https://doi.org/10.31489/2025M4/107-124 *Research article*

Analysis and classification of fixed points of operators on a simplex

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This paper investigates the dynamical behavior of Lotka–Volterra type operators defined on the four and five dimensional simplexes, focusing on their fixed points and structural representation through directed graphs (tournaments). For several classes of such operators, we derive algebraic and combinatorial conditions under which the configuration of fixed points exhibits transitive, cyclic, or homogeneous structures. Using methods from algebraic graph theory, Lyapunov stability theory, and Young's inequality, explicit criteria are established for the existence, uniqueness, and stability of interior and boundary fixed points. A detailed analysis is provided for the class of operators whose associated skew-symmetric matrices are in general position. The connection between the minors of these matrices and the orientation of arcs in the tournament is clarified, revealing how dynamical transitions correspond to changes in tournament type. Furthermore, we demonstrate that under certain parameter regimes, fixed points coincide with evolutionarily stable strategies (ESS) in replicator dynamics, thus bridging discrete population models and evolutionary game theory. The obtained results enrich the theory of quadratic stochastic and Lotka–Volterra operators, providing new insights into nonlinear mappings on simplexes, combinatorial dynamics, and applications to models of interacting populations.

Keywords: Lotka–Volterra mapping, simplex dynamics, fixed points, replicator dynamics, evolutionary stability, directed graphs, tournaments, cyclic structures, Lyapunov function, nonlinear systems.

2020 Mathematics Subject Classification: 37B25, 37C25, 37C27.

Introduction

A number of applied studies are devoted to the investigation of dynamical systems — both continuous and discrete — as well as systems involving fractional-order derivatives [1–3]. To this day, all three types of systems remain relevant; however, they differ in the methods of analysis and in the nature of the results obtained [4–6]. The application areas of such models are wide-ranging and include medicine (covering problems in epidemiology, oncology, and population genetics), ecology, economics, computer virology, and many others [7–9]. Building on these applications, we now turn to the theoretical foundations of a particular class of discrete dynamical systems — the so-called quadratic stochastic operators — which play a central role in many models, especially in population genetics and game dynamics.

Let us start by recalling the known facts that we will rely on in the article, as well as recalling the works of some authors on its topic. It is known that [10], a $(m-1)$-dimensional standard simplex in $\mathbb{R}^m$ is defined as the relation

$$S^{m-1} = \{x = (x_1, ..., x_m) : \ x_i \geq 0, \ \sum_{i=1}^{m} x_i = 1\} \subset \mathbb{R}^m.$$

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This research was funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (No. BR 27100483 "Development of predictive exploration technologies for identifying ore-prospective areas based on data analysis from the unified subsurface user platform "Minerals.gov.kz" using artificial intelligence and remote sensing methods").

Received: 26 July 2025; *Accepted:* 15 September 2025.

It is easy to verify that S^{m-1} is a convex and compact subset of $\mathbb{R}^m$.

A class of mappings defined on S^{m-1} known as *quadratic stochastic operators* was introduced by Bernstein [11] and further developed by R.N. Ganikhodzhaev in [12, 13]. Such mappings are defined by a set of coefficients $P_{ij,k}$ for $i, j, k = 1, \ldots, m$, satisfying the conditions

$$P_{ij,k} = P_{ji,k} \geq 0, \quad \sum_{k=1}^{m} P_{ij,k} = 1,$$

and act according to the equations

$$x'_k = (Vx)_k = \sum_{i,j=1}^{m} P_{ij,k} x_i x_j, \quad k = 1, \ldots, m.$$

This mapping was introduced by R.N. Ganikhodzhaev in [12].

Definition 1. A quadratic stochastic mapping is called a Lotka–Volterra mapping if the inheritance coefficients satisfy the condition $P_{ij,k} = 0$ for all $k \notin \{i, j\}$.

It is known (see [14]) that any Lotka–Volterra mapping defined on S^{m-1} can be represented as

$$x_k^{'} = x_k(1 + \sum_{i=1}^{m} a_{ki} x_i), \quad k = 1, \ldots, m, \tag{1}$$

where

$$a_{ki} = \begin{cases} 2P_{ik,k} - 1, & \text{if } i \neq k, \\ 0, & \text{if } i = k, \end{cases} \quad \text{with } |a_{ki}| \leq 1, \quad k, i = 1, \ldots, m. \tag{2}$$

Here, $A = (a_{ki})$ is a real skew-symmetric matrix, satisfying $A = -A^T$, where A^T denotes the transpose of A.

Definition 2. [15] A skew-symmetric matrix is called a matrix of general position if all of its principal minors of even order are nonzero.

Since $a_{ki} = -a_{ik}$, all off-diagonal entries are antisymmetric. In particular, $a_{ki} \neq 0$ for $i \neq k$ if and only if the corresponding $P_{ij,k} \neq \frac{1}{2}$.

It is known that each skew-symmetric matrix in general position can be associated with a complete-oriented graph (tournament) [15].

Let $A = (a_{ki})$ be a skew-symmetric matrix in general position associated with Lotka–Volterra mapping (1), where the coefficients satisfy conditions (2). We place m points on a plane and label them $1, 2, \ldots, m$. For each pair of distinct indices $i \neq k$, we draw a directed edge from vertex i to vertex k if $a_{ik} > 0$ (equivalently, $a_{ki} < 0$).

This construction defines a well-posed directed graph. We then call the constructed graph the tournament of dynamic system (1) with the skew-symmetric matrix $A = (a_{ki})$ and denote it by T_m.

A directed graph is called a tournament if, for every pair of distinct vertices i and k, exactly one of the edges (i, k) or (k, i) is present. A graph in which every two vertices are connected by an edge is called a complete graph. If each edge of a complete graph is assigned a direction, the resulting directed graph is a tournament [16–18].

Two tournaments are said to be isomorphic if there exists a bijection between their vertex sets that preserves the direction of all edges.

It is known that there are 12 pairwise non-isomorphic tournaments with 5 vertices [17].

A tournament is called *strong* if, for any two vertices, there exists a directed path from one to the other. Among the 12 tournaments with 5 vertices, 6 are strong [15].

A tournament is said to be *transitive* if it contains no strong subtournaments. Equivalently, a tournament is transitive if it does not contain any directed cycles. Among the tournaments with 5 vertices, exactly 1 is transitive, 6 are strong, and the remaining 5 are neither strong nor transitive.

Definition 3. [15] A tournament is homogeneous if every sub-tournament is either strong or transitive.

In this paper, we study the structure of the set of fixed points (referred to as the *card of fixed points*) and characterize the fixed points of strong and homogeneous tournaments.

Every face of the simplex S^{m-1} is invariant under the Lotka–Volterra mapping, and the restriction V to this face is also a Lotka-Volterra mapping [12–14].

In recent works [19–21] Lotka–Volterra mappings have been studied from the perspective of dynamical systems, population genetics, and game theory. A particularly fruitful approach is to analyze their fixed points and dynamical behavior via combinatorial structures such as tournaments and their geometric realizations on simplex [22–24]. Lotka–Volterra mappings are popular in modeling the spread of viral diseases. In [25–27], degenerate Lotka–Volterra mappings and their applications were considered.

In this paper, we focus on the structure of the set of fixed points — referred to as the *card of fixed points* — for various types of Lotka–Volterra operators V. We pay special attention to operators corresponding to strong and homogeneous tournaments. Also explore conditions for the existence of fixed points on the interior and the faces of the simplex, as well as criteria for their stability and evolutionary significance.

Additionally, we establish links with replicator dynamics and evolutionary game theory, including conditions under which fixed points of the system can be interpreted as evolutionary stable strategies (ESS).

1 *Card of fixed points*

Introduce the following notation:

$$P_\alpha = \{x \in \Gamma_\alpha : A_\alpha x \geq 0\}, \qquad Q_\alpha = \{x \in \Gamma_\alpha : A_\alpha x \leq 0\},$$

where Γ_α denotes the face of the simplex S^{m-1} corresponding to the index set $\alpha \subset I = \{1, 2, \dots, m\}$, and A_α is the submatrix of A corresponding to the indices in α.

It is known [14], each of the sets P_α and Q_α contains a unique fixed point. In some cases, it is possible that $P_\alpha = Q_\alpha$.

The set of all fixed points of the operator V, $\mathrm{Fix}(V) = \{x \in S^{m-1} : Vx = x\}$ can be represented as a set of points in a plane. For each $\alpha \subset I$, the fixed point P_α is connected to the fixed point Q_α by a directed arc pointing from P_α to Q_α. The resulting directed graph is called the *card of fixed points* of the operator V, and is denoted by G_V [14, 15].

Definition 4. Two fixed points (vertices of the graph G_V) $x(\alpha)$ and $x(\beta)$ are called *adjacent* if the following conditions hold:

1. $|\alpha| = |\beta|$,
2. $|\alpha \cap \beta| = |\alpha| - 1$,

where $|\alpha|$ denotes the number of elements in $\alpha \subset I = \{1, 2, \dots, m\}$.

In other words, $x(\alpha)$ and $x(\beta)$ correspond to faces of the same dimension and their supports differ by exactly one index.

For example, all vertices of the simplex (corresponding to one-element subsets) are pairwise adjacent. However, the fixed points $x(\{2, 3, 5\})$ and $x(\{1, 2, 4\})$ are not adjacent.

Theorem 1. Any two adjacent vertices in the graph G_V are connected by a directed arc.

Proof. Let $x(\alpha)$ and $x(\beta)$ be adjacent vertices of G_V, corresponding to the subsets $\alpha, \beta \subset I = \{1, 2, \ldots, m\}$. By definition of adjacency, $|\alpha| = |\beta|$, and $|\alpha \cap \beta| = |\alpha| - 1$. Let $\gamma = \alpha \cup \beta$, so that $|\gamma| = |\alpha| + 1$.

Let us denote $\gamma = \{i_1, i_2, \ldots, i_t\}$, with $t = |\gamma|$. Then, without loss of generality, we may assume

$$\alpha = \{i_2, i_3, \ldots, i_t\}, \quad \beta = \{i_1, i_2, \ldots, i_{t-1}\}.$$

Now consider the restriction of the mapping V to the face $\Gamma_\gamma \subset S^{m-1}$. Since $x(\alpha)$ and $x(\beta)$ lie in Γ_γ, we consider the action of the submatrix A_γ from the skew-symmetric matrix A on the face Γ_γ.

Recall the property of Lotka–Volterra mappings on invariant faces: for a fixed point $x \in \Gamma_\gamma$,

$$\operatorname{supp} x \cap \operatorname{supp}(A_\gamma x) = \varnothing, \quad \operatorname{supp} x \cup \operatorname{supp}(A_\gamma x) = \gamma.$$

That is, the nonzero coordinates of $A_\gamma x$ are complementary to the support of x within γ.

Applying this to $x(\alpha)$, which has support $\alpha = \{i_2, \ldots, i_t\}$, we obtain that $(A_\gamma x(\alpha))_{i_1} \neq 0$, and all other coordinates of $A_\gamma x(\alpha)$ vanish. Similarly, since $\beta = \{i_1, \ldots, i_{t-1}\}$, the only nonzero coordinate of $A_\gamma x(\beta)$ is $(A_\gamma x(\beta))_{i_t} \neq 0$.

We now consider the signs of these nonzero coordinates. If

$$\operatorname{sign}(A_\gamma x(\alpha))_{i_1} \cdot \operatorname{sign}(A_\gamma x(\beta))_{i_t} < 0,$$

then, the directions of the corresponding arcs go from one to the other, and $x(\alpha)$ and $x(\beta)$ form a directed pair (P_α, Q_α), meaning they are connected by an arc in G_V.

If the signs are the same, then both $x(\alpha)$ and $x(\beta)$ would have outgoing arcs in the same direction on the face Γ_γ, which contradicts the uniqueness of the sink (i.e., the unique point with all incoming arcs) in the fixed point diagram on Γ_γ.

Hence, in either case, the pair $(x(\alpha), x(\beta))$ must be connected by a directed arc in G_V. □

2 Main results

Consider the general form of the Lotka–Volterra operator V_1:

$$V_1 : \begin{cases} x_1' = x_1(1 - a_{12}x_2 - a_{13}x_3 - a_{14}x_4 + a_{15}x_5), \\ x_2' = x_2(1 + a_{12}x_1 - a_{23}x_3 - a_{24}x_4 - a_{25}x_5), \\ x_3' = x_3(1 + a_{13}x_1 + a_{23}x_2 - a_{34}x_4 - a_{35}x_5), \\ x_4' = x_4(1 + a_{14}x_1 + a_{24}x_2 + a_{34}x_3 - a_{45}x_5), \\ x_5' = x_5(1 - a_{15}x_1 + a_{25}x_2 + a_{35}x_3 + a_{45}x_4). \end{cases} \tag{3}$$

The operator V_1 corresponds to the strong and homogeneous tournament shown in Figure 1.

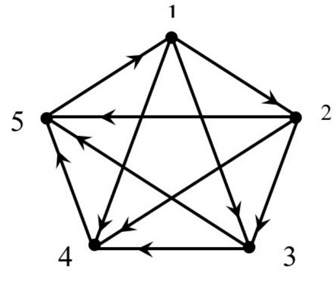

Figure 1. The tournament associated with the operator V_1

The corresponding skew-symmetric matrix $A_1 = (a_{ij})$ associated with mapping (3) has the form:

$$A_1 = \begin{pmatrix} 0 & -a_{12} & -a_{13} & -a_{14} & a_{15} \\ a_{12} & 0 & -a_{23} & -a_{24} & -a_{25} \\ a_{13} & a_{23} & 0 & -a_{34} & -a_{35} \\ a_{14} & a_{24} & a_{34} & 0 & -a_{45} \\ -a_{15} & a_{25} & a_{35} & a_{45} & 0 \end{pmatrix}.$$

In order for the operator V_1 to correspond to a matrix in general position, it is required that all even-order principal minors of the matrix A_1 be nonzero.

For second-order minors, the condition $a_{ki} > 0$ ensures their positivity. Calculating the principal minors of order four (there are five such minors), we obtain:

$$\Delta_1^{11} = (a_{23}a_{45} + a_{25}a_{34} - a_{24}a_{35})^2, \quad \Delta_2^{22} = (a_{15}a_{34} + a_{14}a_{35} - a_{13}a_{45})^2,$$
$$\Delta_3^{33} = (a_{15}a_{24} + a_{14}a_{25} - a_{12}a_{45})^2, \quad \Delta_4^{44} = (a_{15}a_{23} + a_{13}a_{25} - a_{12}a_{35})^2,$$
$$\Delta_5^{55} = (a_{14}a_{23} + a_{12}a_{34} - a_{13}a_{24})^2.$$

Since the matrix A_1 is in general position, all even-order principal minors are nonzero, i.e., $\Delta_i^{ii} \neq 0$ for all $i = 1, \ldots, 5$.

Let us define the expressions inside the squares as:

$$\Delta_1 = a_{23}a_{45} + a_{25}a_{34} - a_{24}a_{35},$$
$$\Delta_2 = a_{15}a_{34} + a_{14}a_{35} - a_{13}a_{45},$$
$$\Delta_3 = a_{15}a_{24} + a_{14}a_{25} - a_{12}a_{45},$$
$$\Delta_4 = a_{15}a_{23} + a_{13}a_{25} - a_{12}a_{35},$$
$$\Delta_5 = a_{14}a_{23} + a_{12}a_{34} - a_{13}a_{24}.$$

Theorem 2. If $\Delta_2, \Delta_3, \Delta_4 > 0$, then the card of the fixed point operator V_1 is transitive (Figure 2)

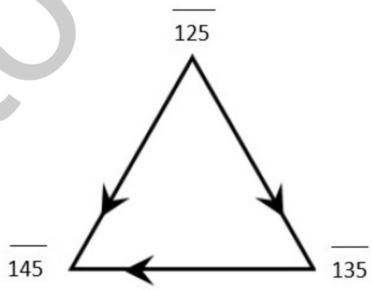

Figure 2. The transitive card of the fixed point

Proof. As shown in Figure 1, the tournament contains three cyclic triples: $\overline{125}, \overline{135}, \overline{145}$. These correspond to the following fixed points:

$$M_{125} = \left(\frac{a_{25}}{a_{12} + a_{15} + a_{25}}, \frac{a_{15}}{a_{12} + a_{15} + a_{25}}, 0, 0, \frac{a_{12}}{a_{12} + a_{15} + a_{25}} \right),$$
$$M_{135} = \left(\frac{a_{35}}{a_{13} + a_{15} + a_{35}}, 0, \frac{a_{15}}{a_{13} + a_{15} + a_{35}}, 0, \frac{a_{13}}{a_{13} + a_{15} + a_{35}} \right),$$
$$M_{145} = \left(\frac{a_{45}}{a_{14} + a_{15} + a_{45}}, 0, 0, \frac{a_{15}}{a_{14} + a_{15} + a_{45}}, \frac{a_{14}}{a_{14} + a_{15} + a_{45}} \right),$$

where all coefficients are assumed to be positive.

Now, define the following functions:

$$\varphi_{125}(x) = (x_1^{a_{25}} x_2^{a_{15}} x_5^{a_{12}})^{\frac{1}{a_{12}+a_{15}+a_{25}}}, \quad \varphi_{135}(x) = (x_1^{a_{35}} x_3^{a_{15}} x_5^{a_{13}})^{\frac{1}{a_{13}+a_{15}+a_{35}}},$$

$$\varphi_{145}(x) = (x_1^{a_{45}} x_4^{a_{15}} x_5^{a_{14}})^{\frac{1}{a_{14}+a_{15}+a_{45}}}.$$

We now apply "Young's inequality" [28], which states that for any $c_k \geq 0$, $p_k \geq 0$, such that $\sum\limits_{k=1}^{m} p_k = 1$, the following holds:

$$\prod_{k=1}^{m} c_k^{p_k} \leq \sum_{k=1}^{m} c_k p_k.$$

Using this, one derives the following estimates:

$$\varphi_{125}(Vx) \leq \frac{\varphi_{125}(x)}{\Delta_{125}} \left(\Delta_{125} - \Delta_4 x_3 - \Delta_3 x_4\right), \tag{4}$$

$$\varphi_{135}(Vx) \leq \frac{\varphi_{135}(x)}{\Delta_{135}} \left(\Delta_{135} + \Delta_4 x_2 - \Delta_2 x_4\right), \tag{5}$$

$$\varphi_{145}(Vx) \leq \frac{\varphi_{145}(x)}{\Delta_{145}} \left(\Delta_{145} + \Delta_3 x_2 + \Delta_2 x_3\right). \tag{6}$$

Here, the constants are:

$$\Delta_{125} = a_{12} + a_{15} + a_{25}, \quad \Delta_{135} = a_{13} + a_{15} + a_{35}, \quad \Delta_{145} = a_{14} + a_{15} + a_{45}.$$

We now determine the directions of arcs between the fixed points:

1. "Between M_{125} and M_{135}": In inequalities (4) and (5), the term involving Δ_4 appears with opposite signs. If $\Delta_4 > 0$, then in (4) this term decreases $\varphi_{125}(Vx)$, while in (5) it increases $\varphi_{135}(Vx)$. This implies the direction of the fixed-point flow is $M_{125} \to M_{135}$.

2. "Between M_{135} and M_{145}": In inequalities (5) and (6), Δ_2 appears with opposite signs. If $\Delta_2 > 0$, this implies the direction $M_{135} \to M_{145}$.

3. "Between M_{125} and M_{145}": Comparing (4) and (6), if $\Delta_3 > 0$, the sign of the corresponding term shows the direction $M_{125} \to M_{145}$.

As a result, all three fixed points are connected in a consistent directed order:

$$M_{125} \to M_{135} \to M_{145} \leftarrow M_{125},$$

and the resulting subgraph forms a transitive triangle, as shown in Figure 2. □

Let $Vx = x$, i.e., x is a fixed point of the mapping. The eigenvalues of the Jacobian matrix at the fixed point are found as the solutions of the characteristic equation:

$$\det(J(x) - \lambda E) = 0, \tag{7}$$

where $J(x)$ is the Jacobian matrix of the mapping V evaluated at the fixed point x, and E is the identity matrix.

The nature of the fixed point can be characterized based on the eigenvalues of the Jacobian. To do this, we first introduce some definitions regarding the classification of fixed points [29].

To investigate the nature of fixed points of the mapping, we introduce the following definitions from [29].

Definition 5. A fixed point is called an *attractor* if all eigenvalues of the Jacobian matrix (i.e., the solutions of equation (7)) have modulus strictly less than one.

Definition 6. A fixed point is called a *repeller* if all eigenvalues of the Jacobian matrix have modulus strictly greater than one.

Definition 7. A fixed point is called a *saddle point* if the spectrum of the Jacobian contains eigenvalues with modulus both less than and greater than one. In other words, it is neither an attractor nor a repeller.

Corollary 1. If $\Delta_2, \Delta_3, \Delta_4 > 0$, then the fixed point M_{125} of the operator V_1 is a repeller, the fixed point M_{145} is an attractor, and the fixed point M_{135} is a saddle point.

Proof. Using equation (7), we compute the eigenvalues of the Jacobian matrix at each fixed point. Let us denote the diagonal entries of the Jacobian matrix at a general point x as:

$$\begin{aligned}
t_1 &= 1 - a_{12}x_2 - a_{13}x_3 - a_{14}x_4 + a_{15}x_5,\\
t_2 &= 1 + a_{12}x_1 - a_{23}x_3 - a_{24}x_4 - a_{25}x_5,\\
t_3 &= 1 + a_{13}x_1 + a_{23}x_2 - a_{34}x_4 - a_{35}x_5,\\
t_4 &= 1 + a_{14}x_1 + a_{24}x_2 + a_{34}x_3 - a_{45}x_5,\\
t_5 &= 1 - a_{15}x_1 + a_{25}x_2 + a_{35}x_3 + a_{45}x_4.
\end{aligned}$$

Then the Jacobian matrix J takes the form:

$$J = \begin{pmatrix}
t_1 & -a_{12}x_1 & -a_{13}x_1 & -a_{14}x_1 & -a_{15}x_1\\
a_{12}x_2 & t_2 & -a_{23}x_2 & -a_{24}x_2 & -a_{25}x_2\\
a_{13}x_3 & a_{23}x_3 & t_3 & -a_{34}x_3 & -a_{35}x_3\\
a_{14}x_4 & a_{24}x_4 & a_{34}x_4 & t_4 & -a_{45}x_4\\
-a_{15}x_5 & a_{25}x_5 & a_{35}x_5 & a_{45}x_5 & t_5
\end{pmatrix}.$$

Substituting the coordinates of the fixed point M_{125} into J, we obtain:

$$J(M_{125}) = \begin{pmatrix}
1 & -\frac{a_{12}a_{25}}{\Delta_{125}} & -\frac{a_{13}a_{25}}{\Delta_{125}} & -\frac{a_{14}a_{25}}{\Delta_{125}} & \frac{a_{15}a_{25}}{\Delta_{125}}\\
\frac{a_{12}a_{15}}{\Delta_{125}} & 1 & -\frac{a_{23}a_{15}}{\Delta_{125}} & -\frac{a_{24}a_{15}}{\Delta_{125}} & -\frac{a_{25}a_{15}}{\Delta_{125}}\\
0 & 0 & 1+\frac{a_{13}a_{25}+a_{23}a_{15}-a_{35}a_{12}}{\Delta_{125}} & 0 & 0\\
0 & 0 & 0 & 1+\frac{a_{14}a_{25}+a_{24}a_{15}-a_{45}a_{12}}{\Delta_{125}} & 0\\
-\frac{a_{15}a_{12}}{\Delta_{125}} & \frac{a_{25}a_{12}}{\Delta_{125}} & \frac{a_{35}a_{12}}{\Delta_{125}} & \frac{a_{45}a_{12}}{\Delta_{125}} & 1
\end{pmatrix},$$

where $\Delta_{125} = a_{12} + a_{15} + a_{25}$.

From this matrix, two eigenvalues are immediately identified as: $\lambda_1 = 1 + \frac{\Delta_4}{\Delta_{125}}$, $\lambda_2 = 1 + \frac{\Delta_3}{\Delta_{125}}$, corresponding to the diagonal entries.

The remaining eigenvalues are obtained from the characteristic equation for the 3×3 leading principal minor:

$$\begin{vmatrix}
1-\lambda & -\frac{a_{12}a_{25}}{\Delta_{125}} & \frac{a_{15}a_{25}}{\Delta_{125}}\\
\frac{a_{12}a_{15}}{\Delta_{125}} & 1-\lambda & -\frac{a_{25}a_{15}}{\Delta_{125}}\\
-\frac{a_{15}a_{12}}{\Delta_{125}} & \frac{a_{25}a_{12}}{\Delta_{125}} & 1-\lambda
\end{vmatrix} = 0.$$

Solving it, we find: $\lambda_{3,4} = 1 \pm i\sqrt{\frac{a_{12}a_{15}a_{35}}{\Delta_{125}}}$, $\lambda_5 = 1$. Thus, the spectrum of the Jacobian at M_{125} is:

$$\sigma(J(M_{125})) = \left\{1,\ 1+\frac{\Delta_4}{\Delta_{125}},\ 1+\frac{\Delta_3}{\Delta_{125}},\ 1 \pm i\sqrt{\frac{a_{12}a_{15}a_{35}}{\Delta_{125}}}\right\}.$$

Similarly, we have:

$$\sigma(J(M_{135})) = \left\{1,\ 1-\frac{\Delta_4}{\Delta_{135}},\ 1+\frac{\Delta_2}{\Delta_{135}},\ 1\pm i\sqrt{\frac{a_{13}a_{15}a_{35}}{\Delta_{135}}}\right\},$$
$$\sigma(J(M_{145})) = \left\{1,\ 1-\frac{\Delta_3}{\Delta_{145}},\ 1-\frac{\Delta_2}{\Delta_{145}},\ 1\pm i\sqrt{\frac{a_{14}a_{15}a_{45}}{\Delta_{145}}}\right\}.$$

Assuming $\Delta_2, \Delta_3, \Delta_4 > 0$, we observe:

– for M_{125}: all real parts of the eigenvalues are strictly greater than 1. Hence, M_{125} is a "repeller";

– for M_{135}: one eigenvalue has real part greater than 1, another less than 1. Hence, M_{135} is a "saddle point";

– for M_{145}: all real parts of the eigenvalues are less than 1. Hence, M_{145} is an "attractor". □

Theorem 3. If $\Delta_2, \Delta_4 > 0$ and $\Delta_3 < 0$, then the fixed point card of the operator V_1 is cyclic and, in addition to the fixed points M_{125}, M_{135}, and M_{145}, contains an internal fixed point with all five coordinates nonzero (see Figure 3).

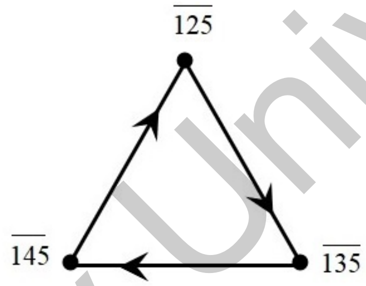

Figure 3. Cyclic structure of the fixed point graph with an additional internal fixed point

Proof. The cyclic structure of the fixed point graph G_V follows from Theorem 2, which characterizes the orientation of arcs between the fixed points M_{125}, M_{135}, and M_{145} depending on the signs of Δ_2, Δ_3, and Δ_4.

When $\Delta_2, \Delta_4 > 0$ and $\Delta_3 < 0$, the inequalities derived in Theorem 2 imply the formation of a cycle:

$$M_{125} \to M_{135} \to M_{145} \to M_{125}.$$

Let $\alpha = \{1, 2, 3, 4, 5\}$ denote the full support. Then Γ_α is the interior of the simplex S^4.

Since M_{125}, M_{135}, and M_{145} form a cyclic triple, none of them can serve as the sink (i.e., the unique fixed point Q_α) of the face Γ_α. By the uniqueness of such a point ([15], it follows that Γ_α must contain an additional fixed point M_α, which lies strictly inside the simplex. Hence, all coordinates of M_α are nonzero.

Therefore, under the stated conditions, the graph G_V acquires a cyclic structure and includes an internal fixed point with full support. □

Next, we consider another representative of the Lotka–Volterra mapping and the corresponding tournament.

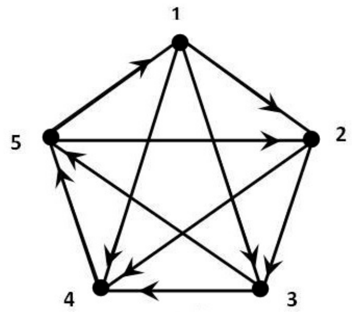

Figure 4. A strong, homogeneous tournament with four cyclic triples

Figure 4 illustrates a strong homogeneous tournament containing four cyclic triples. This tournament corresponds to the Lotka–Volterra operator V_2, defined by:

$$V_2:\begin{cases} x_1^{'}=x_1(1-a_{12}x_2-a_{13}x_3-a_{14}x_4+a_{15}x_5),\\ x_2^{'}=x_2(1+a_{12}x_1-a_{23}x_3-a_{24}x_4+a_{25}x_5),\\ x_3^{'}=x_3(1+a_{13}x_1+a_{23}x_2-a_{34}x_4-a_{35}x_5),\\ x_4^{'}=x_4(1+a_{14}x_1+a_{24}x_2+a_{34}x_3-a_{45}x_5),\\ x_5^{'}=x_5(1-a_{15}x_1-a_{25}x_2+a_{35}x_3+a_{45}x_4). \end{cases} \tag{8}$$

The corresponding skew-symmetric matrix A_2 associated with this operator is given by:

$$A_2=\begin{pmatrix} 0 & -a_{12} & -a_{13} & -a_{14} & a_{15}\\ a_{12} & 0 & -a_{23} & -a_{24} & a_{25}\\ a_{13} & a_{23} & 0 & -a_{34} & -a_{35}\\ a_{14} & a_{24} & a_{34} & 0 & -a_{45}\\ -a_{15} & -a_{25} & a_{35} & a_{45} & 0 \end{pmatrix}.$$

If we compute all principal minors of order four of the skew-symmetric matrix A_2, we obtain squares of certain expressions. Let these expression denoted by $\Delta_i \neq 0$, for $i = 1, \ldots, 5$:

$$\Delta_1=a_{24}a_{35}-a_{23}a_{45}+a_{25}a_{34},\quad \Delta_2=a_{14}a_{35}-a_{13}a_{45}+a_{15}a_{34},$$
$$\Delta_3=a_{14}a_{25}-a_{15}a_{24}+a_{12}a_{45},\quad \Delta_4=a_{12}a_{35}-a_{15}a_{23}+a_{13}a_{25},$$
$$\Delta_5=a_{12}a_{34}-a_{13}a_{24}+a_{14}a_{23}.$$

The Lotka–Volterra operator V_2 defined in equation (8) admits four cyclic triples: $\overline{135}$, $\overline{145}$, $\overline{235}$, and $\overline{245}$. These cyclic triples correspond to strong sub-tournaments of the tournament on the 4-simplex S^4 (see Figure 4), each containing a unique internal fixed point.

These fixed points are given by:

$$M_{135}=\left(\frac{a_{35}}{a_{13}+a_{15}+a_{35}},\ 0,\ \frac{a_{15}}{a_{13}+a_{15}+a_{35}},\ 0,\ \frac{a_{13}}{a_{13}+a_{15}+a_{35}}\right),$$

$$M_{145}=\left(\frac{a_{45}}{a_{14}+a_{15}+a_{45}},\ 0,\ 0,\ \frac{a_{15}}{a_{14}+a_{15}+a_{45}},\ \frac{a_{14}}{a_{14}+a_{15}+a_{45}}\right),$$

$$M_{235}=\left(0,\ \frac{a_{35}}{a_{23}+a_{25}+a_{35}},\ \frac{a_{25}}{a_{23}+a_{25}+a_{35}},\ 0,\ \frac{a_{23}}{a_{23}+a_{25}+a_{35}}\right),$$

$$M_{245} = \left(0,\ \frac{a_{45}}{a_{24}+a_{25}+a_{45}},\ 0,\ \frac{a_{25}}{a_{24}+a_{25}+a_{45}},\ \frac{a_{24}}{a_{24}+a_{25}+a_{45}}\right),$$

where all coefficients a_{ij} are assumed to be strictly positive.

For the operator V_2, applying Young's inequality yields the following estimates:

$$\begin{aligned}
\varphi_{135}(Vx) &\leq \frac{\varphi_{135}(x)}{\Delta_{135}}\left(\Delta_{135} + \Delta_4 x_2 - \Delta_2 x_4\right),\\
\varphi_{145}(Vx) &\leq \frac{\varphi_{145}(x)}{\Delta_{145}}\left(\Delta_{145} + \Delta_3 x_2 + \Delta_2 x_3\right),\\
\varphi_{235}(Vx) &\leq \frac{\varphi_{235}(x)}{\Delta_{235}}\left(\Delta_{235} - \Delta_4 x_3 - \Delta_3 x_4\right),\\
\varphi_{245}(Vx) &\leq \frac{\varphi_{245}(x)}{\Delta_{245}}\left(\Delta_{245} - \Delta_4 x_3 - \Delta_3 x_4\right),
\end{aligned}$$

for all $x \in S^4$, where

$$\Delta_{135} = a_{13}+a_{15}+a_{35}, \quad \Delta_{145} = a_{14}+a_{15}+a_{45}, \quad \Delta_{235} = a_{23}+a_{25}+a_{35}, \quad \Delta_{245} = a_{24}+a_{25}+a_{45}.$$

If the second and fourth even-order principal minors of the skew-symmetric matrix A_2 are nonzero, then A_2 is said to be in general position. In this case, the card of fixed points of the operator V_2 has the structure shown in Figure 5.

Figure 5. The card of fixed points for the mapping V_2

In the card of fixed points, no directions are initially indicated, as the orientations on the faces of the simplex depend on the signs of the expressions Δ_i, for $i = 1, 2, 3, 4, 5$.

The orientation of a graph refers to assigning a direction (arrow) to each of its edges, i.e., specifying an order for every pair of adjacent vertices. A directed graph, or digraph, is one in which no two vertices are connected by a pair of edges pointing in opposite directions. Thus, every orientation of an undirected graph yields a digraph [17].

For a graph with four vertices, there are $2^4 = 16$ possible orientations. Among these 16 digraphs, some are isomorphic — that is, structurally identical up to a relabeling of vertices. There are exactly four non-isomorphic directed graphs with four vertices that contain a directed cycle. These are illustrated in Figure 6.

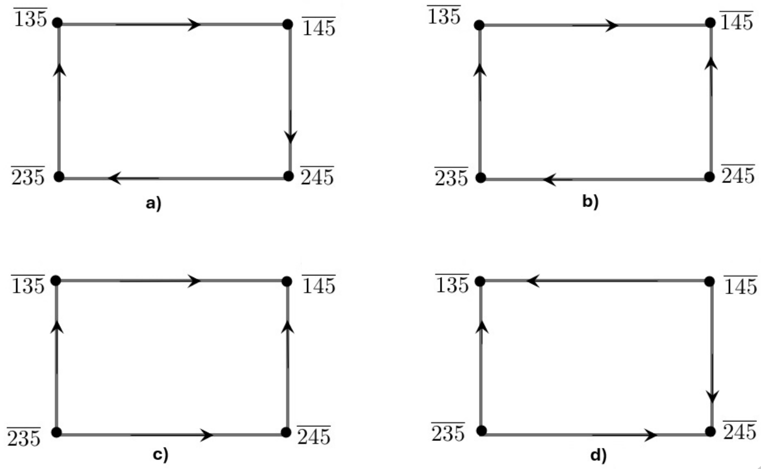

Figure 6. The four non-isomorphic directed graphs

Theorem 4. Let the following conditions hold:

1. If $\Delta_1, \Delta_4 < 0$, then the card of fixed points of the operator V_2 has the structure shown in Figure 6, case a).
2. If $\Delta_1, \Delta_3, \Delta_4 < 0$, then the card of fixed points of the operator V_2 has the structure shown in Figure 6, case b).
3. If $\Delta_3, \Delta_4 < 0$, then the card of fixed points of the operator V_2 has the structure shown in Figure 6, case c).

The proof of Theorem 4 follows directly from Theorems 2 and 3.

Theorem 5. If A_2 is a skew-symmetric matrix in general position, then the card of fixed points of the operator V_2 cannot take the form shown in Figure 6 case d).

Proof. The fact that the card of fixed points of the operator V_2 cannot take the form shown in Figure 6, case d) follows from a uniqueness fakt stated in [15]. Specifically, if the skew-symmetric matrix is in general position, then the sets of points P and Q are each unique [13, 15].

However, in the fixed point diagram shown in Figure 6 case d), there are two P-points, namely $(\overline{145}, \overline{235})$, and two Q-points, namely $(\overline{135}, \overline{245})$, which contradicts this uniqueness. □

Let us consider the mapping $V_3 : S^4 \to S^4$ defined by the following system of equations:

$$V_3 : \begin{cases} x_1^{'} = x_1(1 + a_{12}x_2 + a_{13}x_3 - a_{14}x_4 - a_{15}x_5), \\ x_2^{'} = x_2(1 - a_{12}x_1 + a_{23}x_3 + a_{24}x_4 - a_{25}x_5), \\ x_3^{'} = x_3(1 - a_{13}x_1 - a_{23}x_2 + a_{34}x_4 + a_{35}x_5), \\ x_4^{'} = x_4(1 + a_{14}x_1 - a_{24}x_2 - a_{34}x_3 + a_{45}x_5), \\ x_5^{'} = x_5(1 + a_{15}x_1 + a_{25}x_2 - a_{35}x_3 - a_{45}x_4), \end{cases}$$

where the coefficients satisfy the conditions $0 < a_{ki} \leq 1$ for all i, k.

The strong, homogeneous tournament corresponding to this operator is illustrated in Figure 7.

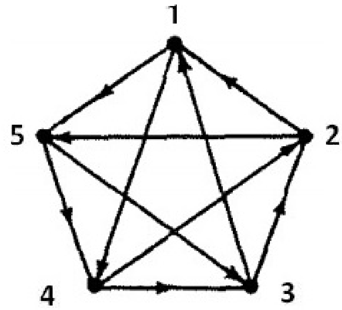

Figure 7. The strong, homogeneous tournament corresponding to the operator V_3

It has five cyclic triples: $\overline{124}, \overline{134}, \overline{135}, \overline{235}, \overline{245}$, each of whose corresponding faces contains exactly one fixed point:

$$M_{124} = \left(\frac{a_{24}}{a_{12}+a_{14}+a_{24}},\ \frac{a_{14}}{a_{12}+a_{14}+a_{24}},\ 0,\ \frac{a_{12}}{a_{12}+a_{14}+a_{24}},\ 0\right),$$
$$M_{134} = \left(\frac{a_{34}}{a_{13}+a_{14}+a_{34}},\ 0,\ \frac{a_{14}}{a_{13}+a_{14}+a_{34}},\ \frac{a_{13}}{a_{13}+a_{14}+a_{34}},\ 0\right),$$
$$M_{135} = \left(\frac{a_{35}}{a_{13}+a_{15}+a_{35}},\ 0,\ \frac{a_{15}}{a_{13}+a_{15}+a_{35}},\ 0,\ \frac{a_{13}}{a_{13}+a_{15}+a_{35}}\right),$$
$$M_{235} = \left(0,\ \frac{a_{35}}{a_{23}+a_{25}+a_{35}},\ \frac{a_{25}}{a_{23}+a_{25}+a_{35}},\ 0,\ \frac{a_{23}}{a_{23}+a_{25}+a_{35}}\right),$$
$$M_{245} = \left(0,\ \frac{a_{45}}{a_{24}+a_{25}+a_{45}},\ 0,\ \frac{a_{25}}{a_{24}+a_{25}+a_{45}},\ \frac{a_{24}}{a_{24}+a_{25}+a_{45}}\right).$$

We use the following notation:

$$\begin{gathered} \Delta_1 = a_{24}a_{35} - a_{23}a_{45} + a_{25}a_{34}, \quad \Delta_2 = a_{14}a_{35} - a_{15}a_{34} + a_{13}a_{45}, \\ \Delta_3 = a_{14}a_{25} - a_{12}a_{45} + a_{15}a_{24}, \quad \Delta_4 = a_{12}a_{35} - a_{15}a_{23} + a_{13}a_{25}, \\ \Delta_5 = a_{13}a_{24} - a_{12}a_{34} + a_{14}a_{23}. \end{gathered} \tag{9}$$

For the operator V_3, we also apply Young's inequality and obtain the following estimates:

$$\varphi_{124}(Vx) \leq \frac{\varphi_{124}(x)}{\Delta_{124}}\left(\Delta_{124} - \Delta_5 x_3 - \Delta_3 x_5\right),$$
$$\varphi_{134}(Vx) \leq \frac{\varphi_{134}(x)}{\Delta_{134}}\left(\Delta_{134} + \Delta_5 x_2 + \Delta_2 x_5\right),$$
$$\varphi_{135}(Vx) \leq \frac{\varphi_{135}(x)}{\Delta_{135}}\left(\Delta_{135} + \Delta_4 x_2 - \Delta_2 x_4\right),$$
$$\varphi_{235}(Vx) \leq \frac{\varphi_{235}(x)}{\Delta_{235}}\left(\Delta_{235} - \Delta_4 x_1 + \Delta_1 x_4\right),$$
$$\varphi_{245}(Vx) \leq \frac{\varphi_{245}(x)}{\Delta_{245}}\left(\Delta_{245} + \Delta_3 x_1 - \Delta_1 x_3\right).$$

for all $x \in S^4$, where

$$\Delta_{124} = a_{12} + a_{14} + a_{24}, \quad \Delta_{134} = a_{13} + a_{14} + a_{34}, \quad \Delta_{135} = a_{13} + a_{15} + a_{35},$$
$$\Delta_{235} = a_{23} + a_{25} + a_{35}, \quad \Delta_{245} = a_{24} + a_{25} + a_{45}.$$

Theorem 6. Let the quantities $\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5$ be defined as in (9). Then:

1. If $\Delta_1, \Delta_2, \Delta_3, \Delta_4 < 0$ and $\Delta_5 > 0$, then the fixed point card of the operator V_3 contains a Hamiltonian cycle, and the operator admits an internal fixed point with all five coordinates nonzero (see Figure 8, case a)).
2. If $\Delta_2, \Delta_3, \Delta_4 < 0$ and $\Delta_1, \Delta_5 > 0$, then the fixed point card of the operator V_3 takes the form shown in Figure 8, case b).
3. If $\Delta_2, \Delta_4 < 0$ and $\Delta_1, \Delta_3, \Delta_5 > 0$, then the fixed point card of the operator V_3 takes the form shown in Figure 8, case c).
4. If $\Delta_1 \cdot \Delta_2 \cdot \Delta_3 \cdot \Delta_4 \cdot \Delta_5 \neq 0$, then the fixed point card of the operator V_3 cannot take the form shown in Figure 8, case d).

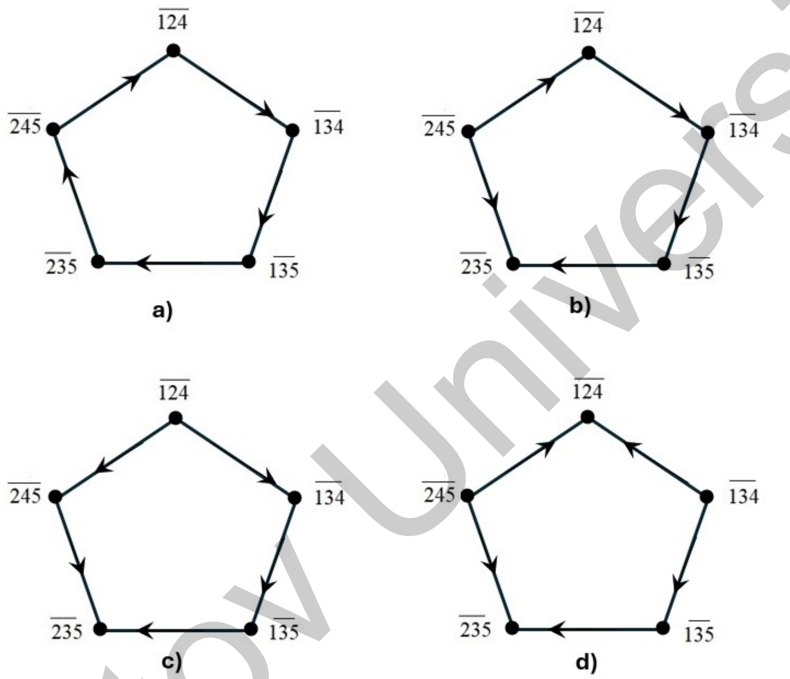

Figure 8. Possible cards of fixed points of the operator V_3

The proof of Theorem 6 follows directly from Theorems 2 and 3.

Theorem 6 characterizes the types of fixed point configurations of the operator V_3 depending on the signs of the expressions Δ_i. In particular, case 1. indicates the existence of an internal fixed point. The following lemma makes this statement precise.

Lemma 1. Let the operator $V_3 : S^4 \to S^4$ be defined by the system

$$V_3(x)_k = x_k \left(1 + \sum_{i=1}^{5} a_{ki} x_i\right), \quad k = 1, \ldots, 5,$$

where $a_{ki} = -a_{ik}$, $x_i \geq 0$, $\sum_{i=1}^{5} x_i = 1$, and Δ_i are the fourth-order principal minors of the skew-symmetric matrix $A = (a_{ij})$. Then:

1. If $\Delta_1, \Delta_2, \Delta_3, \Delta_4 < 0$ and $\Delta_5 > 0$, then the operator V_3 has at least one internal fixed point $x^* \in \operatorname{int}(S^4)$.
2. If at least three of the values Δ_i are positive, then there are no internal fixed points.

Proof. Consider the directed graph (tournament) G_{V_3} corresponding to the operator V_3, where the vertices represent the coordinates x_i, and the direction of the edges is determined by the sign of the coefficients a_{ki}.

1. Existence of an internal point. According to results by Hofbauer J. and Ganikhodzhaev R. [13, 19], if the tournament G_{V_3} contains a Hamiltonian cycle, then the operator V_3 has at least one internal fixed point. This behavior occurs when the fixed points on the faces (e.g., $M_{124}, M_{134}, M_{135}, M_{235}, M_{245}$) are connected by directed transitions forming a cycle. The conditions $\Delta_1, \Delta_2, \Delta_3, \Delta_4 < 0$ and $\Delta_5 > 0$ ensure the required orientation of the transitions between faces, forming a Hamiltonian cycle.

2. Non-existence of an internal point. If at least three of the values Δ_i are positive, the structure of G_{V_3} does not contain a full directed cycle (it becomes either transitive or splits into sub-tournaments). This implies that all trajectories of V_3 are attracted to fixed points on the boundary faces of the simplex, and internal fixed points are either unstable or do not exist. □

3 Connection with replicator dynamics and evolutionary stability

The Lotka–Volterra operators considered in this paper are structurally close to replicator dynamics from evolutionary game theory. In both models, the trajectories are confined to the standard simplex S^{m-1}, and fixed points correspond to stationary population states.

3.1 Replicator dynamics and stability

The replicator equation for a population with m strategies and payoff matrix $A = (a_{ij})$ has the form [30–32]: $\dot{x}_i = x_i\left((Ax)_i - x^\top Ax\right)$, where $x \in S^{m-1}$, and $(Ax)_i$ denotes the fitness of strategy i. A point $x^* \in S^{m-1}$ is a fixed point if all strategies present in x^* have equal fitness: $(Ax^*)_i = x^{*\top}Ax^*$ for all $x_i^* > 0$.

3.2 Evolutionarily stable strategy (ESS)

A point $x^* \in S^{m-1}$ is called an *evolutionarily stable strategy (ESS)* if the following two conditions are satisfied:

1. x^* is a Nash equilibrium: $x^{*\top}Ax^* \geq x^\top Ax^*$ for all $x \in S^{m-1}$;
2. if $x \neq x^*$ and $x^{*\top}Ax = x^{*\top}Ax^*$, then $x^\top Ax < x^{*\top}Ax$.

This means that small deviations from x^* result in lower fitness for mutants, and strategy x^* cannot be invaded.

3.3 Analogy with Lotka–Volterra operators

Consider the discrete Lotka–Volterra operator:

$$x'_k = x_k\left(1 + \sum_{i=1}^{m} a_{ki}x_i\right), \quad k = 1, \ldots, m.$$

After normalization and transition to continuous time, this system approximates the replicator form: $\dot{x}_k = x_k\left(\sum\limits_{i=1}^{m} a_{ki}x_i - \Phi(x)\right)$, where $\Phi(x)$ is the average fitness. This supports the interpretation of coefficients a_{ij} as measures of fitness differences or interactions between strategies.

Thus, interior fixed points of the operator V, i.e., those with all coordinates positive, can be interpreted as candidates for ESS.

3.4 Classification of fixed points

Let $M_\alpha \subset S^4$ be a fixed point associated with a face Γ_α defined by a cyclic triple. Then:

- if all eigenvalues of the Jacobian matrix at M_α have modulus less than one, the point is *asymptotically stable* and may be ESS;
- if the point is a saddle or repeller, then it cannot be evolutionarily stable.

Proposition 1. Let x^* be a fixed point of a Lotka–Volterra operator V. Then:

- if x^* is a strict local maximum of a potential function (if one exists), then x^* is an ESS;
- if x^* is a saddle or repeller, then it is not evolutionarily stable.

As an example, we can consider the operator V_2. Under the conditions $\Delta_2, \Delta_3, \Delta_4 < 0$, $\Delta_1, \Delta_5 > 0$, the fixed point structure corresponds to Figure 8, case b), where there exists a unique interior fixed point. If the eigenvalues of the Jacobian matrix at this point all have modulus less than one, the point is asymptotically stable and can be interpreted as an ESS.

The connection with replicator dynamics provides a biological interpretation of the behavior of Lotka–Volterra operators. Attracting interior fixed points behave as stable combinations of strategies or species, while saddle points correspond to unstable ecological or strategic equilibria.

4 Conclusion

In this work, we analyzed the structure of the set of fixed points — referred to as the *card of fixed points* — for Lotka–Volterra type operators defined on the standard simplex S^{m-1}. By associating these nonlinear maps with skew-symmetric matrices in general position, we established a correspondence between the dynamical system and directed graphs, particularly focusing on strong and homogeneous tournaments.

This graph-theoretical interpretation allowed us to classify the qualitative behavior of the system based on the topology of the corresponding tournament - including the presence of Hamiltonian cycles and internal fixed points. Analytical conditions were derived using the signs of even-order principal minors Δ_i, which determine the number and nature of fixed points. Additionally, Young's inequality was applied to obtain upper estimates for the evolution of invariant functions defined on simplex faces.

Beyond theoretical significance, the results of this study find direct applications in several domains where discrete population dynamics are modeled. In evolutionary biology, Lotka–Volterra operators serve as simplified models of frequency-dependent selection, where fixed points correspond to evolutionarily stable strategies (ESS). Interior fixed points represent coexistence states, while saddle points and repellers describe unstable or metastable configurations.

In socio-economic systems, such as market competition, opinion dynamics, or resource allocation, agent interactions can also be described using skew-symmetric structures. In this context, the tournament representation reflects dominance, influence, or preference relations. Therefore, the topological classification of fixed point cards provides insights into long-term system behavior based on interaction patterns.

The proposed approach can be further extended to systems with noise, spatial heterogeneity, or adaptive responses, making it a promising tool for modeling complex real-world phenomena. Future directions may include the development of algorithms to infer tournament structure from empirical data and applying the derived stability criteria to detect equilibrium configurations in evolutionary and economic games.

Acknowledgments

This research was funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (No. BR 27100483 "Development of predictive exploration technologies for identifying ore-prospective areas based on data analysis from the unified subsurface user platform "Minerals.gov.kz" using artificial intelligence and remote sensing methods").

Author Contributions

- **Dilfuza Bahramovna Eshmamatova** developed the mathematical model, formulated the main definitions, proved the key results, wrote the introduction and conclusion, interpreted the results in terms of evolutionary dynamics, contributed to the applications section.
- **Mokhbonu Akram khizi Tadzhieva** conducted the theoretical analysis, proved the key results, and derived the main estimates.

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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