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On the formulation and investigation of a boundary value problem for a third-order equation of a parabolic-hyperbolic type

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In the paper a novel boundary value problem for a third-order partial differential equation (PDE) of a parabolic-hyperbolic type, within a pentagonal domain consisting of both parabolic and hyperbolic regions was investigated. Such equations are pivotal in modeling complex physical phenomena across diverse fields such as physics, engineering, and finance due to their ability to encapsulate a wide range of dynamics through their mixed-type nature. By employing a constructive solution approach, we demonstrate the unique solvability of the posed problem. The significance of this study lies in its extension of the mathematical framework for understanding and solving higher-order mixed PDEs in complex geometrical domains, thus offering new avenues for theoretical and applied research in mathematical physics and related disciplines.

Keywords: differential equations, parabolic-hyperbolic type, a third-order parabolic-hyperbolic type.

2020 Mathematics Subject Classification: 35K30, 35L20, 35R05, 35J25.

Introduction

The study of non-classical equations of mathematical physics refers to the investigation of partial differential equations (PDEs) that exhibit behaviors beyond the standard classifications of parabolic, hyperbolic, and elliptic equations. These equations are often referred to as non-classical or degenerate equations. At present, the study of non-classical equations of mathematical physics is being intensively developed — equations of mixed, composite and mixed-composite types. One of the main reasons is the emergence of applied applications of boundary value problems posed for equations of these types. Many problems in physics, technology, mechanics and other areas require the study of such equations.

First, they began to study second-order mixed equations of the elliptic-hyperbolic type. The Italian mathematician Tricomi began to study fundamental studies of equations of such types in the 1920s.

After that, we began to study many different problems for equations of these types. A review of theoretical and applied research is given in the works and books of A.V. Bitsadze, L. Bers, M.M. Smirnov, as well as, in Uzbekistan, in the books of M.S. Salokhitdinov, T.D. Juraev.

Research into equations of elliptic-parabolic, parabolic-hyperbolic types began in the 1950s and 1960s. In 1959, I.M. Gelfand [1] pointed out the need for joint consideration of equations in one part of the domain of parabolic, and the other part of hyperbolic types. He gives an example related to the movement of gas in a channel surrounded by a porous medium: in the channel, the movement of gas is described by the wave equation, and outside it — by the diffusion equation.

Then, in the 1970s and 1980s, they began to study various problems for equations of the third and higher orders of the parabolic-hyperbolic type. Such problems were studied mainly by T.D. Dzhuraev and his students (for example, see [2, 3]).

At present, the study of various boundary value problems for equations of the third and higher orders of the parabolic-hyperbolic type has been developed on a broad scale (for example, see [4–15]).

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1 Formulation of the problem

In this article, we consider one boundary value problem for a third-order parabolic-hyperbolic type equation of the form

$$\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}+c\right)(Lu)=0 \tag{1}$$

in the pentagonal region G of the plane xOy, where $G=G_1\cup G_2\cup G_3\cup J_1\cup J_2$,

$$Lu=\begin{cases} u_{xx}-u_y, & (x,y)\in D_1,\\ u_{xx}-u_{yy}, & (x,y)\in D_i,\quad i=2,3,\end{cases}$$

$c\in R$, and G_1 is a rectangle with vertices at points $A(0;0)$, $B(1;0)$, $B_0(1,1)$, $A_0(0,1)$; G_2 — triangle with vertices at points B, $C(0,-1)$, $D(-1,0)$; G_3 — rectangle with vertices at points A, D, $D_0(-1,1)$, A_0; J_1 — open segment with vertices at points B, D; J_2 — an open segment with vertices at points A, A_0.

The equation (1) is a special case of the equation $\left(a\frac{\partial}{\partial x}+b\frac{\partial}{\partial y}+c\right)(Lu)=0$ when $\gamma=\frac{b}{a}=-1$, that is, the angular coefficient of the characteristic of the operator $a\frac{\partial}{\partial x}+b\frac{\partial}{\partial y}$ is equal to $\gamma=-1$.

For the equation (1), the following problem is posed:

Problem 1. It is required to find the function $u(x,y)$ which is 1) continuous in $\overline{G}$ and in the domain of $G\backslash J_1\backslash J_2$ has continuous derivatives involved in the equation (1), and u_x and u_y are continuous in G up to part of the boundary of the domain G specified in the boundary conditions; 2) satisfies the equation (1) in the domain $G\backslash J_1\backslash J_2$; 3) satisfies the following boundary conditions:

$$u(1,y)=\varphi_1(y),\quad 0\le y\le 1, \tag{2}$$

$$u(-1,y)=\varphi_2(y),\quad 0\le y\le 1,$$

$$u_x(1,y)=\varphi_3(y),\quad 0\le y\le 1,$$

$$u|_{BC}=\psi_1(x),\quad 0\le x\le 1, \tag{3}$$

$$u|_{DF}=\psi_2(x),\quad -1\le x\le -\frac{1}{2}, \tag{4}$$

$$\left.\frac{\partial u}{\partial n}\right|_{BC}=\psi_3(x),\quad -1\le x\le 0; \tag{5}$$

4) satisfies the following gluing conditions on the lines of type changing:

$$u(x,+0)=u(x,-0)=T(x),\quad -1\le x\le 1, \tag{6}$$

$$u_y(x,+0)=u_y(x,-0)=N(x),\quad -1\le x\le 1, \tag{7}$$

$$u_{yy}(x,+0)=u_{yy}(x,-0)=M(x),\quad -1\le x\le 1, \tag{8}$$

$$u(+0,y)=u(-0,y)=\tau_3(y),\quad 0\le y\le 1, \tag{9}$$

$$u_x(+0,y)=u_x(-0,y)=\nu_3(y),\quad 0\le y\le 1, \tag{10}$$

$$u_{xx}(+0,y)=u_{xx}(-0,y)=\mu_3(y),\quad 0\le y\le 1, \tag{11}$$

where

$$T(x)=\begin{cases}\tau_1(x), & if\quad 0\le x\le 1,\\ \tau_2(x), & if\quad -1\le x\le 0;\end{cases}$$

$N(x)=\begin{cases}\nu_1(x), & if\quad 0\le x\le 1,\\ \nu_2(x), & if\quad -1\le x\le 0;\end{cases}$ $M(x)=\begin{cases}\mu_1(x), & if\quad 0<x<1,\\ \mu_2(x), & if\quad -1<x<0,\end{cases}$ $\varphi_i,\ \psi_i\ (1,2,3)$ are given sufficiently smooth functions, τ_i, ν_i, μ_i $(i=1,\,2,\,3)$ are unknown yet sufficiently smooth functions, n is an internal normal to the line $x-y=1$, and the point F has coordinates $F\left(-1/2,-1/2\right)$.

2 *Studying of the Problem*

Theorem 2.1. If $\varphi_1,\ \varphi_2 \in C^3[0,1],\ \varphi_3 \in C^2[0,1],\ \psi_1 \in C^3[0,1],\ \psi_2 \in C^3[-1,-1/2],\ \psi_3 \in C^2[0,1]$, and the matching conditions $\varphi_1(0) = \psi_1(1)$ fulfilled, $\psi_2(-1) = \varphi_2(0)$, then Problem 1 is uniquely solvable.

Proof. We will prove the theorem by constructing the solution. To do this, we will rewrite the equation (1) as

$$u_{1xx} - u_{1y} = \omega_1(x+y)\exp{(cy)}\,,\ \ (x,\, y) \in G_1, \tag{12}$$

$$u_{ixx} - u_{iyy} = \omega_i(x+y)\exp{(cy)}\,,\ \ (x,\, y) \in G_i,\ \ \ i = 2,\, 3, \tag{13}$$

where the notation $u(x,\, y) = u_i\,(x,\, y)\,,\ (x,\, y) \in G_i\ \ \left(i = \overline{1,\, 3}\right)$, functions $\omega_i\,(x+y)\,,\ \ i = \overline{1,\, 3}$ are unknown sufficiently smooth functions to be determined.

The study will be carried out first in the domain G_2. The solution of the equation (13) $(i = 2)$ satisfying the conditions (6), (7) can be represented as

$$\begin{aligned} &u_2(x,y) = \tfrac{1}{2}[T(x+y) + T(x-y)] + \\ &+ \tfrac{1}{2}\int_{x-y}^{x+y} N(t)dt - \tfrac{1}{2}\int_0^y \exp(c\eta)d\eta \int_{x-y+\eta}^{x+y-\eta} \omega_2(\xi+\eta)d\xi. \end{aligned} \tag{14}$$

Substituting (14) into the condition (5), after some calculations, we find

$$\omega_2(x+y) = -\sqrt{2}\psi_3'(\frac{x+y+1}{2})\exp\left[-\frac{c}{2}(x+y-1)\right],\ -1 \le x+y \le 1.$$

Further, substituting (14) into the condition (3), after some simplifications, we obtain the first relation between the unknown functions $T(x)$ and $N(x)$ on the line J_1 of type changing:

$$T'(x) + N(x) = \alpha_1(x),\ -1 \le x \le 1, \tag{15}$$

where $\alpha_1(x) = \psi_1'(\frac{x+1}{2}) + \omega_2(x)\int_0^{(x-1)/2} e^{c\eta}d\eta$.

If we take into account the representation of the function $T(x)$, then for $-1 \le x \le 0$ the equation (15) has the form

$$\tau_2'(x) + \nu_2(x) = \alpha_1(x),\ \ \ -1 \le x \le 0. \tag{16}$$

Now, substituting (14) into (4), after some transformations, we have

$$\tau_2'(x) - \nu_2(x) = \delta_1(x),\ \ \ -1 \le x \le 0, \tag{17}$$

where $\delta_1(x) = \psi_2'(\frac{x-1}{2}) + \int_0^{-(x+1)/2} e^{c\eta}\omega_2(x+2\eta)d\eta$.

From (16) and (17) we obtain

$$\tau_2(x) = \frac{1}{2}\int_{-1}^{x}\left[\alpha_1(t) + \delta_1(t)\right]dt + \psi_2(-1),\ \ \ \nu_2(x) = \frac{1}{2}\left[\alpha_1(x) - \delta_1(x)\right].$$

For $0 \le x \le 1$ from (15), we have the first relation between the unknown functions $\tau_1(x)$ and $\nu_1(x)$ on the line J_1 of type changing in the following form:

$$\tau_1'(x) + \nu_1(x) = \alpha_1(x),\ \ \ 0 \le x \le 1. \tag{18}$$

Passing to the limit at $y \to 0$ in the equation (13) $(i = 2)$, we will find the second relation between the unknown functions $\tau_1(x)$ and $\mu_1(x)$ on J_1:

$$\tau_1''(x) - \mu_1(x) = \omega_2(x),\ \ \ 0 \le x \le 1. \tag{19}$$

The equation (1) in the domain G_1 can be rewritten as

$$u_{1xxx} - u_{1xy} - u_{1xxy} + u_{1yy} + cu_{1xx} - cu_{1y} = 0.$$

Passing to the limit at $y \to 0$ in the last equation, we obtain the third relation between the unknown functions $\tau_1(x)$, $\nu_1(x)$ and $\mu_1(x)$ on the line of type changing J_1:

$$\tau_1'''(x) - \nu_1'(x) - \nu_1''(x) + \mu_1(x) + c\tau_1''(x) - c\nu_1(x) = 0, \quad 0 \leq x \leq 1. \tag{20}$$

Eliminating the functions $\nu_1(x)$ and $\mu_1(x)$ from the equations (18), (19) and (20) and integrating the resulting equation from 0 to x, we arrive at the equation

$$\tau_1''(x) + (1 + \frac{c}{2})\tau_1'(x) + \frac{c}{2}\tau_1(x) = \alpha_2(x) + k_1, \quad 0 \leq x \leq 1, \tag{21}$$

where $\alpha_2(x) = \frac{1}{2}\alpha_1'(x) + \frac{1}{2}\alpha_1(x) + \frac{1}{2}\int_0^x \left[\omega_2(t) + c\alpha_1(t)\right] dt$, and k_1 is still unknown constant.

When solving the equation (21), we consider the following cases:

1°. $c \neq 2, \ c \neq 0$;

2°. $c = 2$;

3°. $c = 0$.

In the case 1° it is easy to see that the solution of the equation (21) satisflying the conditions

$$\begin{array}{l} \tau_1(0) = \frac{1}{2}\int_{-1}^0 \left[\alpha_1(t) + \delta_1(t)\right] dt + \psi_2(-1), \\ \tau_1'(0) = \frac{1}{2}\left[\alpha_1(0) + \delta_1(0)\right], \\ \tau_1(1) = \varphi_1(0) \end{array} \tag{22}$$

has the form

$$\tau_1(x) = \frac{2}{2-c}\int_0^x \left[e^{\frac{c}{2}(t-x)} - e^{t-x}\right]\alpha_2(t)dt +$$

$$+\frac{2k_1}{2-c}\left[\frac{2}{c}(1 - e^{-\frac{c}{2}x}) - (1 - e^{-x})\right] + k_2e^{-x} + k_3e^{-\frac{c}{2}x},$$

where $k_3 = \dfrac{1}{2-c}\left\{\displaystyle\int_{-1}^0 \left[\alpha_1(t) + \delta_1(t)\right] dt + 2\psi_2(-1) + \alpha_1(0) + \delta_1(0)\right\}$,

$$k_2 = \frac{1}{c-2}\left\{\frac{c}{2}\int_{-1}^0 \left[\alpha_1(t) + \delta_1(t)\right] dt + c\psi_2(-1) + \alpha_1(0) + \delta_1(0)\right\},$$

$$k_1 = \left[\frac{c}{2}(1 - e^{-\frac{c}{2}}) - (1 - e^{-1})\right]^{-1}\left\{\frac{2-c}{2}\left[\varphi_1(0) - \right.\right.$$

$$\left.\left. -k_2e^{-1} + k_3e^{-\frac{c}{2}}\right] - \int_0^1 \left[e^{\frac{c}{2}(t-1)} - e^{t-1}\right]\alpha_2(t)dt\right\}.$$

Also, in the case 2°, one can show that the solution of solving the equation (21) satisfying the conditions (22), has the following form

$$\tau_1(x) = \int_0^x (x-t)e^{t-x}\alpha_2(t)dt + k_1\left[1 - (x+1)e^{-x}\right] + (k_2 + k_3x)e^{-x},$$

where $k_2 = \frac{1}{2}\int_{-1}^0 \left[\alpha_1(t) + \delta_1(t)\right] dt + \psi_2(-1)$, $k_3 = k_2 + \frac{1}{2}\left[\alpha_1(0) + \delta_1(0)\right]$,

$$k_1 = \frac{1}{e-2}\left[\varphi_1(0)e - k_2 - k_3 - \int_0^1 (1-t)e^t\alpha_2(t)dt\right].$$

Moreover, for the case 3°, the solution of (21) satisflying (22) defined by

$$\tau_1(x)=\int_0^x e^{t-x}\alpha_3(t)dt+k_1(x-1+e^{-x})+k_2(1-e^{-x})+k_3e^{-x},$$

where $\alpha_3(x)=\int_0^x\alpha_2(t)dt$, $k_3=\frac{1}{2}\int_{-1}^0\left[\alpha_1(t)+\delta_1(t)\right]dt+\psi_2(-1)$, $k_2=k_3+\frac{1}{2}\left[\alpha_1(0)+\delta_1(0)\right]$,

$$k_1=\varphi_1(0)e-k_2(e-1)-k_3-\int_0^1 e^t\alpha_3(t)dt.$$

Now, we consider the G_3. Let us introduce the notation:

$$\omega_3(x+y)=\begin{cases}\omega_{31}(x+y), & -1\le x+y\le 0,\\ \omega_{32}(x+y), & 0\le x+y\le 1\,.\end{cases}$$

Then, passing to the limit at $y\to 0$, in the equations (13) $(i=2)$ and (13) $(i=3)$ due to (6)–(8), we get

$$\omega_{31}(x)=\omega_2(x),\quad -1\le x\le 0.$$

Now, we consider the following problem:

$$\begin{cases}u_{3xx}-u_{3yy}=\omega_3(x+y)e^{cy},\\ u_3(x,\,0)=\tau_2(x),\quad u_{3y}(x,\,0)=\nu_2(x),\quad -1\le x\le 0,\\ u_3(-1,\,y)=\varphi_2(y),\quad u_3(0,\ ,y)=\tau_3(y),\quad 0\le y\le 1.\end{cases}$$

The solution to this problem will be sought in the form

$$u_3(x,\,y)=u_{31}(x,\,y)+u_{32}(x,\,y)+u_{33}(x,y),\tag{23}$$

where $u_{31}(x,\,y)$ is the solution of the problem

$$\begin{cases}u_{31xx}-u_{31yy}=0,\\ u_{31}(x,\,0)=\tau_2(x),\quad u_{31y}(x,\,0)=0,\ -1\le x\le 0,\\ u_{31}(-1,\,y)=\varphi_2(y),\quad u_{31}(0,\,y)=\tau_3(y),\ 0\le y\le 1;\end{cases}\tag{24}$$

$u_{32}(x,\,y)$ is the solution of the problem

$$\begin{cases}u_{32xx}-u_{32yy}=0,\\ u_{32}(x,\,0)=0,\ u_{32y}(x,\,0)=\nu_2(x),\ -1\le x\le 0,\\ u_{32}(-1,\,y)=0,\ u_{32}(0,\,y)=0,\ 0\le y\le 1;\end{cases}\tag{25}$$

$u_{33}(x,\,y)$ is the solution of the problem

$$\begin{cases}u_{33xx}-u_{33yy}=\omega_3(x+y)e^{cy},\\ u_{33}(x,\,0)=0,\ u_{33y}(x,\,0)=0,\ -1\le x\le 0,\\ u_{33}(-1,\,y)=0,\ u_{33}(0,\,y)=0,\ 0\le y\le 1.\end{cases}\tag{26}$$

Using the continuation method, we find solutions to the problems (24)–(26). The solutions can be represented as follows

$$u_{31}(x,\,y)=\frac{1}{2}\left[T_2(x+y)+T_2(x-y)\right],\tag{27}$$

where $T_2(x) = \begin{cases} 2\varphi_2(-1-x) - \tau_2(-2-x), & -2 \le x \le -1, \\ \tau_2(x), & -1 \le x \le 0, \\ 2\tau_3(x) - \tau_2(-x), & 0 \le x \le 1; \end{cases}$

$$u_{32}(x,\, y) = \frac{1}{2}\int_{x-y}^{x+y} N_2(t)dt, \tag{28}$$

where $N_2(x) = \begin{cases} -\nu_2(-2-x), & -2 \le x \le -1, \\ \nu_2(x), & -1 \le x \le 0, \\ -\nu_2(-x), & 0 \le x \le 1; \end{cases}$

$$u_{33}(x,\, y) = -\frac{1}{2}\int_0^y e^{c\eta}d\eta \int_{x-y+\eta}^{x+y-\eta} \Omega_3(\xi+\eta)d\xi. \tag{29}$$

Using the condition $u_{33}(-1,\, y) = 0$, after some transformations, from (29), we obtain

$$\frac{1}{2}\int_{-1-y}^{y-1} e^{\frac{c}{2}(y+1+z)}\Omega_3(z)dz = -\omega_{31}(y-1)\int_0^y e^{c\eta}d\eta. \tag{30}$$

Hence, by differentiating (30), we find

$$\begin{aligned} \Omega_{31}(-1-y) = c\omega_{31}(y-1)\textstyle\int_0^y e^{c\eta}d\eta - \\ -2\omega_{31}'(y-1)\textstyle\int_0^y e^{c\eta}d\eta - 3\omega_{31}(y-1)e^{cy}. \end{aligned}$$

Now, using condition $u_{33}(0,\, y) = 0$, after some transformations, from (30), we have

$$\omega_{32}(y)\int_0^y e^{c\eta}d\eta = -\int_{-y}^{y} e^{\frac{c}{2}(y+z)}\Omega_3(z)dz.$$

Substituting (27), (28) and (29) into (23), we get

$$\begin{aligned} u_3(x,\, y) = \tfrac{1}{2}\left[T_2(x+y) + T_2(x-y)\right] + \\ +\tfrac{1}{2}\int_{x-y}^{x+y} N_2(t)dt - \tfrac{1}{2}\int_0^y e^{c\eta}d\eta \int_{x-y+\eta}^{x+y-\eta} \Omega_3(\xi+\eta)d\xi. \end{aligned}$$

Differentiating this solution with respect to x and tending x to zero, and also taking into account the condition (10), after some transformations, we have the following relation:

$$\nu_3(y) = \tau_3'(y) + \tau_2'(-y) - \nu_2(-y) + \frac{1}{2}\int_{-y}^{y} e^{\frac{c}{2}(y+z)}\Omega_3(z)dz. \tag{31}$$

Passing to the limit at $x \to 0$ in the equations (12) and (13) ($i = 2$) and taking into account (9) and (11), we obtain

$$\mu_3(y) - \tau_3'(y) = \omega_{11}(y)\exp(-cy),\;\; \mu_3(y) - \tau_3''(y) = \omega_{32}(y)e^{cy}.$$

Eliminating the function $\mu_3(y)$ from these equations, we find

$$\omega_{32}(y) = \omega_{11}(y) - \left[\tau_3''(y) - \tau_3'(y)\right]e^{-cy}. \tag{32}$$

Now, passing to the limit at $y \to 0$ in the equation (13) and taking (6) and (7) into account after replacing x with $x+y$, we obtain

$$\omega_{11}(x+y) = \tau_1''(x+y) - \nu_1(x+y),\;\; 0 \le x+y \le 1, \tag{33}$$

where $\omega_1(x+y) = \begin{cases} \omega_{11}(x+y), & 0 \le x+y \le 1, \\ \omega_{12}(x+y), & 1 \le x+y \le 2. \end{cases}$

Finally, by substituting (33) into (32), we arrive at the relation

$$\omega_{32}(y) = \tau_1''(y) - \nu_1(y) - \left[\tau_3''(y) - \tau_3'(y)\right] e^{-cy}, \tag{34}$$

and substituting (34) into (31), after some calculations, we get

$$\nu_3(y) = \frac{1}{2}\tau_3'(y) - \frac{c-2}{4}\int_0^y e^{\frac{c}{2}(y-z)}\tau_3'(z)dz + \beta_1(y),$$

where

$$\beta_1(y) = \tau_2'(-y) - \nu_2(-y) + \frac{1}{2}\nu_1(0)e^{\frac{c}{2}y} +$$

$$+\frac{1}{2}\int_{-y}^0 e^{\frac{c}{2}(y+z)}\omega_{31}(z)dz + \frac{1}{2}\int_0^y e^{\frac{c}{2}(y+z)}\left[\tau_1''(z) - \nu_1(z)\right]dz.$$

Now, we consider the domain G_1. The solution of the equation (12) satisfying the conditions (2), (6), (9) has the form

$$u_1(x,\, y) = \left[\int_0^y \tau_3(\eta)G_\xi(x,\, y;\ ,0,\, \eta)d\eta - \right.$$

$$-\int_0^y \varphi_1(\eta)G_\xi(x,\, y;\, 1,\, \eta)d\eta + \int_0^1 \tau_1(\xi)G(x,\, y;\, \xi,\, 0)d\xi -$$

$$-\int_0^y e^{c\eta}d\eta\int_0^{1-\eta}\omega_{11}(\xi+\eta)G(x,\, y;\, \xi,\, \eta)d\xi -$$

$$-\int_0^y e^{c\eta}d\eta\int_{1-\eta}^1 \omega_{12}(\xi+\eta)G(x,\, y;\, \xi,\, \eta)d\xi.$$

Differentiating this solution by x and passing x to zero and to one, we obtain the following relations

$$\begin{aligned}
\nu_3(y) = &-\int_0^y \tau_3'(\eta)N(0,y;0,\eta)d\eta + \\
&+\int_0^y \varphi_1'(\eta)N(0,y;1,\eta)d\eta + \int_0^1 \tau_1'(\xi)N(0,y;\xi,0)d\xi + \\
&+\int_0^y e^{c\eta}d\eta\int_0^{1-\eta}\left[\tau_1''(\xi+\eta) - \nu_1(\xi+\eta)\right]N_\xi(0,y;\xi,\eta)d\xi + \\
&+\int_0^y e^{c\eta}d\eta\int_{1-\eta}^1 \omega_{12}(\xi+\eta)N_\xi(0,\, y;\, \xi,\, \eta)d\xi,
\end{aligned}$$

$$\begin{aligned}
\varphi_3(y) = &-\int_0^y \tau_3'(\eta)N(1,y;0,\eta)d\eta + \\
&+\int_0^y \varphi_1'(\eta)N(1,y;1,\,\eta)d\eta + \int_0^1 \tau_1'(\xi)N(1,\, y;\, \xi,\, 0)d\xi + \\
&+\int_0^y e^{c\eta}d\eta\int_0^{1-\eta}\left[\tau_1''(\xi+\eta) - \nu_1(\xi+\eta)\right]N_\xi(1,y;\xi,\eta)d\xi + \\
&+\int_0^y e^{c\eta}d\eta\int_{1-\eta}^1 \omega_{12}(\xi+\eta)N_\xi(1,y;\xi,\eta)d\xi.
\end{aligned} \tag{35}$$

Here and at the top of the functions $G(x,\, y;\, \xi,\, \eta)$ and $N(x,\, y;\, \xi,\, \eta)$ have the form:

$$\left.\begin{matrix} G(x,y;\xi,\eta) \\ N(x,y;\xi,\eta) \end{matrix}\right\} = \frac{1}{2\sqrt{\pi(y-\eta)}}\sum_{n=-\infty}^{+\infty}\left\{\exp\left[-\frac{(x-\xi-2n)^2}{4(y-\eta)}\right] \mp \exp\left[-\frac{(x+\xi-2n)^2}{4(y-\eta)}\right]\right\}.$$

They are Green's functions of the first and second boundary value problems for the heat equation. Substituting (32) into (33), after some transformations, we have the equation

$$\tau_3'(y) + \int_0^y \tau_3'(\eta) K_1(y,\eta) d\eta + \int_0^y K_2(y,\eta)\omega_{12}(1+\eta) d\eta = g_1(y). \tag{36}$$

And differentiating the equation (35) after some calculations, we obtain the Volterra integral equation of the second kind with respect to $\omega_{12}(1+y)$:

$$\omega_{12}(1+y) + \int_0^y K_3(y,\eta)\omega_{12}(1+\eta) d\eta + \int_0^y \tau_3'(\eta) K_4(y,\eta) d\eta = g_2(y), \tag{37}$$

where $K_1(y,\eta)$, $K_2(y,\eta)$, $K_3(y,\eta)$, $K_4(y,\eta)$, $g_1(y)$, $g_2(y)$ are known functions, and $K_1(y,\eta)$, $K_3(y,\eta)$ have a weak singularity $(\frac{1}{2})$, and $K_2(y,\eta)$, $K_4(y,\eta)$, $g_1(y)$, $g_2(y)$ are continuous functions.

Solving the system of equations (36), (37), we find the functions $\tau_3'(y)$, $\omega_{12}(1+y)$ and thus, the functions $\nu_3(y)$, $\omega_{32}(y)$, $u_1(x,\, y)$, $u_3(x,\, y)$.

Remark 1. The case when $-1 < \gamma < 0$, the problem is investigated by dividing the domain G_1 into n parts whose heights of the first $n-1$ domains are equal to $-\frac{b}{a}$, and the last – no more than $-\frac{b}{a}$. The problem is solved in each domain sequentially, similar to the case of $\gamma = -1$.

Remark 2. In [4, 11], a number of boundary value problems for more general equations of the third and fourth orders of parabolic-hyperbolic type in a domain with a single line of type change were considered.

Conclusion

This work presents the formulation and comprehensive analysis of a boundary value problem for a third-order parabolic-hyperbolic PDE within a geometrically intricate pentagonal domain. Through the development of a constructive method for the equation's solution, we have established its unique solvability. Our findings enrich the theoretical underpinnings of mixed-type equations and extend the toolkit for addressing boundary value problems in domains with complex geometries. This research not only advances our understanding of parabolic-hyperbolic equations of third order but also has potential implications for their application in modeling multifaceted physical systems and phenomena. Future studies may explore the application of these findings in practical scenarios and the investigation of similar problems in higher-dimensional spaces or with more complex boundary conditions.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Параболалық-гиперболалық типтегі үшінші ретті теңдеу үшін шекаралық шама есебін тұжырымдау және зерттеу туралы

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Мақалада параболалық және гиперболалық аймақтардан тұратын бесбұрышты аймақтағы параболалық-гиперболалық типтегі үшінші ретті ішінара туындылардағы теңдеудің жаңа шекті есебі зерттелген. Мұндай теңдеулер физика, инженерия және қаржы сияқты әртүрлі салалардағы күрделі физикалық құбылыстарды модельдеуде шешуші рөл атқарады, олардың аралас типтегі табиғатына орай динамиканың кең ауқымын инкапсуляциялау қабілетіне байланысты. Шешудің конструктивті тәсілін қолдана отырып, біз қойылған есептің біржақты шешілуін көрсетеміз. Бұл зерттеудің маңыздылығы математикалық физика мен сабақтас пәндердегі теориялық және қолданбалы зерттеулерге жаңа мүмкіндіктер ашатын күрделі геометриялық салалардағы жоғары ретті жартылай дифференциалдық теңдеулерді түсіну және шешу үшін математикалық негізді кеңейту болып табылады.

Кілт сөздер: параболалық-гиперболалық типтегі дифференциалдық теңдеулер, үшінші ретті параболалық-гиперболалық тип.

О постановке и исследовании краевой задачи для уравнения третьего порядка параболо-гиперболического типа

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В статье исследована новая краевая задача для уравнения в частных производных третьего порядка параболо-гиперболического типа в пятиугольной области, состоящей как из параболических, так и из гиперболических областей. Такие уравнения играют решающую роль в моделировании сложных физических явлений в различных областях, таких как физика, инженерия и финансы, благодаря их способности инкапсулировать широкий диапазон динамики из-за своей природы смешанного типа. Используя конструктивный подход к решению, мы демонстрируем однозначную разрешимость поставленной задачи. Значимость этого исследования заключается в расширении математической основы для понимания и решения смешанных уравнений в частных производных высшего порядка в сложных геометрических областях, что открывает новые возможности для теоретических и прикладных исследований в математической физике и смежных дисциплинах.

Ключевые слова: дифференциальные уравнения параболо-гиперболического типа, параболо-гиперболический тип третьего порядка.

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