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Boundary value problem for the time-fractional wave equation

M.T. Kosmakova[1], A.N. Khamzeyeva[1,*], L.Zh. Kasymova[2]

[1]*Institute of Applied Mathematics, Karaganda Buketov University, Karaganda, Kazakhstan;*
[2]*Abylkas Saginov Karaganda Technical University, Karaganda, Kazakhstan*
(E-mail: svetlanamir578@gmail.com, aiymkhamzeyeva@gmail.com, l.kasymova2017@mail.ru)

In the article, the boundary value problem for the wave equation with a fractional time derivative and with initial conditions specified in the form of a fractional derivative in the Riemann-Liouville sense is solved. The definition domain of the desired function is the upper half-plane (x,t). To solve the problem, the Fourier transform with respect to the spatial variable was applied, then the Laplace transform with respect to the time variable was used. After applying the inverse Laplace transform, the solution to the transformed problem contains a two-parameter Mittag-Leffler function. Using the inverse Fourier transform, a solution to the problem was obtained in explicit form, which contains the Wright function. Next, we consider limiting cases of the fractional derivative's order which is included in the equation of the problem.

Keywords: fractional derivative, Laplace transform, Fourier transform, Mittag-Leffler function, Wright function.

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Introduction

The mathematical apparatus of fractional order integrodifferentiation plays a significant role in various fields of science and engineering, including physics, biology, economics, etc [1]. Its application makes it possible to more accurately model and analyze phenomena that cannot be described by classical differential equations or integrals. Applications include: modeling the dynamics of complex systems with long-term dependence and memory, such as financial markets, environmental systems, communication networks, etc., analysis of nonlinear processes and phenomena, including diffusion, thermal conductivity, wave propagation, etc., solving optimization and control problems under conditions of uncertainty and changing conditions.

Fractional derivatives can be interpreted as a way to account for memory effects and temporal nonlocality in systems. In the classical differential model, all changes in the system instantly affect its state. However, in reality, many systems have memories and histories that influence their future behavior. Fractional order derivatives take this memory into account, allowing the modeling of systems with long-term dependencies and time delays in response to external influences. In addition, they can also take into account spatial correlations and coordinate nonlocality in systems where the influence on the state at a given point in space depends not only on neighboring points, but also on more distant ones [2].

Fractional derivative equations are a way to describe the evolution of physical systems with losses. They can model systems in which energy, mass, or other physical quantities are lost over time or space. The fractional derivative usually characterizes the degree of loss or dissipation in the system. For example, in diffusion processes, fractional derivatives can describe an anomalous distribution of particles due to long-term correlations or heterogeneity of the medium. Wave processes with losses can also be described using fractional derivatives, which makes it possible to take into account energy

*Corresponding author. *E-mail: aiymkhamzeyeva@gmail.com*
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dissipation in the system [3]. In mathematical modeling of continuous media with memory, equations arise describing a new type of wave motion that occupies an intermediate position between ordinary diffusion and classical waves [4, 5].

A loaded differential equation is an equation with a loaded term, which can contain differential or integrodifferential operators. This loaded term can be expressed as a function containing both the variables themselves and their derivatives.

Loaded equations allow you to model more complex physical or mathematical systems that cannot always be described by simple equations. For example, in problems of mathematical physics or control theory, loaded differential equations can be used to take into account the influence of external factors or additional conditions on the dynamics of the system.

Such equations play an important role in research related to the theory of boundary value problems, stability and control of dynamic systems, as well as in other areas of science and engineering where adequate consideration of the load on the system under study is required. In [6], the class of flat problems on the effect of moving loads on the surface of an aminated plate is studied. However, the presence of a loaded operator is accompanied by some difficulties during research, since it is not always possible to use direct research methods. For problems with loads, adaptation and development of specialized numerical methods are required [7]. All this emphasizes both the theoretical and practical significance of studying various boundary value problems for loaded differential equations. It is obvious that the presence of a loaded term gives rise to new, still unexplored problems in the theory of boundary value problems, therefore there is a need to develop new methods for solving the evolving theory of loaded differential equations [8].

Loaded differential equations can be considered as weak or strong perturbations of differential equations. In some cases, boundary value problems remain correct in natural classes of functions, where the loaded term is interpreted as a weak perturbation [9]. If the uniqueness of the solution to the boundary value problem is violated, then the load can be considered as a strong perturbation [10]. It turns out that the nature of the load (weak or strong perturbation) depends both on the order of the derivatives included in the loaded (perturbed) part of the operator, and on the manifold on which the trace of the desired function is specified.

The study of boundary value problems with loaded terms, presented in the form of integrals or fractional derivatives, can lead to different results depending on the specifics of the equation and the conditions of the problem. There may also be difficulties associated with the analysis and evaluation of integral operators in the resulting integral equations, since their kernels contain special functions. In [11, 12], the intervals for changing the order of the fractional derivative, that is contained in the loaded term, are determined, for which the theorems of existence and uniqueness of solutions to boundary value problems and arising integral equations are valid. We also note that the boundary value problems of heat conduction and the Volterra integral equations arising in their study with singularities in the kernel, similar to the singularities in this paper, were considered in [13, 14].

Also, integral equations with singularities in the kernel arise when studying boundary value problems in non-cylindrical domains that degenerate into a point at the initial moment of time [15–20].

Fractional derivatives in equations add new aspects and difficulties in the study of boundary value problems, since they take into account not only the previous state of the system, but also its history. The fractional order differentiation operation is a combination of differentiation and integration operations. Recently, work has appeared on the study of inverse boundary value problems with a load of fractional order. In [21], the inverse problem with a nonlinear gluing condition for a loaded equation of parabolic-hyperbolic type is studied for solvability. The problem is reduced to the study of the nonlinear Fredholm integral equation of the second kind. In [22], as an application of the analyticity of the solution, the uniqueness of an inverse problem in determining the fractional orders in the multi-term time-fractional diffusion equations from one interior point observation is established.

In this article, the boundary value problem for the fractional wave equation was solved, and two limiting cases were considered. The article is structured as follows. In Section 1, we introduce some necessary definitions and mathematical preliminaries of fractional calculus, special functions and boundary value problems which will be needed in the forthcoming Sections. The problem statement for the Riemann-Liouville fractional derivative wave equation in the upper half-plane (x,t) is given in Section 2. The initial conditions are given as a fractional derivative. Solving the problem is the content of Section 3: the Fourier transform for a spatial variable was consistently applied, followed by the Laplace transform for a temporal variable, the inverse Laplace transform and the inverse Fourier transform. Next, the limiting cases of the order of the fractional derivative are considered in Section 4. In the last Section the main result is formulated.

1 Preliminaries

Definition 1. [23] Let $f(t) \in L_1[a,b]$. Then, the Riemann-Liouville integral of the order β is defined as follows

$$ {}_rD_{a,t}^{-\beta} f(t) = \frac{1}{\Gamma(\beta)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-\beta}} d\tau, \quad \beta, a \in R, \quad \beta > 0. \tag{1}$$

Definition 2. Let $f(t) \in L_1[a,b]$. Then, the Riemann-Liouville derivative of the order β is defined as follows

$$ {}_rD_{a,t}^{\beta} f(t) = \frac{1}{\Gamma(n-\beta)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\beta-n+1}} d\tau, \quad \beta, a \in R, \quad n-1 < \beta < n. \tag{2}$$

From formula (2) it follows that

$$ {}_rD_{a,t}^{0} f(t) = f(t), \quad {}_rD_{a,t}^{n} f(t) = f^{(n)}(t), \quad n \in N.$$

Taking into account formula (1), formula (2) can be rewritten as

$$ {}_rD_{a,t}^{\beta} f(t) = \frac{d^n}{dt^n} {}_rD_{a,t}^{\beta-n} f(t), \quad \beta, a \in R, \quad n-1 < \beta < n.$$

The entire function of the form

$$ E_{\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\lambda n + \mu)}, \quad \lambda > 0, \quad \mu \in C, \tag{3}$$

is called the Mittag-Leffler function.

The entire function of the form

$$ \phi(\lambda, \mu; z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(\lambda n + \mu)}, \quad \lambda > -1, \quad \mu \in C, \tag{4}$$

is called the Wright function.

The formula for the integral Laplace transform of the Mittag-Leffler function is valid [24]

$$ L\left[t^{\gamma-1} E_{a,\gamma}(\lambda t^a)\right] = \frac{s^{a-\gamma}}{s^a - \lambda}, \quad |\lambda| < |s|^a, \quad a > 0, \quad \gamma > -1. \tag{5}$$

Also the formula for the integral Laplace transform of the Wright function is valid [25]

$$ L\left[t^{\beta-1} \phi(\rho, \beta, -\lambda t^{\rho})\right] = s^{-\beta} \exp(-\lambda s^{-\rho}), \quad -1 < \rho < 0, \quad \lambda > 0. \tag{6}$$

2 Statement of the problem

In the domain $\Omega = \{(x,t) \mid -\infty < x < +\infty;\ t > 0\}$ find a solution to the problem:

$$D_{0t}^{\alpha} u(x,t) - u_{xx}(x,y) = f(x,t), \tag{7}$$

$$D_{0t}^{\alpha-1} u|_{t=0} = \varphi(x); \quad D_{0t}^{\alpha-2} u|_{t=0} = \psi(x), \quad \lim_{x\to\infty} u(x,t) = 0, \tag{8}$$

where $D_{0t}^{\alpha} f(t)$ is the Riemann-Liouville derivative of an order $\alpha \in (1;2)$.

We call a function $u(x,t)$ a *regular solution to equation (7) in the domain* G if $t^{1-\mu}u(x,t) \in C\left(\overline{G}\right)$ for some $\mu > 0$; in G, $u(x,t)$ has continuous derivatives with respect to x of the first and second order; the functions $D_{0t}^{\alpha-1}u(x,t)$ and $D_{0t}^{\alpha-2}u(x,t)$ are continuously differentiable as functions of t for a fixed x at interior points of G; and $u(x,t)$ satisfies equation (7) at all points of G.

3 Solving the problem

We apply Fourier transform to problem (7)-(8) with respect to the variable x:

$$D_{0t}^{\alpha} U(p,t) + p^2 U(p,t) = F(p,t), \tag{9}$$

$$D_{0t}^{\alpha-1} U|_{t=0} = \bar{\varphi}(p), \quad D_{0t}^{\alpha-2} U|_{t=0} = \bar{\psi}(p), \tag{10}$$

where $F(p,t); \bar{\varphi}(p); \bar{\psi}(p)$ are the Fourier images of input data in problem (7)-(8).

Let's apply Laplace transform to equation (9) with respect to the variable t taking into account conditions (10). Then we obtain

$$s^{\alpha}\bar{u}(p,s) - \bar{\varphi}(p) - s\bar{\psi}(p) + p^2\bar{u}(p,s) = \bar{f}(p,s),$$

where $\bar{f}(p,s)$ is the image of the function $F(p,t)$, or

$$\bar{u}(p,s) = \frac{\bar{f}(p,s)}{s^{\alpha}+p^2} + \frac{\bar{\varphi}(p)}{s^{\alpha}+p^2} + \frac{s}{s^{\alpha}+p^2}\bar{\psi}(p). \tag{11}$$

Applying the inverse Laplace transform to (11) with respect to the variable s and taking into account formula (5), we get

$$U(p,t) = \left((t^{\alpha-1}E_{\alpha,\alpha}(-p^2t^{\alpha})) * F(p,t)\right)(t) +$$

$$+ t^{\alpha-1}E_{\alpha,\alpha}(-p^2t^{\alpha})\bar{\varphi}(p) + t^{\alpha-2}E_{\alpha,\alpha-1}(-p^2t^{\alpha})\bar{\psi}(p), \tag{12}$$

where $E_{\lambda,\mu}(z)$ is the Mittag-Leffler function (3) and $*$ is the convolution operation.

Applying the inverse Fourier transform to (12) with respect to the variable p, we obtain

$$u(x,t) = \int_0^t \int_{-\infty}^{+\infty} G_1(x-\xi,\tau) f(\xi, t-\tau) d\xi d\tau + \int_{-\infty}^{+\infty} G_1(x-\xi,\tau)\varphi(\xi)d\xi +$$

$$+ \int_{-\infty}^{+\infty} G_2(x-\xi,\tau)\psi(\xi)d\xi, \tag{13}$$

where

$$G_1(x,t) = \frac{1}{\pi}\int_0^{+\infty} t^{\alpha-1}E_{\alpha,\alpha}(-p^2t^{\alpha})\cos{(px)}dp;$$

$$G_2(x,t) = \frac{1}{\pi}\int_0^{+\infty} t^{\alpha-2} E_{\alpha,\alpha-1}(-p^2 t^\alpha)\cos(px)dp.$$

While derivation of formula (13) the well-known formula for the inverse Fourier transform with respect to the function $f(p)$ was used

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} e^{-ipx} f(p)dp = \frac{1}{\pi}\int_0^{+\infty} f(p)\cos(px)dp.$$

The function $G_1(x,t)$ was found in [24; 141]

$$G_1(x,t) = \frac{1}{2} t^{\frac{\alpha}{2}-1}\phi\left(-\frac{\alpha}{2}, \frac{\alpha}{2}; -\frac{|x|}{t^{\frac{\alpha}{2}}}\right),$$

where $\phi(\lambda,\mu;z)$ is the Wright function (4), according to the following scheme.

Let's apply Laplace transform to the second term in (13) with respect to the variable t and use formula:

$$L[t^{\gamma-1}E_{a,\gamma}(\lambda t^a)] = \frac{s^{a-\gamma}}{s^a - \lambda}$$

with $a = \alpha,\ \gamma = \alpha - 1,\ \lambda = -p^2$.

Subtracting the last integral and taking into account formula 3.723 from [26], we get

$$g_1(x,s) = \frac{1}{2} s^{-\frac{\alpha}{2}} \exp(-|x| s^{\frac{\alpha}{2}}). \tag{14}$$

Applying the inverse Laplace transform to (14) with respect to the variable s taking into account formula (6) with $\lambda = x,\ \beta = \frac{\alpha}{2},\ \rho = -\frac{\alpha}{2},\ \lambda \in (1;2)$, we get

$$G_1(x,t) = \frac{1}{2} t^{\frac{\alpha}{2}-1}\phi\left(-\frac{\alpha}{2}; \frac{\alpha}{2}; -|x| t^{-\frac{\alpha}{2}}\right). \tag{15}$$

Similarly, applying Laplace transform to the third term in (13) with respect to the variable t taking into account formula (6) with $k = 0,\ a = \alpha,\ b = \alpha - 1,\ \lambda = p^2$, we obtain

$$g_2(x,s) = \frac{1}{\pi}\int_0^{\infty} \frac{s\cos px}{s^\alpha + p^2} dp = \frac{1}{2} s^{1-\frac{\alpha}{2}} \exp(-|x| s^{\frac{\alpha}{2}}).$$

Applying the inverse Laplace transform and taking into account the formula (6) with $\lambda = x$, $\rho = -\frac{\alpha}{2}$, $\beta = -\frac{\alpha}{2}$, we get

$$G_2(x,t) = \frac{1}{2} t^{\frac{\alpha}{2}-2}\phi\left(-\frac{\alpha}{2}; \frac{\alpha}{2} - 1; -|x| t^{-\frac{\alpha}{2}}\right). \tag{16}$$

Substituting (15) and (16) into (13), we obtain a solution to the original problem (7)-(8):

$$u(x,t) = \frac{1}{2}\int_0^t \int_{-\infty}^{+\infty} \tau^{\frac{\alpha}{2}-1}\phi\left(-\frac{\alpha}{2}, \frac{\alpha}{2}; -\frac{|x-\xi|}{\tau^{\frac{\alpha}{2}}}\right) f(\xi, t-\tau) d\xi d\tau +$$

$$+\frac{1}{2}\int_{-\infty}^{+\infty} t^{\frac{\alpha}{2}-1}\phi\left(-\frac{\alpha}{2}, \frac{\alpha}{2}; -\frac{|x-\xi|}{t^{\frac{\alpha}{2}}}\right)\varphi(\xi)d\xi + \frac{1}{2}\int_{-\infty}^{+\infty} t^{\frac{\alpha}{2}-2}\phi\left(-\frac{\alpha}{2}, \frac{\alpha}{2} - 1; -\frac{|x-\xi|}{t^{\frac{\alpha}{2}}}\right)\psi(\xi)d\xi.$$

4 Limiting cases

Let's consider the limiting cases of the fractional derivative's order α.
I. $\alpha = 1$. Then problem (7)-(8) will take the form:

$$u_t - u_{xx} = f(x,t), \tag{17}$$

$$u|_{t=0} = \varphi(x), \tag{18}$$

$$\int_0^t u(x,\tau)d\tau|_{t=0} = \psi(x). \tag{19}$$

In the domain Ω the solution of problem (17)-(18) has the form [27]:

$$u(x,t) = \int_{-\infty}^{\infty} \varphi(\xi)G(x,\xi,t)d\xi + \int_0^t \int_{-\infty}^{\infty} f(\xi,\tau)G(x,\xi,t-\tau)d\xi d\tau, \tag{20}$$

where

$$G(x,\xi,t) = \frac{1}{2\sqrt{\pi t}} \exp\left(-\frac{(x-\xi)^2}{4t}\right). \tag{21}$$

We show that condition (19) is the overdetermination condition for $\alpha = 1$ in problem (17)-(18).
The solution of problem (17)-(18) has the form (20).
By virtue of Fubini's theorem, we have:

$$\int_0^t u(x,\tau)d\tau = \int_{-\infty}^{\infty} \varphi(\xi) \int_0^t G(x,\xi,t)\, d\tau d\xi + \int_{-\infty}^{\infty}\int_0^t f(\xi,\theta) \int_0^t G(x,\xi,\tau-\theta)\, d\tau d\theta d\xi,$$

where function $G(x,\xi,t)$ is defined by formula (21).
We calculate it using formula 3.471(2) [26; 354]

$$\int_0^t G(x,\xi,\tau)d\tau = \int_0^t \frac{1}{2\sqrt{\pi\tau}} \exp\left(-\frac{(x-\xi)^2}{4\tau}\right) d\tau = \sqrt{\frac{2}{x-\xi}}\, t^{\frac{3}{4}}\, exp\left(-\frac{(x-\xi)^2}{8\tau}\right) W_{-\frac{3}{4},\frac{1}{4}}\left(\frac{(x-\xi)^2}{4(t-\tau)}\right),$$

and

$$\int_\theta^t G(x,\xi,\tau-\theta)d\tau = \int_0^{t-\theta} G(x,\xi,\lambda)d\lambda = \sqrt{\frac{2}{x-\xi}}(t-\theta)^{\frac{3}{4}} exp\left(-\frac{(x-\xi)^2}{8(t-\tau)}\right) W_{-\frac{3}{4},\frac{1}{4}}\left(\frac{(x-\xi)^2}{4(t-\tau)}\right),$$

where $W_{\alpha,\beta}(z)$ is the Whittaker function [26; 1073].
Since for large values of z [26; 1075]

$$W_{\alpha,\beta} \sim e^{-\frac{z}{2}} z^{\alpha} \left(1 + \sum_{k=1}^{\infty} \frac{(\beta^2 - (\alpha-\frac{1}{2})^2)(\beta^2 - (\alpha-\frac{3}{2})^2)...(\beta^2 - (\alpha-k+\frac{1}{2})^2)}{k! z^{\alpha}}\right)$$

and with given $\lim_{t\to 0} \frac{(x-\xi)^2}{4t}$ and $0 < \theta < t$, then

$$\lim_{t\to 0} \int_0^t G(x,\xi,\tau)d\tau = \left\| z = \frac{(x-\xi)^2}{4t} \Rightarrow t = \frac{(x-\xi)^2}{4t} \right\| =$$

$$= \lim_{z\to\infty} \frac{\sqrt{2}}{\sqrt{|x-\xi|}} \frac{\sqrt{|x-\xi|}}{\sqrt{2} z^{\frac{3}{4}}} \exp\left(-\frac{z}{4}\right) z^{\frac{3}{4}} = \lim_{z\to\infty} e^{-\frac{z}{4}} = 0.$$

Similarly

$$\lim_{t\to 0}\int_0^t G(x,\xi,\tau-\theta)d\tau = 0.$$

Then the condition $D_{0t}^{\alpha-2}u|_{t=0} = \psi(x)$ in problem (17)-(18) is excess when $\alpha = 1$.
On the other hand, for $\alpha = 1$ out of (15)-(16) we have [28; 9]

$$G_1(x,t) = \frac{1}{2\sqrt{t}}\phi\left(-\frac{1}{2},\frac{1}{2};-\frac{|x|}{\sqrt{t}}\right) = \frac{1}{2\sqrt{\pi t}}\exp\left(-\frac{x^2}{4t}\right).$$

Then, for $\alpha = 1$, the solution of (20) coincides with (21).
II. $\alpha = 2$. The problems (7)-(8) become as follows:

$$u_t - u_{xx} = f(x,t),\ u|_{t=0} = \varphi(x),\ u_t|_{t=0} = \varphi(x).$$

The solution has the form [27; 258]

$$u(x,t) = \frac{1}{2}[\psi(x-t)+\psi(x+t)] + \frac{1}{2}\int_{x-t}^{x+t}\varphi(\xi)d\xi + \frac{1}{2}\int_0^t\int_{x-(t-\tau)}^{x+(t-\tau)} f(\xi,\tau)d\xi d\tau. \qquad (22)$$

On the other hand, for $\alpha = 2$ out of (15) we consider, that the function

$$G_1(x,t) = \frac{1}{2}\phi\left(-1,1;\frac{x}{t}\right)$$

doesn't exist.

Let's apply $\alpha = 2$ to (8).

$$U(p,t) = ((tE_{2,2}(-p^2t^2)) * F(p,t))(t) + tE_{2,2}(-p^2t^2)\overline{\varphi}(p) + tE_{2,1}(-p^2t^2)\overline{\psi}(p).$$

Known that $\sin z = zE_{2,2}(-z^2)$, $\cos z = E_{2,1}(-z^2)$. And we consider $z = pt$. Then

$$U(p,t) = \left(\frac{1}{p}\sin(pt) * F(p,t)\right)(t) + \frac{1}{p}\sin(pt)\overline{\varphi}(p) + \cos(pt)\overline{\psi}(p).$$

Apply the inverse Laplace transform.

Since

$$\sin(pt) = \frac{1}{2i}(e^{ipt} - e^{-ipt}), \cos(pt) = \frac{1}{2}(e^{ipt} + e^{-ipt}),$$

then

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{1}{2ip}((e^{ipt} - e^{-ipt}) * F(p,t))(t)e^{ipx}dp +$$

$$+\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{1}{2ip}((e^{ipt} - e^{-ipt})\overline{\varphi}(p)e^{ipx}dp + \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{1}{2}((e^{ipt} + e^{-ipt})\overline{\psi}(p)e^{ipx}dp.$$

Note that

$$\frac{1}{ip}((e^{ip(x+t)} - e^{-ip(x-t)}) = \int_{x-t}^{x+t} e^{ip\eta}d\eta.$$

Then, given the convolution formula with respect to the variable t, we get

$$u(x,t) = \frac{1}{4\pi}\int_0^t\left\{\int_{-\infty}^{+\infty}\int_{x-(t-\tau)}^{x+(t-\tau)} e^{ip\eta}d\eta F(p,\tau)dp\right\}d\tau +$$

$$+\frac{1}{4\pi}\int_{-\infty}^{+\infty}\int_{x-t}^{x+t} e^{ip\eta}\overline{\varphi}(p)d\eta dp+\frac{1}{4\pi}\int_{-\infty}^{+\infty}(e^{ip(x+t)}+e^{ip(x-t)})\overline{\psi}(p)dp.$$

Changing the order of integration in the first and second integrals and considering that

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ip\eta}F(p,t)dp=f(\eta,t),$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ip\eta}\overline{\varphi}(p)dp=\varphi(\eta),$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ip(x\pm t)}\overline{\psi}(p)dp=\psi(x\pm t)$$

are the originals of the function, we finally get

$$u(x,t)=\frac{1}{2}\int_0^t\int_{x-(t-\tau)}^{x+(t-\tau)} f(\eta,\tau)d\eta d\tau+\frac{1}{2}\int_{x-t}^{x+t}\varphi(\eta)d\eta+\frac{1}{2}[\psi(x+t)+\psi(x-t)].$$

Same as formula (22).

5 *The main result*

So, the following theorem has been proven.

Theorem 1. Let the function $u(x,t)$ be a regular solution to equation (7), and satisfies the conditions (8). Then for any point $(x,t)\in\Omega$ and $\alpha\in[1;2]$ the relation holds

$$u(x,t)=\frac{1}{2}\int_0^t\int_{-\infty}^{+\infty}\tau^{\frac{\alpha}{2}-1}\phi\left(-\frac{\alpha}{2},\frac{\alpha}{2};-\frac{|x-\xi|}{\tau^{\frac{\alpha}{2}}}\right)f(\xi,t-\tau)d\xi d\tau+$$

$$+\frac{1}{2}\int_{-\infty}^{+\infty}t^{\frac{\alpha}{2}-1}\phi\left(-\frac{\alpha}{2},\frac{\alpha}{2};-\frac{|x-\xi|}{t^{\frac{\alpha}{2}}}\right)\varphi(\xi)d\xi+\frac{1}{2}\int_{-\infty}^{+\infty}t^{\frac{\alpha}{2}-2}\phi\left(-\frac{\alpha}{2},\frac{\alpha}{2}-1;-\frac{|x-\xi|}{t^{\frac{\alpha}{2}}}\right)\psi(\xi)d\xi, \quad (23)$$

where $\phi(\lambda,\mu;z)$ is the Wright function (4).

Conclusion

It can be shown that the function

$$G(x,t,\xi)=\frac{1}{2}t^{\frac{\alpha}{2}-1}\phi\left(-\frac{\alpha}{2},\frac{\alpha}{2};-\frac{|x-\xi|}{t^{\frac{\alpha}{2}}}\right)$$

is a fundamental solution to the equation

$$D_{0t}^{\alpha}u(x,t)-u_{xx}(x,y)=0,\quad \alpha\in(1;2).$$

In the future, we plan to solve a BVP in which the equation contains a loaded term in the form of a fractional derivative. When solving the problem, we will use the representation of the solution in the form (23). We assume that for certain values of the fractional derivative's order and of the type of manifold on that the load is specified, the uniqueness of the BVP's solution will be violated.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Уақыт бойынша бөлшек туындысы бар толқындық теңдеудің шеткі есебі

М.Т. Космакова[1], А.Н. Хамзеева[1], Л.Ж. Касымова[2]

[1] *Қолданбалы математика институты, Академик Е.А. Бөкетов атындағы Қарағанды университеті, Қарағанды, Қазақстан;*
[2] *Әбілқас Сағынов атындағы Қарағанды техникалық университеті, Қарағанды, Қазақстан*

Мақалада уақыт бойынша бөлшек туындысы бар және Риман-Лиувилл мағынасында бөлшек туынды ретінде берілген бастапқы шарттары бар толқындық теңдеудің шеткі есебі шешілді. Қажетті функцияны анықтау аймағы жоғарғы жартылай жазықтық (x, t) болып табылады. Есепті шешу үшін кеңістіктік айнымалы бойынша Фурье түрлендіруі дәйекті түрде қолданылады, содан кейін уақыт айнымалысы бойынша Лаплас түрлендіріледі. Кері Лаплас түрлендіруін қолданғаннан кейін түрлендірілген есепті шешу Миттаг-Леффлердің екіпараметрлі функциясын қамтиды. Кері Фурье түрлендіруін пайдаланғаннан кейін, Райт функциясын қамтитын тапсырманың шешімі айқын түрде алынады. Әрі қарай, есептің теңдеуіне кіретін бөлшек туынды ретінің шеткі жағдайлары қарастырылған.

Кілт сөздер: бөлшек туынды, Лаплас түрлендіруі, Фурье түрлендіруі, Миттаг-Леффлер функциясы, Райт функциясы.

Краевая задача для волнового уравнения с дробной производной по времени

М.Т. Космакова[1], А.Н. Хамзеева[1], Л.Ж. Касымова[2]

[1] *Институт прикладной математики, Карагандинский университет имени академика Е.А. Букетова, Караганда, Казахстан;*
[2] *Карагандинский технический университет имени Абылкаса Сагинова, Караганда, Казахстан*

В статье решена краевая задача для волнового уравнения с дробной производной по времени и с начальными условиями, заданными в виде дробной производной в смысле Римана-Лиувилля. Областью определения искомой функции является верхняя полуплоскость (x, t). Для решения задачи последовательно применено преобразование Фурье по пространственной переменной, затем — преобразование Лапласа по временной переменной. Решение преобразованной задачи после применения обратного преобразования Лапласа содержит двухпараметрическую функцию Миттаг-Леффлера. После применения обратного преобразования Фурье получено решение поставленной задачи в явном виде, которое содержит функцию Райта. Далее рассмотрены предельные случаи порядка дробной производной, входящей в уравнение задачи.

Ключевые слова: дробная производная, преобразование Лапласа, преобразование Фурье, функция Миттаг-Леффлера, функция Райта.

Author Information[*]

Minzilya Timerbaevna Kosmakova — PhD, Associate professor, Professor, Karaganda Buketov University, 28 Universitetskaya street, Karaganda, 100028, Kazakhstan; e-mail: *svetlanamir578@gmail.com*; https://orcid.org/0000-0003-4070-0215

Aiym Nurlanovna Khamzeyeva (*corresponding author*) — Master of Mathematics, Teacher, Karaganda Buketov University, 28 Universitetskaya street, Karaganda, 100028, Kazakhstan; e-mail: *aiymkhamzeyeva@gmail.com*; https://orcid.org/0009-0009-1254-8077

Laila Zhumazhanovna Kasymova — PhD, Senior Lecturer, Abylkas Saginov Karaganda Technical University, 56 Ave. Nursultan Nazarbayev, Karaganda, 100027, Kazakhstan; e-mail: *l.kasymova2017@mail.ru*; https://orcid.org/0000-0002-4696-867X

[*]The author's name is presented in the order: First, Middle and Last Names.