_{p-1,r}(t_{p-1},\widehat{\lambda}), \quad p \neq r, \quad p \neq r+1,$$

$$q_{p,p}\big(\widehat{\lambda}\big) = -I + z_{p-1,p}\big(t_{p-1},\widehat{\lambda}\big), \quad q_{p,p-1}\big(\widehat{\lambda}\big) = I + z_{p-1,p-1}\big(t_{p-1},\widehat{\lambda}\big), \quad p = \overline{2,N}, \ r = \overline{1,N},$$

where $z_i[t,\widehat{\lambda}] = \big(z_{1,i}(t,\widehat{\lambda}), \dots, z_{N,i}(t,\widehat{\lambda})\big)$, $i = \overline{1,N}$, is the solution to the special Cauchy problem (10), (11) for $\lambda = \widehat{\lambda}$.

3 An algorithm for solving problem (6)–(9)

Assume that the conditions A, B hold and problem (10), (11) is well-posedness. For the initial approximation of the solution to problem (6)–(9), we take a pair $(\lambda^{(0)}, v[t,\lambda^{(0)}])$ and find the sequence $(\lambda^{(k)}, v[t,\lambda^{(k)}])$, $k \in \mathbb{N}$, according to the following algorithm:

Step 1. a) Employing the values of elements of the function system $v[t,\lambda^{(0)}]$, we compose the n vector and the $n \times n$ matrices:

$$Q_*\big(\Delta_N;\lambda^{(0)}, v[t,\lambda^{(0)}]\big) = \begin{pmatrix} g\big[\lambda_1^{(0)}, \lambda_N^{(0)} + v_N\big(T,\lambda^{(0)}\big)\big] \\ \lambda_1^{(0)} + v_1\big(t_1,\lambda^{(0)}\big) - \lambda_2^{(0)} \\ \dots \\ \lambda_{N-1}^{(0)} + v_{N-1}\big(t_{N-1},\lambda^{(0)}\big) - \lambda_N^{(0)} \end{pmatrix},$$

$$P_{ri}^{(0)}(t) = A(t)\sigma_{ri} + \sum_{k=1}^{m} \varphi_k(t) \int_{t_{i-1}}^{t_i} \psi_k(\tau) \frac{\partial f_k(\tau, v_i(\tau,\lambda^{(0)}) + \lambda_i^{(0)})}{\partial x} d\tau, \quad t \in [t_{r-1}, t_r], \quad r, i = \overline{1,N},$$

$$\Psi_{kj}^{(0)}(t) = \psi_k(t) \frac{\partial f_k(t, v_j(t,\lambda^{(0)}) + \lambda_j^{(0)})}{\partial x}, \quad t \in [t_{r-1}, t_r], \quad r, j = \overline{1,N}.$$

b) By solving N special matrix Cauchy problems for the system of linear IDEs

$$\frac{dz_{ri}}{dt} = A(t) z_{ri} + \sum_{k=1}^{m} \varphi_k(t) \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \Psi_{kj}^{(0)}(t) z_{ji}(\tau) d\tau + P_{ri}^{(0)}(t), \quad t \in [t_{r-1}, t_r],$$

$$z_{ri}(t_{r-1}) = 0, \quad r, i = \overline{1,N},$$

we find the function systems

$$z_i\big[t,\lambda^{(0)}\big] = \big(z_{1i}(t,\lambda^{(0)}), \dots, z_{Ni}(t,\lambda^{(0)})\big), \quad i = \overline{1,N}.$$

c) Construct the Jacobi matrix $\dfrac{\partial Q_*\big(\Delta_N;\lambda^{(0)}, v[t,\lambda^{(0)}]\big)}{\partial \lambda}$ by formula (19), where

$$q_{1,1}\big(\lambda^{(0)}\big) = g'_v\Big[\lambda_1^{(0)}, \lambda_N^{(0)} + v_N\big(T,\lambda^{(0)}\big)\Big] + g'_w\Big[\lambda_1^{(0)}, \lambda_N^{(0)} + v_N\big(T,\lambda^{(0)}\big)\Big] z_{N,1}\big(T,\lambda^{(0)}\big),$$

$$q_{1,s}\big(\lambda^{(0)}\big) = g'_w\Big[\lambda_1^{(0)}, \lambda_N^{(0)} + v_N\big(T,\lambda^{(0)}\big)\Big] z_{N,s}\big(T,\lambda^{(0)}\big), \quad s = \overline{2, N-1},$$

$$q_{1,N}\big(\lambda^{(0)}\big) = g'_w\Big[\lambda_1^{(0)}, \lambda_N^{(0)} + v_N\big(T,\lambda^{(0)}\big)\Big] \times \Big[I + z_{N,N}\big(T,\lambda^{(0)}\big)\Big],$$

$$q_{p,r}\big(\lambda^{(0)}\big) = z_{p-1,r}\big(t_{p-1},\lambda^{(0)}\big), \quad p \neq r, \quad p \neq r+1,$$

$$q_{p,p}\big(\lambda^{(0)}\big) = -I + z_{p-1,p}\big(t_{p-1},\lambda^{(0)}\big), \quad q_{p,p-1}\big(\lambda^{(0)}\big) = I + z_{p-1,p-1}\big(t_{p-1},\lambda^{(0)}\big), \quad p = \overline{2,N}, \quad r = \overline{1,N}.$$

Solve the system of linear algebraic equations

$$\frac{\partial Q_*\big(\Delta_N;\lambda^{(0)}, v[t,\lambda^{(0)}]\big)}{\partial \lambda} \Delta\lambda = -\frac{1}{\alpha} Q_*(\Delta_N;\lambda^{(0)}, v[t,\lambda^{(0)}]), \quad \Delta\lambda \in R^{nN},$$

for some $\alpha \geq 1$ and find $\Delta\lambda^{(0)}$. Determine $\lambda^{(1)}$ as follows:

$$\lambda^{(1)} = \lambda^{(0)} + \Delta\lambda^{(0)}.$$

d) Choose the function system $v\big[t,\lambda^{(0)}\big]$ as an initial guess solution to problem (10), (11) for $\lambda = \lambda^{(1)}$, and by iterative process [22; 53] find the function system $v\big[t,\lambda^{(1)}\big]$.

Step 2. a) Employing the values of elements of the function system $v[t,\lambda^{(1)}]$, we compose the n vector and the $n \times n$ matrices:

$$Q_*\big(\Delta_N;\lambda^{(1)},v[t,\lambda^{(1)}]\big) = \begin{pmatrix} g\big[\lambda_1^{(1)},\lambda_N^{(1)} + v_N\big(T,\lambda^{(1)}\big)\big] \\ \lambda_1^{(1)} + v_1\big(t_1,\lambda^{(1)}\big) - \lambda_2^{(1)} \\ \dots \\ \lambda_{N-1}^{(1)} + v_{N-1}\big(t_{N-1},\lambda^{(1)}\big) - \lambda_N^{(1)} \end{pmatrix},$$

$$P_{ri}^{(1)}(t) = A(t)\sigma_{ri} + \sum_{k=1}^{m}\varphi_k(t)\int_{t_{i-1}}^{t_i}\psi_k(\tau)\frac{\partial f_k(\tau,v_i(\tau,\lambda^{(1)}) + \lambda_i^{(1)})}{\partial x}d\tau, \quad t \in [t_{r-1},t_r], \quad r,i = \overline{1,N},$$

$$\Psi_{kj}^{(1)}(t) = \psi_k(t)\frac{\partial f_k(t,v_j(t,\lambda^{(1)}) + \lambda_j^{(1)})}{\partial x}, \quad t \in [t_{r-1},t_r], \quad r,j = \overline{1,N}.$$

b) By solving N special matrix Cauchy problems for the system of linear IDEs

$$\frac{dz_{ri}}{dt} = A(t)z_{ri} + \sum_{k=1}^{m}\varphi_k(t)\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\Psi_{kj}^{(1)}(t)z_{ji}(\tau)d\tau + P_{ri}^{(1)}(t), \quad t \in [t_{r-1},t_r],$$

$$z_{ri}(t_{r-1}) = 0, \quad r,i = \overline{1,N},$$

we find the function systems

$$z_i\big[t,\lambda^{(1)}\big] = \big(z_{1i}(t,\lambda^{(1)}),\dots,z_{Ni}(t,\lambda^{(1)})\big), \quad i = \overline{1,N}.$$

c) Construct the Jacobi matrix $\dfrac{\partial Q_*\big(\Delta_N;\lambda^{(1)},v[t,\lambda^{(1)}]\big)}{\partial\lambda}$ by formula (19), where

$$q_{1,1}\big(\lambda^{(1)}\big) = g'_v\Big[\lambda_1^{(1)},\lambda_N^{(1)} + v_N\big(T,\lambda^{(1)}\big)\Big] + g'_w\Big[\lambda_1^{(1)},\lambda_N^{(1)} + v_N\big(T,\lambda^{(1)}\big)\Big]z_{N,1}\big(T,\lambda^{(1)}\big),$$

$$q_{1,s}\big(\lambda^{(1)}\big) = g'_w\Big[\lambda_1^{(1)},\lambda_N^{(1)} + v_N\big(T,\lambda^{(1)}\big)\Big]z_{N,s}\big(T,\lambda^{(1)}\big), \quad s = \overline{2,N-1},$$

$$q_{1,N}\big(\lambda^{(1)}\big) = g'_w\Big[\lambda_1^{(1)},\lambda_N^{(1)} + v_N\big(T,\lambda^{(1)}\big)\Big] \times \Big[I + z_{N,N}\big(T,\lambda^{(0)}\big)\Big],$$

$$q_{p,r}\big(\lambda^{(1)}\big) = z_{p-1,r}\big(t_{p-1},\lambda^{(1)}\big), \quad p \neq r, \quad p \neq r+1, \quad q_{p,p}\big(\lambda^{(1)}\big) = -I + z_{p-1,p}\big(t_{p-1},\lambda^{(1)}\big),$$

$$q_{p,p-1}\big(\lambda^{(1)}\big) = I + z_{p-1,p-1}\big(t_{p-1},\lambda^{(1)}\big), \quad p = \overline{2,N}, \quad r = \overline{1,N}.$$

Solve the system of linear algebraic equations

$$\frac{\partial Q_*\big(\Delta_N;\lambda^{(1)},v[t,\lambda^{(1)}]\big)}{\partial\lambda}\Delta\lambda = -\frac{1}{\alpha}Q_*(\Delta_N;\lambda^{(1)},v[t,\lambda^{(1)}]), \quad \Delta\lambda \in R^{nN},$$

for some $\alpha \geq 1$ and find $\Delta\lambda^{(1)}$. Determine $\lambda^{(2)}$ as follows:

$$\lambda^{(2)} = \lambda^{(1)} + \Delta\lambda^{(1)}.$$

d) Choose the function system $v\big[t,\lambda^{(1)}\big]$ as an initial guess solution to problem (10), (11) for $\lambda = \lambda^{(2)}$, find the function system $v\big[t,\lambda^{(2)}\big]$.

Continuing this process, in the kth step of the algorithm we get a pair $\big(\lambda^{(k)},v\big[t,\lambda^{(k)}\big]\big)$, $k = 1,2,\dots$. The conditions of Theorem 2 [22; 53] and Theorem 2 ensure the convergence of this sequence to $\big(\lambda^*,v\big[t,\lambda^*\big]\big)$, the solution to problem (6)–(9), as $k \to \infty$.

Given $\varepsilon > 0$, the iterative process should be terminated if $\|\Delta\lambda^{(k)}\| < \varepsilon$. Theorem 2 yields an alternative termination criterion. If conditions of this theorem are fulfilled, then the inequality $\|\lambda^* - \lambda^{(k)}\| \leq \gamma^*\|Q_*(\Delta_N,\lambda^{(k)},v[t,\lambda^{(k)}])\|$ is true. Therefore, the iterative process terminates if $\gamma^*\|Q_*(\Delta_N,\lambda^{(k)},v[t,\lambda^{(k)}])\| < \varepsilon$.

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Фредгольм сызықты емес интегралдық-дифференциалдық теңдеуі үшін шеттік есептің шешілімділігі туралы

Мақалада Фредгольм интегралдық-дифференциалдық теңдеуі үшін сызықты емес шеттік есепті шешудің конструктивті әдісі ұсынылған. Д.С. Джумабаевтың параметрлеу әдісін қолдана отырып, қарастырылып отырған есеп ішкі интервалдардағы параметрлі сызықты емес интегралдық-дифференциалдық теңдеулер жүйесі үшін эквивалентті шеттік есепке келтірілген. Фредгольм сызықты емес интегралдық-дифференциалдық теңдеуіне параметрлеу әдісін қолданған кезде аралық есеп параметрлі сызықты емес интегралдық-дифференциалдық теңдеулер жүйесі үшін арнайы Коши есебі болып табылады. Арнайы Коши есебінің шешімін шеттік шартқа және бастапқы есептің шешімін бөліктеудің ішкі нүктелеріндегі үзіліссіздік шарттарына қою арқылы параметрлерге қатысты сызықты емес алгебралық теңдеулер жүйесі құрылады. Бұл жүйенің шешілімділігі бастапқы шеттік есептің шешілімділігін қамтамасыз ететіндігіне негізделген. Параметрлерге қатысты алгебралық теңдеулер жүйесін және арнайы Коши есебін шешу үшін итерациялық әдістер қолданылады. Қарастырылып отырған шеттік есепті шешу алгоритмі ұсынылған.

Кілт сөздер: сызықты емес Фредгольм интегралдық-дифференциалдық теңдеуі, шеттік есеп, итерациялық процесс, оқшауланған шешім, алгоритм, Джумабаевтың параметрлеу әдісі.

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О разрешимости краевой задачи для нелинейного интегро-дифференциального уравнения Фредгольма

В статье предложен конструктивный метод решения нелинейной краевой задачи для интегро-дифференциального уравнения Фредгольма. С помощью метода параметризации Д.С. Джумабаева рассматриваемая задача преобразована в эквивалентную краевую задачу для системы нелинейных интегро-дифференциальных уравнений с параметрами на подынтервалах. При применении метода параметризации к нелинейному интегро-дифференциальному уравнению Фредгольма промежуточной задачей является специальная задача Коши для системы нелинейных интегро-дифференциальных уравнений с параметрами. Путем подстановки решения специальной задачи Коши с параметрами в граничное условие и условия непрерывности решения исходной задачи в точках внутреннего разбиения строится система нелинейных алгебраических уравнений по параметрам. Доказано, что разрешимость этой системы обеспечивает существование решения исходной краевой задачи. Итерационные методы использованы как для решения построенной системы алгебраических уравнений по параметрам, так и для решения специальной задачи Коши. Приведен алгоритм решения рассматриваемой краевой задачи.

Ключевые слова: нелинейное интегро-дифференциальное уравнение Фредгольма, краевая задача, специальная задача Коши, итерационный процесс, изолированное решение, алгоритм, метод параметризации Джумабаева.

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