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Deformations of the three-dimensional Lie algebra $\mathfrak{sl}(2)$

Deformation is one of key questions of the structural theory of algebras over a field. Especially, it plays a important role in the classification of such algebras. In odd characteristics of algebraically closed fields, local deformations of classical Lie algebras are completely described. Local deformations are also known for classical Lie algebras with a homogeneous root system over an algebraically closed field of characteristic 2, except for the three-dimensional Lie algebra $\mathfrak{sl}(2)$. In the characteristic 2, deformations of Lie algebras with an non-homogeneous root system are calculated only for Lie algebras of small ranks. In this paper we investigate deformations of the three-dimensional classical Lie algebra $\mathfrak{sl}(2)$ over an algebraically closed field k of characteristic $p = 2$. We also describe three-dimensional two-sided Alia algebras associated with Lie algebra $\mathfrak{sl}(2)$ in the characteristics 2 and 3. It is proved that, in characteristic 2, the space of local deformations of the Lie algebra $\mathfrak{sl}(2)$ is five-dimensional. The structural specialty of the second cohomology space of the adjoint representation of the Lie algebra $\mathfrak{sl}(2)$ are analyzed. In particular, the subspace of cosets of restricted cocycles is described. It is proved that the subspace of classes of restricted cocycles is two-dimensional and the corresponding local deformations are restricted Lie algebras in the sense of Jacobson. It was found that a family of simple three-dimensional unrestricted Lie algebras correspond to unrestricted non-trivial cocycles. In characteristics 2 and 3, three-dimensional two-sided Alia algebras that are non-isomorphic to the Lie algebra $\mathfrak{sl}(2)$ are constructed. In the process of the study, a complete description of the space of all derivations of the Lie algebra $\mathfrak{sl}(2)$ is obtained.

Keywords: Lie algebra, module, representation, derivation, outer derivation, deformation, restricted deformation, cohomology, cocycle, commutative cocycle, Alia algebra.

Introduction

Over an algebraically closed field of characteristic zero, classical Lie algebras are rigid. Deformations of classical Lie algebras in positive characteristics were studied in [1–10]. In [11–14] deformations of Cartan type Lie algebras are studied.

In this paper deformations of the three-dimensional classical Lie algebra $\mathfrak{g} = \mathfrak{sl}(2)$ over an algebraically closed field k of characteristic $p = 2$ are calculated. It is well-known that, in the case when $p > 2$, the Lie algebra $\mathfrak{sl}(2)$ is rigid [6]. We prove that in characteristic 2 the Lie algebra $\mathfrak{sl}(2)$ admits a five-dimensional space of local deformations (Theorem 1). To prove Theorem 1, we use information on the structure of the space of outer derivations of the Lie algebra $\mathfrak{g}$. In section 1 we give a complete description of the space of outer differentiations of the Lie algebra $\mathfrak{g}$ (Proposition 1). The dimension of the space of outer derivations of the Lie algebra $\mathfrak{g}$ was previously calculated in [15]. In section 2 the spaces of usual and restricted second cohomologies of the Lie algebra $\mathfrak{g}$ with coefficients in the adjoint representation are calculated. According to the general theory of deformation, a necessary condition for the deformation of a Lie algebra is the non-triviality of its second cohomology with coefficients in the adjoint representation. However, in the general case, the correspondence between the 2-cocycle classes of the second cohomology for the adjoint representation and the deformations of the Lie algebra is not one-to-one [8]. In this connection, we prove that the parametrizability of local deformations of a restricted Lie algebra $\mathfrak{g}$ by elements of the second cohomology $H^2(\mathfrak{g}, \mathfrak{g})$ (Lemma 1). By restricted local deformations of a Lie algebra $\mathfrak{g}$ we mean deformations corresponding to elements of the second restricted cohomology $H^2_*(\mathfrak{g}, \mathfrak{g})$. A restricted cohomology of a restricted Lie algebra was first introduced by Hochschild in [16]. In the last section 3 the space of commutative cocycles with coefficients in k is calculated (Proposition 2). Commutative cocycles play an important role in the structural theory of two-sided Alia algebras [17–19] and in the second cohomology groups of current Lie algebras [20]. An algebra $(A, \circ)$ is called a two-sided Alia algebra if the identities

$$[a, b] \circ c + [b, c] \circ a + [c, a] \circ b = 0,$$

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$$a \circ [b,c] + b \circ [c,a] + c \circ [a,b] = 0$$

hold, where $[a,b] = a \circ b - b \circ a$ is the usual commutator. The two-sided Alia algebra is Lie-admissible, i.e. the space A becomes a Lie algebra respect to the multiplication $[a,b] = a \circ b - b \circ a$. The Lie algebra deformed by commutative cocycles is a two-sided Alia algebra non-isomorphic to the Lie algebra itself. It was proved in [17] that if $p \neq 2, 3$, then among the classical Lie algebras only the three-dimensional Lie algebra $\mathfrak{sl}(2)$ admits commutative cocycles. Commutative cocycles for some important classes of Lie algebras such as current Lie algebras, Kac–Moody algebras, finite-dimensional semi-simple algebras were studied in [21].

Let $\mathfrak{g}$ be a Lie algebra over an algebraically closed field k of characteristic p. The space of i-dimensional cochains $C^i(\mathfrak{g},\mathfrak{g})$ of an ordinary cochain complex is defined as the space of skew-symmetric poly-linear functions $\psi : \bigwedge^i(\mathfrak{g}) \to \mathfrak{g}$ with differential d defined by

$$d\psi(l_1, l_2, \cdots, l_{i+1}) = \sum_{j=1}^{i+1} (-1)^j [l_j, \psi(l_1, \cdots, \widehat{l_j}, \cdots, l_{i+1})] +$$

$$\sum_{p<q} (-1)^{p+q} \psi([l_p, l_q], \cdots, \widehat{l_p}, \cdots, \widehat{l_q}, \cdots, l_{i+1}),$$

where $l_1, l_2, \cdots, l_{i+1} \in \mathfrak{g}$, and the notation $\widehat{l_j}$ means that the element l_j should be omitted.

Let

$Z^i(\mathfrak{g},\mathfrak{g}) = Ker\, d \cap C^i(\mathfrak{g},\mathfrak{g})$ is the space of i-dimensional cocycles,

$B^i(\mathfrak{g},\mathfrak{g}) = Im\, d \cap C^i(\mathfrak{g},\mathfrak{g})$ is the i-dimensional cochains,

and $H^i(\mathfrak{g},\mathfrak{g}) = Z^i(\mathfrak{g},\mathfrak{g})/B^i(\mathfrak{g},\mathfrak{g})$ is the i-dimensional cohomologies.

If $\omega \in Z^1(\mathfrak{g},\mathfrak{g})$, $\psi \in Z^2(\mathfrak{g},\mathfrak{g})$ then

$$-\omega([l_1, l_2]) - [l_1, \omega(l_2)] + [l_2, \omega(l_1)] = 0 \text{ for all } l_1, l_2 \in \mathfrak{g}, \tag{1}$$

$$-\psi([l_1, l_2], l_3) + \psi([l_1, l_3], l_2) - \psi([l_2, l_3], l_1) - \tag{2}$$

$$[l_1, \psi(l_2, l_3)] + [l_2, \psi(l_1, l_3)] - [l_3, \psi(l_1, l_2)] = 0 \text{ for all } l_1, l_2, l_3 \in \mathfrak{g}.$$

Let now $\mathfrak{g} = \mathfrak{sl}(2)$. Choose a basis $\{e, h, f\}$ of the Lie algebra $\mathfrak{g}$ with the multiplication table $[h,e] = 2e$, $[h,f] = -2f$, $[e,f] = h$. In the dual space $\mathfrak{g}^*$ we choose the dual basis $\{e^*, h^*, f^*\}$ for the basis $\{e,h,f\}$. We identify the space $C^i(\mathfrak{g},\mathfrak{g})$ with the space $\bigwedge^i(\mathfrak{g}^*) \otimes \mathfrak{g}$. The cohomological class of the cocycle $\psi \in Z^i(\mathfrak{g},\mathfrak{g})$ is denoted by $[\psi]$.

1 Derivations

Proposition 1. Let $\mathfrak{g} = \mathfrak{sl}_2(k)$ be the three-dimensional classical Lie algebra over an algebraically closed field k of characteristic $p = 2$. Then the following isomorphisms of the vector spaces over k hold:

(a) $Z^1(\mathfrak{g},\mathfrak{g}) \cong \langle \omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6 \rangle_k$;

(b) $H^1(\mathfrak{g},\mathfrak{g}) \cong \langle [\omega_1], [\omega_2], [\omega_3], [\omega_4] \rangle_k$,

where $\omega_1 = e^* \otimes e + h^* \otimes h$, $\omega_2 = e^* \otimes f$, $\omega_3 = f^* \otimes e$, $\omega_4 = f^* \otimes f + h^* \otimes h$, $\omega_5 = e^* \otimes h$, $\omega_6 = f^* \otimes h$.

Proof. (a) First we prove that the cochains $\omega_1, \omega_2, \cdots, \omega_6$ are cocycles. To do this, it is sufficient to check condition (1) for the basis elements e, h, f. Indeed, since

$$-\omega_1([e,h]) - [e, \omega_1(h)] + [h, \omega_1(e)] = -2\omega_1(e) - [e,h] + [h,e] = -2e - 2e + 2e = 0,$$

$$-\omega_1([e,f]) - [e, \omega_1(f)] + [f, \omega_1(e)] = -\omega_1(h) + [f,e] = -h - h = 0,$$

$$-\omega_1([h,f]) - [h, \omega_1(f)] + [f, \omega_1(h)] = 2\omega_1(f) + [f,h] = 2f = 0,$$

then ω_1 is a cocycle. Similarly, the condition (1) is easily verified for other cochains.

Let

$$\omega = x_1 e^* \otimes e + x_2 e^* \otimes h + x_3 e^* \otimes f + y_1 h^* \otimes e + y_2 h^* \otimes h + y_3 h^* \otimes f$$

$$+ z_1 f^* \otimes e + z_2 f^* \otimes h + z_3 f^* \otimes f \in Z^1(\mathfrak{g},\mathfrak{g}),$$

where $x_j, y_j, z_j \in k$. The following implications hold:

$$-\omega([e,h]) - [e, \omega(h)] + [h, \omega(e)] = 0 \implies y_3 = 0,$$

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$$-\omega([e,f]) - [e,\omega(f)] + [f,\omega(e)] = 0 \Longrightarrow y_1 = y_3 = 0,\ x_1 + y_2 + z_3 = 0,$$
$$-\omega([h,f]) - [h,\omega(f)] + [f,\omega(h)] = 0 \Longrightarrow y_1 = 0.$$

Therefore, from the condition (1) it follows that $y_1 = y_3 = 0$ and $x_1 + y_2 + z_3 = 0$. These equalities form a linear system with respect to $x_j, y_j, z_j \in k$. The rank of this system is 3, so it has a six-dimensional space of solutions. As a basis of $Z^1(\mathfrak{g},\mathfrak{g})$, one can choose cocycles $\omega_1, \omega_2, \cdots, \omega_6$.

(*b*) Let $\omega = \sum_{j=1}^{6} a_j\omega_j \in Z^1(\mathfrak{g},\mathfrak{g})$, where $a_j \in k$. Suppose that $\omega \in B^1(\mathfrak{g},\mathfrak{g})$. Then for the basis elements of Li algebra the equalities

$$\omega(e) = [e, b_1e + b_2h + b_3h],\ \omega(h) = [h, b_1e + b_2h + b_3h],\ \omega(f) = [f, b_1e + b_2h + b_3h]$$

hold, where $b_j \in k$. From these equalities it follows that $a_1 = a_2 = a_3 = a_4 = 0$, $a_5 = c_1$, $a_6 = c_3$. Thus, the cocycles $\omega_1, \omega_2, \omega_3, \omega_4$ are linearly independent and its cosets form the basis of the space $H^1(\mathfrak{g},\mathfrak{g})$.

The proof of Proposition 1 is complete.

2 Local deformations

Theorem 1. Let $\mathfrak{g} = \mathfrak{sl}(2)$ be the three-dimensional classical Lie algebra over an algebraically closed field k of characteristic $p = 2$. Then the following isomorphisms of vector spaces over k hold:

(*a*) $Z^2(\mathfrak{g},\mathfrak{g}) \cong \langle\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6, \psi_7, \psi_8\rangle_k$;

(*b*) $H^2(\mathfrak{g},\mathfrak{g}) \cong \langle[\psi_1], [\psi_2], [\psi_3], [\psi_4], [\psi_5]\rangle_k$;

(*c*) $H^2_*(\mathfrak{g},\mathfrak{g}) \cong \langle[\psi_5], [\psi_6]\rangle_k$,

where $\psi_1 = e^* \wedge h^* \otimes e + h^* \wedge f^* \otimes f$, $\psi_2 = e^* \wedge h^* \otimes f$, $\psi_3 = h^* \wedge f^* \otimes e$, $\psi_4 = e^* \wedge f^* \otimes f$, $\psi_5 = e^* \wedge f^* \otimes e$, $\psi_6 = e^* \wedge h^* \otimes h$, $\psi_7 = h^* \wedge f^* \otimes h$, $\psi_8 = e^* \wedge f^* \otimes h$.

Proof. (*a*) Since for any cochain $\psi \in C^2(\mathfrak{g},\mathfrak{g})$,

$$-\psi([e,h],f) + \psi([e,f],h) - \psi([h,f],e) = 0$$

and

$$-[e,\psi_1(h,f)] + [h,\psi_1(e,f)] - [f,\psi_1(e,h)] = [e,f] - [f,e] = 0,$$
$$-[e,\psi_2(h,f)] + [h,\psi_2(e,f)] - [f,\psi_2(e,h)] = -[f,f] = 0,$$
$$-[e,\psi_3(h,f)] + [h,\psi_3(e,f)] - [f,\psi_3(e,h)] = -[e,e] = 0,$$
$$-[e,\psi_4(h,f)] + [h,\psi_4(e,f)] - [f,\psi_4(e,h)] = [h,f] = 0,$$
$$-[e,\psi_5(h,f)] + [h,\psi_5(e,f)] - [f,\psi_5(e,h)] = [h,e] = 0,$$
$$-[e,\psi_6(h,f)] + [h,\psi_6(e,f)] - [f,\psi_6(e,h)] = -[f,h] = 0,$$
$$-[e,\psi_7(h,f)] + [h,\psi_7(e,f)] - [f,\psi_7(e,h)] = -[e,h] = 0,$$
$$-[e,\psi_8(h,f)] + [h,\psi_8(e,f)] - [f,\psi_8(e,h)] = [h,h] = 0,$$

then by (2), the cochains $\psi_1, \psi_2, \cdots, \psi_8$ are cocycles.

Let

$$\psi = x_1e^* \wedge h^* \otimes e + x_2e^* \wedge h^* \otimes h + x_3e^* \wedge h^* \otimes f + y_1e^* \wedge f^* \otimes e + y_2e^* \wedge f^* \otimes h + y_3e^* \wedge f^* \otimes f$$
$$+z_1h^* \wedge f^* \otimes e + z_2h^* \wedge f^* \otimes h + z_3h^* \wedge f^* \otimes f \in Z^2(\mathfrak{g},\mathfrak{g}),$$

where x_j, y_j, $z_j \in k$. Then from the cocycle condition (2) it follows that $x_1 + z_3 = 0$. Therefore, $\dim Z^2(\mathfrak{g},\mathfrak{g}) = 9 - 1 = 8$. The cocycles ψ_j, $j = 1, 2, \cdots, 8$ form the basis of $Z^2(\mathfrak{g},\mathfrak{g})$.

(*b*) Let

$$\psi = \sum_{j=1}^{8} a_j\psi_j \in Z^2(\mathfrak{g},\mathfrak{g})$$

and

$$\omega = b_1e^* \otimes e + b_2e^* \otimes h + b_3e^* \otimes f + b_4h^* \otimes e + b_5h^* \otimes h + b_6h^* \otimes f$$
$$+b_7f^* \otimes e + b_8f^* \otimes h + b_9f^* \otimes f \in C^1(\mathfrak{g},\mathfrak{g}),$$

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where $a_j, b_j \in k$. Then from the condition $\psi = d\omega$ it follows that $a_1 = a_2 = a_3 = 0$, $a_4 = b_6$, $a_5 = b_4$, $a_6 = b_6$, $a_7 = b_4$, $a_8 = b_1 + b_5 + b_9$. Therefore the cocycles

$$\psi_1, \psi_2, \psi_3, \psi_4, \psi_5$$

are linear independent and its cosets form a basis of $H^2(\mathfrak{g}, \mathfrak{g})$.

(c) For the adjoint $\mathfrak{g}$-module $\mathfrak{g}$ there is the following Hochschild exact sequence [16]:

$$0 \to H^1_*(\mathfrak{g}, \mathfrak{g}) \to H^1(\mathfrak{g}, \mathfrak{g}) \xrightarrow{T} S(\mathfrak{g}, \mathfrak{g}^{\mathfrak{g}}) \to H^2_*(\mathfrak{g}, \mathfrak{g}) \to H^2(\mathfrak{g}, \mathfrak{g}) \xrightarrow{D} S(\mathfrak{g}, H^1(\mathfrak{g}, \mathfrak{g})),$$

where

$$S(\mathfrak{g}, V) = \{u : \mathfrak{g} \to V \mid u(\alpha_1 l_1 + \alpha_2 l_2) = \alpha_1^p u(l_1) + \alpha_2^p u(l_2),\ \alpha_1, \alpha_2 \in k,\ l_1, l_2 \in \mathfrak{g}\}$$

is a space of semi-linear maps from $\mathfrak{g}$ to V. The maps T and D defined by

$$T_\omega(l_1) = [l_1^{p-1}, \omega(l_1)] - \omega(l_1^{[p]}),$$

$$D_\psi(l_1)l_2 = \sum_{j=0}^{p-1} (ad\, l_1)^j (\psi(l_1, (ad\, l_1)^{p-1-j}(l_2))) - \psi(l_1^{[p]}, l_2),$$

where $l_1, l_2 \in \mathfrak{g}$, $\omega \in Z^1(\mathfrak{g}, \mathfrak{g})$, and $\psi \in Z^2(\mathfrak{g}, \mathfrak{g})$. In particular, if $p = 2$ then

$$T_\omega(l_1) = [l_1, \omega(l_1)] - \omega(l_1^{[2]}), \tag{3}$$

$$D_\psi(l_1)l_2 = \psi(l_1, [l_1, l_2]) + [l_1, \psi(l_1, l_2)] - \psi(l_1^{[2]}, l_2). \tag{4}$$

If $D_\psi(l_1)$ is a inner derivation for the Lie algebra $\mathfrak{g}$ for some $l_1 \in \mathfrak{g}$ then It is obvious that the image of cocycle coset $[\psi]$ under the map

$$D : H^2(\mathfrak{g}, \mathfrak{g}) \to S(\mathfrak{g}, H^1(\mathfrak{g}, \mathfrak{g}))$$

is trivial.

By (3) and Proposition 1,

$$T_{\omega_1} = T_{\omega_4} = u,\ T_{\omega_2} = \omega_5,\ T_{\omega_3} = \omega_6,$$

where u is a semi-linear map defined by $u(h) = h$, $u(e) = u(f) = 0$. Therefore, $\Im T \cong S(\mathfrak{g}, \mathfrak{g}^{\mathfrak{g}})$. Then from the previous exact sequence it follows that the following sequence is exact:

$$0 \to H^2_*(\mathfrak{g}, \mathfrak{g}) \to H^2(\mathfrak{g}, \mathfrak{g}) \xrightarrow{D} S(\mathfrak{g}, H^1(\mathfrak{g}, \mathfrak{g})). \tag{5}$$

Using (4) and the statement (a) of Proposition 1, we get

$$D_{\psi_1}(h) = \omega_1 + \omega_4,\ D_{\psi_2}(h) = \omega_2,\ D_{\psi_3}(h) = \omega_3,\ D_{\psi_4}(h) = \omega_5,\ D_{\psi_5}(h) = \omega_6.$$

Then, by the statements (a) and (b) of Proposition 1, the maps $D_{\psi_4}(h)$, $D_{\psi_5}(h)$ are inner derivations of $\mathfrak{g}$, and the maps $D_{\psi_1}(h)$, $D_{\psi_2}(h)$, $D_{\psi_3}(h)$ are outer derivations. Therefore, $[\psi_4], [\psi_5] \in H^2_*(\mathfrak{g}, \mathfrak{g})$ and $[\psi_1], [\psi_2]$, $[\psi_3] \notin H^2_*(\mathfrak{g}, \mathfrak{g})$. Then the statement ($c$) follows from the exact sequence (5).

The proof of Theorem 1 is complete.

Lemma 1. The cocycle cosets of the space $H^2(\mathfrak{g}, \mathfrak{g})$ define nontrivial local deformations of the Lie algebra $\mathfrak{g}$.

Proof. By Theorem 1,

$$H^2(\mathfrak{g}, \mathfrak{g}) \cong \langle [\psi_1], [\psi_2], [\psi_3], [\psi_4], [\psi_5] \rangle_k.$$

Denote by $\mathfrak{sl}(2, t_j)$ the local deformation of the Lie algebra $\mathfrak{g}$ corresponding to the cocycle coset $[\psi_j]$. For the basis elements of the Lie algebra $\mathfrak{sl}(2, t_j)$, we also use the notation of the basis elements of the Lie algebra $\mathfrak{g}$. The multiplication of the Lie algebra $\mathfrak{sl}(2, t_j)$ is defined by

$$[l_1, l_2]_{t_j} = [l_1, l_2] + t_j \psi_j(l_1, l_2),\ \ l_1, l_2 \in \mathfrak{sl}(2, t_j),\ t_j \in k.$$

First, we consider the Lie algebra $\mathfrak{sl}(2, t_1)$. It is a simple Lie algebra. Therefore, it corresponds to a nontrivial local deformation of the Lie $\mathfrak{g}$.

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If $j \in \{2,3,4,5\}$ then $\mathfrak{sl}(2,t_j)$ is not a simple Lie algebra. Any homomorphism $\mathfrak{sl}(2,t_j) \longrightarrow \mathfrak{g}$ is not one-to-one. Indeed, if $\phi : \mathfrak{sl}(2,t_j) \longrightarrow \mathfrak{g}$ is a homomorphism of Lie algebras, then $\phi(h) = 0$ if $j = 2,3$, $t_4\phi(h)+\phi(f) = 0$ if $j = 4$, and $t_5\phi(h)+\phi(e) = 0$ if $j = 5$. Hence, the cocycle cosets ψ_2, ψ_3, ψ_4, ψ_5 define nontrivial local deformations of the Lie algebra $\mathfrak{g}$.

The proof of Lemma 1 is complete.

Remark 1. By Theorem 1, $H^2_*(\mathfrak{g},\mathfrak{g})$ is a proper subspace of $H^2(\mathfrak{g},\mathfrak{g})$. This means that not all local deformations of the Lie algebra $\mathfrak{g}$ admit a restricted structure.

3 Deformations by commutative cocycles

Let $\mathfrak{g}$ be a Lie algebra over an algebraically closed field k of characteristic p and V be a $\mathfrak{g}$-module. A bilinear map $\eta : \mathfrak{g} \times \mathfrak{g} \longrightarrow V$ satisfying conditions

$$\eta([l_1,l_2],l_3) + \eta([l_2,l_3],l_1) + \eta([l_3,l_1],l_2) = 0, \qquad (6)$$

$$\eta(l_1,l_2) = \eta(l_2,l_1), \qquad (7)$$

where $l_1, l_2, l_3 \in \mathfrak{g}$, is called *a commutative cocycle with coefficients in V*. Let $Z^2_{com}(\mathfrak{g},M)$ be the space of the commutative cocycles with coefficients in V and $Z^2_{com}(\mathfrak{g}) = Z^2_{com}(\mathfrak{g},k)$. If $p \neq 2,3$, then among classical Lie algebras only a three-dimensional classical Lie algebra $\mathfrak{sl}(2)$ admits commutative cocycles and $\dim Z^2_{com}(\mathfrak{sl}(2)) = 5$ [17, Theorem 1]. In characteristic $p = 2$, from the condition (6) it follows that if $\eta \in Z^2_{com}(\mathfrak{sl}(2))$ then $\eta(h,h) = 0$. Hence, using the condition (7), we get the following

Proposition 2. Let $\mathfrak{g} = \mathfrak{sl}(2)$ be the three-dimensional classical Lie algebra over an algebraically closed field k of characteristic $p = 2$. Then

$$Z^2_{com}(\mathfrak{g}) \cong \langle \eta_i : i = 1,\cdots,5\rangle_k,$$

where

$$\eta_1(e,e) = 1;\ \eta_2(e,h) = \eta_2(h,e) = 1;\ \eta_3(e,f) = \eta_3(f,e) = 1;$$

$$\eta_4(h,f) = \eta_4(f,h) = 1;\ \eta_5(f,f) = 1$$

(not specified components are equal to zero).

Remark 2. A statement similar to Proposition is also true in the case of characteristic $p = 3$. Indeed, according to (6), any commutative cocycle η satisfies the equality $\eta(e,f) = \eta(h,h)$. Therefore, a basic commutative cocycle η_3 may be chosen so that the equalities

$$\eta_3(e,f) = \eta_3(f,e) = \eta_3(h,h) = 1$$

hold.

Remark 3. In the space $\mathfrak{g} = \mathfrak{sl}(2)$ the commutative cocycles of $Z^2_{com}(\mathfrak{g},\mathfrak{g})$ define Lie-admissible two sided Alia algebras non-isomorphic to $\mathfrak{g}$ [17]. Since

$$Z^2_{com}(\mathfrak{g},\mathfrak{g}) \cong Z^2_{com}(\mathfrak{g}) \otimes \mathfrak{g}$$

then, according to Proposition 2 and Remark 2, in characteristics $p = 2,3$ there exist three-dimensional Lie-admissible Alia algebras non-isomorphic to the Lie algebra $\mathfrak{sl}(2)$.

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Үш өлшемді $\mathfrak{sl}(2)$ Ли алгебрасының деформациялары

Деформация өріске қатысты алгебралардың құрылымдық теориясының маңызды сұрақтарының бірі болып табылады. Әсіресе, осындай алгебралардың классификациясындағы оның орны ерекше. Сипаттамасы тақ алгебралық тұйық өрістерде классикалық Ли алгебраларының локальді деформациялары толық есептелген. Сол сияқты, сипаттамасы 2-ге тең өрісте түбірлер жүйесі біртекті классикалық Ли алгебраларының локальді деформациялары да $\mathfrak{sl}(2)$ Ли алгебрасынан басқа жағдайларда белгілі. Сиаттамасы 2-ге тең өрісте түбірлер жүйесі біртекті емес классикалық Ли алгебраларының локальді деформациялары тек рангі төмен Ли алгебраларында есептелген. Мақалада сипаттамасы $p = 2$ алгебралық тұйық k өрісіндегі үш өлшемді классикалық $\mathfrak{sl}(2)$ Ли алгебрасының деформациялары зерттелді. 2 және 3-ке тең өріс сипаттамаларында, $\mathfrak{sl}(2)$ Ли алгебрасымен байланысты үш өлшемді екі жақты Алиа алгебралары қарастырылған. Сипаттамасы 2-ге тең өрісте $\mathfrak{sl}(2)$ Ли алгебрасының локальді деформациялары кеңістігінің бес өлшемді екені дәлелденді. $\mathfrak{sl}(2)$ Ли алгебрасының кіріктірілген модулінің екінші когомологиялар кеңістігінің құрылымдық ерекшеліктері талданды. Дербес жағдайда, шектелген коциклдер кластарының ішкі кеңістігі сипатталды. Шектелген коциклдер кластарының ішкі кеңістігінің екі өлшемді және сәйкесті локальді деформациялардың Джекобсон мағынасында шектелген Ли алгебралары екені дәлелденді. Шектелмеген тривиаль емес коциклдерге жәй үш өлшемді Ли алгебраларының үйірі сәйкес келетіні анықталды. 2 және 3-ке тең сипаттамаларда, $\mathfrak{sl}(2)$ Ли алгебрасына изоморфты емес үш өлшемді екі жақты Алиа алгебралары құрылды. Жүргізілген зерттеулер нәтижесінде $\mathfrak{sl}(2)$ Ли алгебрасының барлық дифференциалдаулар кеңістігінің толық сипаттамасы алынды.

Кілт сөздер: Ли алгебрасы, модуль, көрініс, дифференциалдау, сыртқы дифференциалдау, деформация, шектелген деформация, когомология, коцикл, коммутативті коцикл, Алиа алгебрасы.

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Деформации трехмерой алгебры Ли $\mathfrak{sl}(2)$

Деформация является одним из ключевых вопросов структурной теории алгебр над полем. Особенно важную роль она играет при классификации таких алгебр. В нечетных характеристиках алгебраически замкнутых полей локальные деформации классических алебр Ли описаны полностью. Также известны локальные деформации классических алгебр Ли с однородной системой корней над алгебраически замкнутым полем характеристики 2, кроме трехмерной алгебры Ли $\mathfrak{sl}(2)$. В характеристике 2 деформации алгебр Ли с неоднородной системой корней вычислены только для алгебр Ли малых рангов. В статье изучены деформации трехмерной классической алгебры Ли $\mathfrak{sl}(2)$ над алгебраически замкнутым полем k характеристики $p = 2$. Описаны трехмерные двусторонние алгебры Алиа, связанные с алгеброй Ли $\mathfrak{sl}(2)$ в характеристиках 2 и 3. Доказано, что в характеристике 2 пространство локальных деформаций алгебры Ли $\mathfrak{sl}(2)$ пятимерно. Проанализированы структурные особенности пространства второй когомологии присоединенного представления алгебры Ли $\mathfrak{sl}(2)$. В частности, описано подпространство классов ограниченных коциклов. Доказано, что подпространство классов ограниченных коциклов двумерно и соответствующие локальные деформации являются ограниченными алгебрами Ли в смысе Джекобсона. Выяснено, что неограниченным нетривиальным коциклам соответствует семейство простых трехмерных неограниченных алгебр Ли. В характеристиках 2 и 3 построены трехмерные двусторонние алгебры Алиа, неизоморфные алгебре Ли $\mathfrak{sl}(2)$. В ходе проведенного исследования получено полное описание пространства всех дифференцирований алгебры Ли $\mathfrak{sl}(2)$.

Ключевые слова: алгебра Ли, модуль, представление, дифференцирование, внешнее дифференцирование, деформация, ограниченная деформация, когомология, коцикл, коммутативный коцикл, алгебра Алиа.

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