DOI 10.31489/2022M3/51-58
UDC 517.9

S.Zh. Igisinov*, L.D. Zhumaliyeva, A.O. Suleimbekova, Ye.N. Bayandiyev

M.Kh. Dulaty Taraz Regional University, Taraz, Kazakhstan
(E-mail: igisinovsabit@mail.ru, lzdmath@mail.ru, suleimbekovaa@mail.ru, erikn87@mail.ru)

Estimates of singular numbers (s-numbers) for a class of degenerate elliptic operators

In this paper we study a class of degenerate elliptic equations with an arbitrary power degeneracy on the line. Based on the research carried out in the course of the work, the authors propose methods to overcome various difficulties associated with the behavior of functions from the definition domain for a differential operator with piecewise continuous coefficients in a bounded domain, which affect the spectral characteristics of boundary value problems for degenerate elliptic equations. It is shown the conditions imposed on the coefficients at the lowest terms of the equation, which ensure the existence and uniqueness of the solution. The existence, uniqueness, and smoothness of a solution are proved, and estimates are found for singular numbers (s-numbers) and eigenvalues of the semiperiodic Dirichlet problem for a class of degenerate elliptic equations with arbitrary power degeneration.

Keywords: elliptic operator, boundary value problem, singular numbers, power degeneracy, solution, uniqueness.

1 Introduction. Main results

Let $\Omega = \{(x, y) : -\pi < x < \pi, 0 < y < 1\}$. Consider the following problem

$$Lu = -k(y)u_{xx} - u_{yy} + a(y)u_x + c(y)u = f(x, y) \in L_2(\Omega), \tag{1}$$

$$u(-\pi, y) = u(\pi, y), u_x(-\pi, y) = u_x(\pi, y), \tag{2}$$

$$u(x, 0) = u(x, 1) = 0, \tag{3}$$

where $a(y), c(y)$ are piecewise continuous functions in $[0, 1]$, $k(y) > 0$ as $y \in (0, 1]$ and $k(0) = 0$. Let $C^{\infty}_{0,\pi}(\bar{\Omega})$ be a class of infinitely differentiable finite functions in $\bar{\Omega}$ and satisfying the conditions (2)–(3).

We also denote closure of the operator (1) by the norm of $L_2(\Omega)$ as L.

In the study of the smoothness and approximation properties of solutions to boundary value problems for some nonlinear equations we encounter questions of the spectral properties of linear degenerate elliptic equations. In contrast to elliptic operators, spectral questions for degenerate elliptic operators are poorly understood. Known results on this topic or those close to it in content are contained in the works of M. Smirnov [1], M. Keldysh [2], T. Kalmenov, M. Otelbaev [3], O. Oleinik [4], M. Vishik, V. Grushin [5, 6], and others.

However, in the general case, such traditional questions as asymptotic behavior and estimates of eigenvalues in general are far from complete.

The results of this work are close to those of M.B. Muratbekov [7–10], where differential operators of mixed and hyperbolic types were investigated. In contrast to the above works, here we investigate previously unconsidered degenerate elliptic equations with an arbitrary power-law degeneracy on the degeneracy line.

Definition 1. The function $u \in L_2(\Omega)$ is called a solution to (1)–(3) if there exists a sequence $\{u_k(x, y)\}_{k=1}^{\infty} \subset C^{\infty}_{0,\pi}(\overline{\Omega})$ such that

*Corresponding author.
E-mail: igisinovsabit@mail.ru

$$\|u_k - u\|_2 \to 0, \ \ \|Lu_k - f\|_2 \to 0 \text{ as } k \to \infty.$$

By $C([0,1], L_2(-\pi,\pi))$ we denote the space obtained by completing the set of continuous functions on the interval $[0,1]$ with values in $L_2(-\pi,\pi)$ relative to the norm

$$\|u\|_{C([0,1],L_2)} = \sup_{y\in[0,1]} \left(\int_{-\pi}^{\pi} |u(x,y)|^2 dx \right)^{\frac{1}{2}}$$

and $W_2^1(\Omega)$ is the Sobolev space with norm

$$\|u\|_{2,1,\Omega} = \left[\|u_x\|_2^2 + \|u_y\|_2^2 + \|u\|_2^2\right]^{\frac{1}{2}},$$

where $\|\cdot\|$ is a norm of $L_2(\Omega)$.

Definition 2. [11] Let A be a completely continuous linear operator and $|A| = \sqrt{A^*A}$. Eigenvalues of $|A|$ are called s-numbers of A.

Nonzero s-numbers of L^{-1} will be numbered in descending order, taking into account their multiplicity, so that

$$s_k(L^{-1}) = \lambda_k(|L^{-1}|), k = 1,2,3,...$$

Theorem 1. Let $a(y), c(y)$ are piecewise continuous functions in $[0,1]$ and satisfying the conditions

$$i) a(y) \geq \delta_0 > 0, c(y) \geq \delta > 0.$$

Then there exists a unique solution $u(x,y)$ to (1)–(3) such that

$$\|u\|_{C(0,1),L_2} + \|u\|_{2,1,\Omega} \leq c_0\|f\|_2$$

for all $f \in L_2(\Omega)$, where c_0 is a constant.

Theorem 2. Let the condition i) be fulfilled. Then the estimate

$$c_1\frac{1}{k} \leq s_k \leq c_2\frac{1}{k^{\frac{1}{2}}}, k = 1,2,3,...$$

holds, where c_1, c_2 are constants, $0 < c_1 \leq c_2$, s_k is singular numbers (s-numbers) of L^{-1}.

2 Auxiliary lemmas

Lemma 1. The estimate

$$\|Lu\|_2 \geq c\|u\|_2 \tag{4}$$

holds for all $u \in D(L)$, where c_0 is a constant.

Proof. Let $C_{0,\pi}^{\infty}(\Omega)$. Integrate by parts and taking into account the boundary conditions we have

$$< Lu, u > \geq \int_{\Omega} (u_y^2 + c(y)u^2) dxdy + \int_{\Omega} k(y) u_x^2 dxdy$$

and

$$< Lu, u_x > = \int_{\Omega} a(y) u_x^2 dxdy.$$

From these relations we obtain (4) using the Cauchy inequality with "ϵ"and taking into account the condition i). Lemma 1 is proved.

We denote by l_n the closure of the operator

$$l_n u(y) = -u'' + (n^2 k(y) + ina(y) + c(y))u, n = 0, \pm 1, \pm 2, ...$$

defined on $C_0^\infty[0,1]$, where $C_0^\infty[0,1]$ is a set of infinitely differentiable functions satisfying the conditions (3).

Lemma 2. The estimates

$$\|l_n u\|_{L_2(0,1)} \geq c_1(\|u'\|_{L_2(0,1)} + \|u\|_{L_2(0,1)}), \tag{5}$$

$$\|l_n u\|_{L_2(0,1)} \geq c_2 \|u\|_{C[0,1]}, \tag{6}$$

hold for all $u(y) \in D(l_n)$, where c_1, c_2 are constants.

Proof. Let us compose quadratic form $(l_n u, u)$, $u \in C_0^\infty[0,1]$. Integrating by parts we obtain

$$|(l_n u, u)| = \left| \int_0^1 (l_n u)\bar{u} dy \right| = \left| \int_0^1 (u'^2 + (n^2 k(y) + ina(y) + c(y))|u|^2) dy \right|.$$

Hence, using the inequality $|\alpha + i\beta| \geq max(|\alpha|, |\beta|)$ $(\alpha, \beta \in R)$, the inequality Schwartz and the Cauchy inequality with "$\epsilon > 0$"we obtain

$$\|l_n u\|^2_{L_2(0,1)} \geq n^2 \delta^2 \|u\|^2_{L_2(0,1)},$$

$$c\|l_n u\|^2_{L_2(0,1)} \geq c_3 \int_0^1 (|u'|^2 + c(y)|u|^2) dy + \int_0^1 n^2 k(y)|u|^2) dy. \tag{7}$$

From (7) taking into account $k(y) \geq 0$ we have

$$\|l_n u\|^2_{L_2(0,1)} \geq c_4(\|u\|^2_{L_2(0,1)} + \|u'\|^2_{L_2(0,1)}) \geq c_1 \|u\|^2_{W_2^1(0,1)}.$$

Since the embedding operator of the Sobolev space $W_2^1(0,1)$ to $[0,1]$ is bounded it follows that

$$\|l_n u\|_{L_2(0,1)} \geq c_2(\|u\|_{C[0,1]}$$

which is true for all $u \in D(L)$. Lemma 2 is proved.

Lemma 3. The operator l_n is continuously invertible.

Proof. Taking into account (5) it is enough if we show the density of $D(l_n)$ in $L_2(\Omega)$. Assume the contrary. Let the set $D(l_n)$ is not density in $L_2(0,1)$. Then there exists nonzero element $w \in L_2(0,1)$ such that $(l_n u, w) = 0$ for $u \in D(l_n)$. Hence since the set $D(l_n)$ is not density in $L_2(0,1)$ we obtain that w is a solution to $l_n^* w = -w'' + (n^2 k(y) + c(y))w = 0$. From this equality it follows that $w'' \in L_2(0,1)$ by virtue of the continuous coefficients on [0,1]. Now we show that $w(y)$ satisfies the condition $w(0) = w(1) = 0$. Integrating by parts we obtain

$$0 = (u, l_n^* w) = (l_n u, w) - u'(1)\bar{w}(1) + u'(0)\bar{w}(0)$$

for all $u \in D(l_n)$. Last equality holds if $w(0) = w(1) = 0$. Therefore $w \in D(l_n)$. Then, we obtain

$$\|l_n w\|_{L_2(0,1)} \geq c\|w\|_{L_2(0,1)}$$

same as (5). It is shown that $w = 0$. The resulting contradiction proves the lemma 3.

Lemma 4. The following estimate holds for l_n^{-1}

$$\|l_n\|_{L_2(0,1) \to L_2(0,1)} \geq \frac{1}{n\delta_0}, n = \pm 1, \pm 2, ...$$

Proof. Taking into account the condition i) we have for any function $u \in C_0^\infty[0,1]$

$$|(l_n u, u)| \geq \left| \int_0^1 ina(y)|u|^2 dy \right| \geq |n|\delta_0 \|u\|^2_{L_2(0,1)}.$$

Hence, using the Cauchy inequality we obtain

$$\|l_n u\|_{L_2(0,1)} \geq |n|\delta_0 \|u\|_{L_2(0,1)}.$$

From the last estimates it follows Lemma 4.

3 Proofs of the main theorems

Proof of Theorem 1. The existence and continuity of l_n^{-1} follows from Lemma 3. Let $u_n(y) = (l_n^{-1} f_n)(y)$. By direct verification, we make sure that the function

$$u_k(x,y) = \sum_{n=-k}^{k} u_n(y) e^{inx} = \sum_{n=-k}^{k} (l_n^{-1} f_n)(y) e^{inx}$$

is a solution to (1) with the right side

$$f_k(x,y) = \sum_{n=-k}^{k} f_n(y) e^{inx}$$

which satisfies the condition (2)–(3). Moreover the following equality

$$\|u_k(x,y)\|^2_{L_2(-\pi,\pi)} = 2\pi \sum_{n=-k}^{k} |u_n(y)|^2$$

holds. Then from the estimate (5) it follows that

$$\sup_{y\in[0,1]} \|u_k(x,y)\|^2_{L_2(-\pi,\pi)} = 2\pi \sum_{n=-k}^{k} \sup_{y\in[0,1]} |u_n(y)|^2 \leq$$

$$\leq c_1 2\pi \sum_{n=-k}^{k} \|l_n u\|^2_{L_2(0,1)} \leq c_2 2\pi \sum_{n=-k}^{k} \|f_n(y)\|^2_{L_2(0,1)} = c\|f_k(x,y)\|_2^2. \tag{8}$$

From Lemma 4 we have

$$\|\frac{\partial u_k(x,y)}{\partial x}\|_2^2 = \|\frac{\partial}{\partial x} \sum_{n=-k}^{k} (l_n^{-1} f_n)(y) e^{inx})\|_2^2 = \|in \sum_{n=-k}^{k} (l_n^{-1} f_n)(y) e^{inx}\|_2^2 \leq$$

$$\leq \sum_{n=-k}^{k} |n|^2 \|l_n^{-1}\|^2_{L_2(0,1)\to L_2(0,1)} \|f_n\|^2_{L_2(0,1)} \leq \frac{1}{\delta_0^2} \sum_{n=-k}^{k} \|f_n\|^2_{L_2(0,1)} = \frac{1}{\delta_0^2} \|f_k(x,y)\|_2^2. \tag{9}$$

Similarly, using estimates (5), (6) we obtain

$$\|\frac{\partial u_k(x,y)}{\partial y}\|_2^2 + \|u_k\|_2^2 = \|\frac{\partial}{\partial y} \sum_{n=-k}^{k} (l_n^{-1} f_n)(y) e^{inx})\|_2^2 + \| \sum_{n=-k}^{k} (l_n^{-1} f_n)(y) e^{inx})\|_2^2 \leq$$

$$\leq \sum_{n=-k}^{k} \|f_n\|_2^2 + \sum_{n=-k}^{k} \|f_n\|_2^2 \leq= 2\|f_k(x,y)\|_2^2. \tag{10}$$

It is known that a set of functions

$$f_k(x,y) = \sum_{n=-k}^{k} f_n(y)e^{inx} \ \ (k = 1, 2, ...)$$

is dense in $L_2(\Omega)$. Therefore, we can assume that $\|f_k(x,y) - f(x,y)\|_2 \to 0$ as $k \to \infty$. Then the sequence $\{f_k\}_{k=1}^{\infty}$ is fundamental, and by virtue of estimates (8)–(10)

$$\|u_k(x,y) - u_m(x,y)\|_{C([0,1],L_2)} + \|u_k(x,y) - u_m(x,y)\|_{2,1,\Omega} \leq$$

$$\leq c_6\|f_k(x,y) - f_m(x,y)\|_2 \to 0$$

as $k, m \to \infty$. Hence, since the spaces $C([0,1], L_2(-\pi,\pi))$ and $W_2^1(\Omega)$ are complete, it follows that the sequence $\{u_n(x,y)\}_{k=-\infty}^{\infty}$ has limit $u(x,y)$, for which, by virtue of (8)–(10), the estimate

$$\|u\|_{C([0,1],L_2)} + \|u\|_{2,1,\Omega} \leq c\|f\|_2$$

holds. Theorem 1 is proved.

Let us introduce the sets

$$M = \{u \in L_2(\Omega) : \|Lu\|_2 + \|u\|_2 \leq 1\},$$

$$\widetilde{M}_{c_1} = \{u \in C([0,1], L_2(-\pi,\pi)) : (\|u_x\|_2^2 + \|u_y\|_2^2 + \|u\|_2^2)^{\frac{1}{2}} \leq c_1\},$$

$$\dot{M}_{c_1^{-1}} = \{u \in L_2(\Omega); (\|u_{xx}\|_2^2 + \|u_{yy}\|_2^2 + \|u_x\|_2^2 + \|u_y\|_2^2 + \|u\|_2^2)^{\frac{1}{2}} \leq c_1^{-1}\},$$

where $c_1 > 0$ и $c_1^{-1} > 1$.

The following lemma holds

Lemma 5. Let condition i) be satisfied. Then for some constant $c_1 > 1$ the inclusions

$$\dot{M}_{c_1^{-1}} \subseteq M \subseteq \widetilde{M}_{c_1}$$

hold.

Proof. Let $u(x,y) \in \dot{M}_{c_1^{-1}}$. Then, taking into account condition i), we obtain

$$\|Lu\|_2^2 + \|u\|_2^2 \leq c_2(\|u_{xx}\|_2^2 + \|u_{yy}\|_2^2 + \|u_x\|_2^2 + \|u_y\|_2^2 + \|u\|_2^2)^{\frac{1}{2}} \leq c_1^{-1}c_2,$$

where $c_2 = \max_{y\in[0,1]} \{|k(y)|, |a(y)|, |c(y)|\}$.

Hence, assuming $c_2 = c_1$, we have $\dot{M}_{c_1^{-1}} \subseteq M$.

Let $u \in M$. Then it follows from Theorem 1 that

$$(\|u_x\|_2^2 + \|u_y\|_2^2 + \|u\|_2^2)^{\frac{1}{2}} \leq C(\|Lu\|_2^2 + \|u\|_2^2) \leq C,$$

i.e. $M \subseteq \widetilde{M}_C$. By choosing a constant c_1 such that $c_1 \geq c$ we obtain the assertion of the lemma. Lemma 5 is proved.

Lemma 6. Let condition i) be satisfied. Then the estimates

$$c^{-1}\dot{d}_k \leq d_k \leq c\widetilde{d}_k, \ \ k = 1, 2, ...$$

hold, where $c > 0$ is an any constant, $\widetilde{d}_k, d_k, \dot{d}_k$ are the k-widths of the sets respectively $\widetilde{M}_c, M, \dot{M}_{c^{-1}}$.

The proof of this lemma it follows from Lemma 5 and the properties of the widths.

Let us introduce the functions

$$N(\lambda) = \sum_{d_k > \lambda} 1, \widetilde{N}(\lambda) = \sum_{\widetilde{d}_k > \lambda} 1, \dot{N}(\lambda) = \sum_{\dot{d}_k > \lambda} 1,$$

equal respectively to the number of widths $d_k(M)$, $\widetilde{d}_k$ and $\dot{d}_k$ are greater than $\lambda > 0$. Estimates (8) easily imply the inequalities

$$\dot{N}(c\lambda) \leq N(\lambda) \leq \widetilde{N}(c^{-1}\lambda)$$

Proof of Theorem 2. It is known that the estimates

$$c_0^{-1}\lambda^{-2} \leq \widetilde{N}(\lambda) \leq c_0\lambda^{-2}, \tag{11}$$

$$c_0^{-1}\lambda^{-1} \leq N(\lambda) \leq c_0\lambda^{-1}. \tag{12}$$

hold for the functions $\widetilde{N}(\lambda)$ and $N(\lambda)$. Let $\lambda = \widetilde{d}_k$. Then $\widetilde{N}(\widetilde{d}_k) = k$. Therefore from (11) and (12) it follows

$$C_0^{-1}\frac{1}{\sqrt{k}} \leq \widetilde{d}_k \leq C_0\frac{1}{\sqrt{k}}, \quad C_0^{-1}\frac{1}{k} \leq \dot{d}_k \leq C_0\frac{1}{k}.$$

respectively. Hence, taking into account estimates (7) and the equality $s_{k+1}(L^{-1}) = d_k$ we obtain

$$C_1\frac{1}{k} \leq s_k \leq C_2\frac{1}{k^{\frac{1}{2}}, \quad k = 1, 2, 3, ...}$$

Theorem 2 is proved.

Acknowledgments

This paper was supported by the grant AP08855802 of the Ministry of Education and Science of the Republic of Kazakhstan.

References

1 Смирнов М.М. Вырождающиеся эллиптические и гиперболические уравнения / М.М. Смирнов. — М.: Наука, 1966. — 292 с.

2 Келдыш М.В. О некоторых случаях вырождения уравнений эллиптического типа на границе области / М.В. Келдыш // ДАН СССР. — 1951. — Т. 77. — № 2. — С. 181–183.

3 Кальменов Т.Ш. О гладкости решений одного класса вырождающихся эллиптических уравнений / Т.Ш. Кальменов, М. Отелбаев // Дифференц. уравнения. — 1977. — Т. 13. — №7. — С. 1244–1255.

4 Олейник О.А. О гладкости решений вырождающихся эллиптических и параболических уравнений / О.А. Олейник // ДАН СССР. — 1965. — Т. 163. — №3. — С. 557–580.

5 Вишик М.И. Об одном классе вырождающихся эллиптических уравнений высших порядков /М.И. Вишик, В.В. Грушин // Мат. сб. — 1969. — Т. 79. — №1. — С. 3–36.

6 Вишик М.И. Уравнения 2-го порядка с неотрицательной характеристической формой / М.И. Вишик, В.В. Грушин // Мат. сб. — 1969. — Т. 80. — №4. — С. 445–491.

7 Muratbekov M.B. Estimates of spectrum for a class of mixed type operators / M.B. Muratbekov, M.M. Muratbekov // Differential equations. — 2007. — 43. — P. 143–146.

8 Muratbekov M.B. On existence of the resolvent and discreteness of the spectrum of a class of differential operators of hyperbolic type / M.B. Muratbekov, M.M. Muratbekov, A.M. Abylayeva // Electronic Journal of theory of differential equations. — 2013. — 63. — P. 1–10.

9 Muratbekov M. On discreteness of the spectrum of a class of mixed type singular differential operators / M.B. Muratbekov, T.Sh. Kalmenov, M.M Muratbekov // Complex Variables and Elliptic Equations. — 2015. — 60. — No.12. — P. 1752–1763.

10 Muratbekov M. On the existence of a resolvent and separability for a class of singular hyperbolic type differential operators on an unbounded domain / M. Muratbekov, M. Otelbaev // Eurasian mathematical Journal. — 2016. — 7. — No.1. — P. 50–67.

11 Gohberg I.C. Introduction to the Theory of Linear Nonselfadjoint Operators in Hilbert Space (Translations of mathematical monographs) / I.C. Gohberg, M.G. Krein. — American Mathematical society. — 1969. — Vol.18. — 378 p.

С.Ж. Игисинов, Л.Д. Жумалиева, А.О. Сулеймбекова, Е.Н. Баяндиев

М.Х. Дулати атындағы Тараз өңірлік университеті, Тараз, Қазақстан

Өзгешеленетін эллиптикалық операторлардың бір класы үшін сингулярлық сандарды (s-сандарды) бағалау

Мақалада түзудегі туынды дәрежеде өзгешеленуі бар өзгешеленетін эллиптикалық теңдеулердің бір класы зерттелген. Жұмыс барысында жүргізілген зерттеуге сүйене отырып, авторлар бөлікті-үзіліссіз коэффициенттері бар дифференциалды оператордың анықталу облысындағы функциялардың өзгеруіне байланысты туындайдын шенелген облыстағы шекаралық есептердің спектральдік сипаттамаларына әсер ететін әртүрлі қиындықтарды жеңудің әдістерін ұсынған осы жұмыста шешімнің болуы мен жалғыздығын қамтамасыз ететін теңдеудің кіші мүшелері коэффициенттері үшін қойылған шарттар көрсетілген. Шешімнің бар болуы, жалғыздығы мен тегістігі дәлелденді, сонымен қатар еркін дәрежелі өзгешеленетін эллиптикалық теңдеулердің бір класына қойылған жартылай периодты Дирихле есебінің сингулярлық сандары (s-сандардың) мен меншікті сандарының бағасы алынды.

Кілт сөздер: эллиптикалық оператор, шекаралық есеп, сингулярлық сандар, дәрежелік өзгешелену, шешім, жалғыздық.

С.Ж. Игисинов, Л.Д. Жумалиева, А.О. Сулеймбекова, Е.Н. Баяндиев

Таразский региональный университет имени М.Х. Дулати, Тараз, Казахстан

Оценки сингулярных чисел (s-чисел) для одного класса вырождающихся эллиптических операторов

В статье изучен один класс вырождающихся эллиптических уравнений с произвольным степенным вырождением на прямой. На основе исследований, проведенных в ходе работы, авторами предложены методы, позволяющие преодолеть различные трудности, связанные с поведением функций из области определения дифференциального оператора с кусочно-непрерывными коэффициентами в ограниченной области, которые влияют на спектральные характеристики краевых задач для вырождающихся эллиптических уравнений. Показаны условия, наложенные на коэффициенты при младших членах уравнения, обеспечивающие существование и единственность решения. Доказаны существование, единственность и гладкость решения, а также найдены оценки сингулярных чисел (s-чисел) и собственных чисел полупериодической задачи Дирихле для одного класса вырождающихся эллиптических уравнений с произвольным степенным вырождением.

Ключевые слова: эллиптический оператор, краевая задача, сингулярные числа, степенное вырождение, решение, единственность.

References

1 Smirnov, M.M. (1966). *Vyrozhdaiushchiesia ellipticheskie i giperbolicheskie uravneniia [Degenerate elliptic and hyperbolic equations]*. Moscow: Nauka [in Russian].

2 Keldysh, M.V. (1951). *O nekotorykh sluchaiakh vyrozhdeniia uravnenii ellipticheskogo tipa na granitse oblasti [On some cases of degeneration of equations of elliptic type on the boundary of the domain]. Doklady Akademii nauk SSSR — Proceedings of the Academy of Sciences USSR, 77*(2), 181–183 [in Russian].

3 Kalmenov, T.Sh., & Otelbaev, M. (1977). *O gladkosti reshenii odnogo klassa vyrozhdaiushchikhsia ellipticheskikh uravnenii [On the smoothness of solutions of a class of degenerate elliptic equations]. Differentsialnye uravneniia — Differential Equations, 13*(7), 1244–1255 [in Russian].

4 Oleinik, O.A. (1965). *O gladkosti reshenii vyrozhdaiushchikhsia ellipticheskikh i parabolicheskikh uravnenii [On the smoothness of solutions of degenerating elliptic and parabolic equations]. Doklady Akademii nauk SSSR — Proceedings of the Academy of Sciences USSR, 163*(3), 557–580 [in Russian].

5 Vishik, M.I., & Grushin, V.V. (1969). *Ob odnom klasse vyrozhdaiushchikhsia ellipticheskikh uravnenii vysshikh poriadkov [On a class of degenerate elliptic equations of higher orders]. Matematicheskii sbornik — Mathematical collection, 79*(1), 3–36 [in Russian].

6 Vishik, M.I., & Grushin, V.V. (1969). *Uravneniia 2-go poriadka s neotritsatelnoi kharakteristicheskoi formoi [Second-order equations with nonnegative characteristic form]. Matematicheskii sbornik — Mathematical collection, 80*(4), 445–491 [in Russian].

7 Muratbekov, M.B., & Muratbekov, M.M. (2007). Estimates of spectrum for a class of mixed type operators. *Differential equations, 43*, 143–146.

8 Muratbekov, M.B., Muratbekov, M.M., & Abylayeva, A.M. (2013). On existence of the resolvent and discreteness of the spectrum of a class of differential operators of hyperbolic type. *Electronic Journal of theory of differential equations, 63*, 1–10.

9 Muratbekov, M.B., Kalmenov, T.Sh., & Muratbekov, M.M. (2015). On discreteness of the spectrum of a class of mixed type singular differential operators. *Complex Variables and Elliptic Equations, 60*(12), 1752–1763.

10 Muratbekov, M., & Otelbaev, M. (2016). On the existence of a resolvent and separability for a class of singular hyperbolic type differential operators on an unbounded domain. *Eurasian mathematical journal, 7*(1), 50–67.

11 Gohberg, I.C., & Krein, M.G. (1969). *Introduction to the Theory of Linear Nonselfadjoint Operators in Hilbert Space (Translations of mathematical monographs)*. American Mathematical society.