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About unimprovability the embedding theorems for anisotropic Nikol'skii-Besov spaces with dominated mixed derivates and mixed metric and anisotropic Lorentz spaces

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The embedding theory of spaces of differentiable functions of many variables studies important connections and relationships between differential (smoothness) and metric properties of functions and has wide application in various branches of pure mathematics and its applications. Earlier, we obtained the embedding theorems of different metrics for Nikol'skii-Besov spaces with a dominant mixed smoothness and mixed metric, and anisotropic Lorentz spaces. In this work, we showed that the conditions for the parameters of spaces in the above theorems are unimprovable. To do this, we built the extreme functions included in the spaces from the left sides of the embeddings and not included in the "slightly narrowed" spaces from the spaces in the right parts of the embeddings.

Keywords: anisotropic Lorentz spaces, anisotropic Nikol'skii-Besov spaces, generalized mixed smoothness, mixed metric, embedding theorems.

2020 Mathematics Subject Classification: 46E35.

Introduction

One of the first results related to the theory of embedding of spaces of differentiable functions was a result of S.L. Sobolev [1]. This theory studies important relations of differential (smoothness) properties of functions in various metrics. Further development of this theory is associated with new classes of function spaces defined and studied in the works of S.M. Nikol'skii [2], O.V. Besov [3], P.I. Lizorkin [4], H. Triebel [5], J. Bergh and J. Löfström [6], and many others. The development of this research was determined both by its internal problems and by its applications in the theory of boundary value problems of mathematical physics and approximation theory (see, for example, [7–11]).

In the 1960s, in the works of S.M. Nikol'skii [1], A.D. Dzhabrailov [12] and T.I. Amanov [13] begins the study of spaces with a dominant mixed derivative. Further study of spaces with a dominant mixed derivative which is related with the theory of embedding and interpolation and the theory of approximations is associated with the works of A.P. Uninskij, V.N. Temlyakov, E.D. Nursultanov, D.B. Bazarkhanov, A.S. Romanyuk, G.A. Akishev, K.A. Bekmaganbetov, Ye. Toleugazy and others (see, for example, [14–20]).

In a serie of articles [21–23] we studied various properties of Nikol'skii-Besov spaces with a dominant mixed derivative and with a mixed metric. In these articles, we investigated the interpolation properties of these spaces, obtained limit embedding theorems for these spaces and anisotropic Lorentz spaces, and proved theorems on traces and continuations of functions.

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In the work of K.A. Bekmaganbetov, K.E. Kervenev, Ye. Toleugazy [22], embeddings for Nikol'skii-Besov spaces with a dominant mixed derivative and a mixed metric and anisotropic Lorentz spaces were studied. In this article we are showing that the conditions in the embedding theorems from the work [22] are unimprovable. We build the extreme functions included in the spaces from the left sides of the embeddings and not included in the "slightly narrowed" spaces from spaces in the right parts of the embeddings.

Preliniminaries and auxiliary results

Let $f(\mathbf{x}) = f(x_1, \ldots, x_n)$ be a measurable function defined on $\mathbb{T}^n$. Let multiindexes $\mathbf{1} \leq \mathbf{p} = (p_1, \ldots, p_n) \leq \infty$. A Lebesgue space $L_{\mathbf{p}}(\mathbb{T}^n)$ with mixed metric is the set of functions for which the following quantity is finite

$$\|f\|_{L_{\mathbf{p}}(\mathbb{T}^n)} = \left(\int_{\mathbb{T}} \left(\ldots \left(\int_{\mathbb{T}} |f(x_1,\ldots,x_n)|^{p_1} dx_1\right)^{p_2/p_1} \ldots\right)^{p_n/p_{n-1}} dx_n\right)^{1/p_n}.$$

Here, the expression $\left(\int_{\mathbb{T}} |f(t)|^p dt\right)^{1/p}$ for $p=\infty$ is understood as $\operatorname{ess\,sup}_{t\in\mathbb{T}} |f(t)|$.

For multiple trigonometric series $f(\mathbf{x}) \sim \sum_{\mathbf{k}\in\mathbb{Z}^n} a_{\mathbf{k}} e^{i(\mathbf{k},\mathbf{x})}$ we denote by

$$\Delta_{\mathbf{s}}(f, \mathbf{x}) = \sum_{\mathbf{k}\in\rho(\mathbf{s})} a_{\mathbf{k}}(f) e^{i(\mathbf{k},\mathbf{x})},$$

where $\rho(\mathbf{s}) = \{\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n : 2^{s_i - 1} \leq |k_i| < 2^{s_i}, i = 1, \ldots, n\}$, $(\mathbf{k}, \mathbf{x}) = \sum_{j=1}^{n} k_j x_j$ is the inner product of vectors $\mathbf{k}$ and $\mathbf{x}$.

Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ and $\mathbf{1} \leq \tau = (\tau_1, \ldots, \tau_n) \leq \infty$. The anisotropic Nikol'skii-Besov space with generalized mixed derivates and mixed metric $B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^n)$ is a set of the series $f \sim \sum_{\mathbf{k}\in\mathbb{Z}^{\mathbf{n}}} a_{\mathbf{k}} e^{i(\mathbf{k},\mathbf{x})}$ such that

$$\|f\|_{B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^n)} = \left\| \left\{ 2^{(\alpha,\mathbf{s})} \|\Delta_{\mathbf{s}}(f)\|_{L_{\mathbf{p}}(\mathbb{T}^n)} \right\} \right\|_{l_\tau} < \infty,$$

where $\|\cdot\|_{l_\tau}$ is the norm of a discrete Lebesgue space with mixed metric l_τ.

We will also need the anisotropic Lorentz spaces which introduced by E.D. Nursultanov in [24].

Let $f(\mathbf{x}) = f(x_1, \ldots, x_n)$ be a measurable function defined on $\mathbb{T}^n$. We denote by $f^*(\mathbf{t}) = f^{*_1,\ldots,*_n}(t_1, \ldots, t_n)$ the function obtained from $f(\mathbf{x}) = f(x_1, \ldots, x_n)$ by applying the non-increasing rearrangement successively with respect to each of the variables $x_1, \ldots, x_n$ (the other variables are assumed to be fixed).

Let multiindexes $\mathbf{q} = (q_1, \ldots, q_n)$, $\theta = (\theta_1, \ldots, \theta_n)$ satisfy the conditions: if $0 < q_j < \infty$, then $0 < \theta_j \leq \infty$, if $q_j = \infty$, then $\theta_j = \infty$ for every $j = 1, \ldots, n$. An anisotropic Lorentz space $L_{\mathbf{q}\theta}(\mathbb{T}^n)$ is the set of functions for which the following quantity is finite

$$\|f\|_{L_{\mathbf{q}\theta}(\mathbb{T}^n)} =$$

$$= \left(\int_{\mathbb{T}} \cdots \left(\int_{\mathbb{T}} \left(t_1^{1/q_1} \ldots t_n^{1/q_n} f^{*_1,\ldots,*_n}(t_1, \ldots, t_n)\right)^{\theta_1} \frac{dt_1}{t_1}\right)^{\theta_2/\theta_1} \cdots \frac{dt_n}{t_n}\right)^{1/\theta_n}.$$

The following theorems were obtained in the work [22]:

Theorem A. Let $-\infty < \alpha_{\mathbf{0}} = (\alpha_1^0, \ldots, \alpha_n^0) \leq \alpha_{\mathbf{1}} = (\alpha_1^1, \ldots, \alpha_n^1) < \infty$, $\mathbf{1} \leq \tau = (\tau_1, \ldots, \tau_n) \leq \infty$ and $\mathbf{1} < \mathbf{p_0} = (p_1^0, \ldots, p_n^0), \mathbf{p_1} = (p_1^1, \ldots, p_n^1) < \infty$. Then the embedding

$$B_{\mathbf{p_1}}^{\alpha_{\mathbf{1}}\tau}(\mathbb{T}^n) \hookrightarrow B_{\mathbf{p_0}}^{\alpha_{\mathbf{0}}\tau}(\mathbb{T}^n)$$

holds for $\alpha_{\mathbf{0}} - \mathbf{1}/\mathbf{p_0} = \alpha_{\mathbf{1}} - \mathbf{1}/\mathbf{p_1}$.

Theorem B. Let $\mathbf{1} < \mathbf{p} = (p_1, \ldots, p_n) < \mathbf{q} = (q_1, \ldots, q_n) < \infty$ and $\mathbf{1} \leq \tau = (\tau_1, \ldots, \tau_n) \leq \infty$. Then the embedding

$$B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^n) \hookrightarrow L_{\mathbf{q}\tau}(\mathbb{T}^n)$$

holds for $\alpha = \mathbf{1}/\mathbf{p} - \mathbf{1}/\mathbf{q}$.

Theorem C. Let $\mathbf{1} < \mathbf{q} = (q_1, \ldots, q_n) < \mathbf{p} = (p_1, \ldots, p_n) < \infty$ and $\mathbf{1} \leq \tau = (\tau_1, \ldots, \tau_n) \leq \infty$. Then the embedding

$$L_{\mathbf{q}\tau}(\mathbb{T}^n) \hookrightarrow B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^n)$$

holds for $\alpha = \mathbf{1}/\mathbf{p} - \mathbf{1}/\mathbf{q}$.

Main results

In this work, we show that the conditions for the parameters providing attachments are unimprovable. The proof of these facts in Theorems A–C we carry out by constructing extreme functions.

The following theorem shows that the condition under which the embedding from Theorem A is valid is unimprovable.

Theorem 1. Let $-\infty < \alpha_{\mathbf{0}} = (\alpha_1^0, \ldots, \alpha_n^0) \leq \alpha_{\mathbf{1}} = (\alpha_1^1, \ldots, \alpha_n^1) < \infty$, $\mathbf{1} \leq \tau = (\tau_1, \ldots, \tau_n) \leq \infty$, $\mathbf{1} < \mathbf{p_0} = (p_1^0, \ldots, p_n^0)$, $\mathbf{p_1} = (p_1^1, \ldots, p_n^1) < \infty$ and $\alpha_{\mathbf{0}} - \mathbf{1}/\mathbf{p_0} = \alpha_{\mathbf{1}} - \mathbf{1}/\mathbf{p_1}$, then for arbitrary $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) > \mathbf{0}$ and $\delta = (\delta_1, \ldots, \delta_n) > \mathbf{0}$ there is a function $f_\beta^{(1)} \in B_{\mathbf{p_1}}^{\alpha_{\mathbf{1}}\tau}(\mathbb{T}^n)$ such that $f_\beta^{(1)} \notin B^{(\alpha_{\mathbf{0}}+\varepsilon)\tau}_{\mathbf{p_0}}(\mathbb{T}^n) \cup B_{(\mathbf{p_0}+\delta)}^{\alpha_{\mathbf{0}}\tau}(\mathbb{T}^n)$.

Proof. Taking into account the estimate for the norm of a one-dimensional Dirichlet kernel, we obtain the relation

$$\left\| \sum_{k=2^{\mathbf{s}-1}}^{2^{\mathbf{s}}-1} e^{i(k,\cdot)} \right\|_{L_p(\mathbb{T})} \sim 2^{(1/p',s)}, 1 < p < +\infty.$$

From this relation in the multiple case we have

$$\|\sigma_{\mathbf{s}}(\cdot)\|_{L_{\mathbf{p}}(\mathbb{T}^n)} = \left\| \sum_{\mathbf{k}=2^{\mathbf{s}-1}}^{2^{\mathbf{s}}-1} e^{i(\mathbf{k},\cdot)} \right\|_{L_{\mathbf{p}}(\mathbb{T}^n)} \sim 2^{(\mathbf{1}/\mathbf{p}',\mathbf{s})}. \tag{1}$$

Consider the function $f_\beta^{(1)}(\mathbf{x}) = \sum_{\mathbf{s}=\mathbf{0}}^{\infty} 2^{-(\beta,\mathbf{s})}\sigma_{\mathbf{s}}(\mathbf{x})$, where

$$\alpha_{\mathbf{1}} + \frac{\mathbf{1}}{\mathbf{p_1'}} < \beta < \min\left(\alpha_{\mathbf{0}} + \varepsilon + \frac{1}{\mathbf{p}_0'}, \alpha_{\mathbf{0}} + \frac{\mathbf{1}}{(\mathbf{p_0}+\delta)'}\right).$$

According to estimate (1) we have

$$\left\| f_\beta^{(1)} \right\|_{B_{\mathbf{p_1}}^{\alpha_{\mathbf{1}}\tau}(\mathbb{T}^n)} = \left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty} \left(2^{(\alpha_{\mathbf{1}},\mathbf{s})} \left\| \Delta_{\mathbf{s}}(f_\beta^{(1)}) \right\|_{L_{\mathbf{p_1}}(\mathbb{T}^n)} \right)^{\tau} \right)^{\mathbf{1}/\tau} =$$

$$= \left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty} \left(2^{(\alpha_{\mathbf{1}},\mathbf{s})} 2^{-(\beta,\mathbf{s})} \left\| \sigma_{\mathbf{s}} \right\|_{L_{\mathbf{p_1}}(\mathbb{T}^n)} \right)^{\tau} \right)^{\mathbf{1}/\tau} =$$

$$= \left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty} \left(2^{(\alpha_{\mathbf{1}}-\beta+\frac{\mathbf{1}}{\mathbf{p}_1'},\mathbf{s})} \right)^{\tau} \right)^{\mathbf{1}/\tau} < +\infty,$$

as $\alpha_{\mathbf{1}} + \frac{\mathbf{1}}{\mathbf{p}_1'} - \beta < \mathbf{0}$.

This means that $f_{\beta}^{(1)} \in B_{\mathbf{p_0}}^{\alpha_{\mathbf{0}}\tau}(\mathbb{T}^n)$.

Similarly, we obtain that

$$\left\| f_{\beta}^{(1)} \right\|_{B_{\mathbf{p_0}}^{(\alpha_{\mathbf{0}}+\varepsilon)\tau}(\mathbb{T}^n)} = \left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty} \left(2^{(\alpha_{\mathbf{0}}+\varepsilon,\mathbf{s})} \left\| \Delta_{\mathbf{s}}(f_{\beta}^{(1)}) \right\|_{L_{\mathbf{p_0}}(\mathbb{T}^n)} \right)^{\tau} \right)^{\mathbf{1}/\tau} \geq$$

$$\geq C_1 \left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty} \left(2^{(\alpha_{\mathbf{0}}+\varepsilon,\mathbf{s})} 2^{\left(\frac{\mathbf{1}}{\mathbf{p}_0'}-\beta,\mathbf{s}\right)} \right)^{\tau} \right)^{\mathbf{1}/\tau} =$$

$$= C_1 \left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty} \left(2^{\left(\alpha_{\mathbf{0}}+\varepsilon+\frac{\mathbf{1}}{\mathbf{p}_0'}-\beta,\mathbf{s}\right)} \right)^{\tau} \right)^{\mathbf{1}/\tau} = +\infty,$$

as $\alpha_{\mathbf{0}} + \varepsilon + \frac{\mathbf{1}}{\mathbf{p}_0'} - \beta > \mathbf{0}$. Therefore $f_{\beta}^{(1)} \notin B_{\mathbf{p_0}}^{(\alpha_{\mathbf{0}}+\varepsilon)\tau}(\mathbb{T}^n)$.

Further, we will show that $f_{\beta}^{(1)} \notin B_{(\mathbf{p_0}+\delta)}^{\alpha_{\mathbf{0}}\ \tau}(\mathbb{T}^n)$. We have

$$\left\| f_{\beta}^{(1)} \right\|_{B_{(\mathbf{p_0}+\delta)}^{\alpha_{\mathbf{0}}\ \tau}(\mathbb{T}^n)} = \left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty} \left(2^{(\alpha_{\mathbf{0}},\mathbf{s})} \left\| \Delta_{\mathbf{s}}(f_{\beta}^{(1)}) \right\|_{L_{(\mathbf{p_0}+\delta)}(\mathbb{T}^n)} \right)^{\tau} \right)^{\mathbf{1}/\tau} \geq$$

$$\geq C_2 \left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty} \left(2^{(\alpha_{\mathbf{0}},\mathbf{s})} 2^{\left(\frac{\mathbf{1}}{(\mathbf{p_0}+\delta)'}-\beta,\mathbf{s}\right)} \right)^{\tau} \right)^{\mathbf{1}/\tau} =$$

$$= C_2 \left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty} \left(2^{\left(\alpha_{\mathbf{1}}+\frac{\mathbf{1}}{(\mathbf{p_0}+\delta)'}-\beta,\mathbf{s}\right)} \right)^{\tau} \right)^{\mathbf{1}/\tau} = +\infty,$$

considering that $\alpha_{\mathbf{0}} + \frac{\mathbf{1}}{(\mathbf{p_0}+\delta)'} - \beta > 0$. Therefore $f_{\beta}^{(1)} \notin B_{(\mathbf{p_0}+\delta)}^{\alpha_{\mathbf{0}}\ \tau}(\mathbb{T}^n)$.

Thus, we have shown that $f_{\beta}^{(1)} \notin B_{\mathbf{p_0}}^{(\alpha_{\mathbf{0}}+\varepsilon)\tau}(\mathbb{T}^n) \cup B_{(\mathbf{p_0}+\delta)}^{\alpha_{\mathbf{0}}\ \tau}(\mathbb{T}^n)$.

The proof is complete.

The following theorem shows that the condition under which the embedding of Theorem B is valid, is not improved.

Theorem 2. Let $\mathbf{1} < \mathbf{p} = (p_1, \ldots, p_n) < \mathbf{q} = (q_1, \ldots, q_n) < \infty$ and $\alpha = \mathbf{1}/\mathbf{p} - \mathbf{1}/\mathbf{q}$. Then for an arbitrary $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) > \mathbf{0}$ there is a function $f_{\beta}^{(2)} \in B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^n)$ such that $f_{\beta}^{(2)} \notin L_{\mathbf{q}+\varepsilon,\tau}(\mathbb{T}^n)$.

Proof. First, let's show that $f_{\beta}^{(2)} \in B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^n)$. Consider the function $f_{\beta}^{(2)}(\mathbf{x}) = \sum_{\mathbf{s}=\mathbf{0}}^{\infty} 2^{-(\beta,\mathbf{s})}\sigma_{\mathbf{s}}(\mathbf{x})$, where $\alpha + \frac{\mathbf{1}}{\mathbf{p}'} < \beta \leq \frac{1}{(\mathbf{q}+\varepsilon)'}$.

By analogy with Theorem 1, we have

$$\left\|f_\beta^{(2)}\right\|_{B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^n)} \sim \left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty}\left(2^{(\alpha-\beta+\frac{\mathbf{1}}{\mathbf{p}'},\mathbf{s})}\right)^{\tau}\right)^{\mathbf{1}/\tau} < \infty$$

as $\alpha+\frac{\mathbf{1}}{\mathbf{p}'}-\beta<\mathbf{0}$. It means that $f_\beta^{(2)}\in B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^n)$.

In order to show that $f_\beta^{(2)}\notin L_{\mathbf{q}+\varepsilon,\tau}(\mathbb{T}^n)$ we use Theorem C.

We have

$$\left\|f_\beta^{(2)}\right\|_{L_{\mathbf{q}+\varepsilon,\tau}(\mathbb{T}^n)} \geq \left\|f_\beta^{(2)}\right\|_{B_{\mathbf{p}}^{\frac{\mathbf{1}}{\mathbf{p}}-\frac{\mathbf{1}}{\mathbf{q}+\varepsilon},\tau}(\mathbb{T}^n)} =$$

$$= \left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty}\left(2^{\left(\frac{\mathbf{1}}{\mathbf{p}}-\frac{\mathbf{1}}{\mathbf{q}+\varepsilon},\mathbf{s}\right)}\left\|\Delta_{\mathbf{s}}(f_\beta^{(2)})\right\|_{L_{\mathbf{p}}(\mathbb{T}^n)}\right)^{\tau}\right)^{1/\tau} =$$

$$= C_3\left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty}\left(2^{\left(\frac{\mathbf{1}}{\mathbf{p}}-\frac{\mathbf{1}}{\mathbf{q}+\varepsilon},\mathbf{s}\right)}2^{\left(\frac{\mathbf{1}}{\mathbf{p}'}-\beta,\mathbf{s}\right)}\right)^{\tau}\right)^{1/\tau} =$$

$$= C_3\left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty}\left(2^{\left(\frac{\mathbf{1}}{\mathbf{p}}+\frac{\mathbf{1}}{\mathbf{p}'}-\frac{\mathbf{1}}{\mathbf{q}+\epsilon}-\beta,\mathbf{s}\right)}\right)^{\tau}\right)^{1/\tau} =$$

$$= C_3\left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty}\left(2^{(\mathbf{1}-\frac{\mathbf{1}}{\mathbf{q}+\varepsilon}-\beta,\mathbf{s})}\right)^{\tau}\right)^{1/\tau} = C_3\left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty}\left(2^{\left(\frac{\mathbf{1}}{(\mathbf{q}+\varepsilon)'}-\beta,\mathbf{s}\right)}\right)^{\tau}\right)^{1/\tau} = +\infty,$$

as $\frac{1}{(\mathbf{q}+\varepsilon)'}-\beta>0$. It means that $f_\beta^{(2)}\notin L_{\mathbf{q}+\varepsilon,\tau}(\mathbb{T}^n)$.

The proof is complete.

The following theorem shows that the condition, under which the embedding of Theorem C is valid, is not improved.

Theorem 3. Let $\mathbf{1}<\mathbf{q}=(q_1,\ldots,q_n)<\mathbf{p}=(p_1,\ldots,p_n)<\infty$ and $\alpha=\mathbf{1}/\mathbf{p}-\mathbf{1}/\mathbf{q}$. Then for an arbitrary $\varepsilon=(\varepsilon_1,\ldots,\varepsilon_n)>\mathbf{0}$ and $\delta=(\delta_1,\ldots,\delta_n)>\mathbf{0}$ there is a function $f_\beta^{(3)}\in L_{\mathbf{q},\tau}(\mathbb{T}^n)$ such that $f_\beta^{(3)}\notin B_{\mathbf{p}}^{\alpha+\epsilon,\tau}(\mathbb{T}^n)\cup B_{(\mathbf{p}+\delta)}^{\alpha\tau}(\mathbb{T}^n)$.

Proof. Let's choose a function $f_\beta^{(3)}(\mathbf{x})$ the same as in Theorem 1 with β, satisfying the condition $\frac{\mathbf{1}}{\mathbf{q}'}<\beta\leq\min\left(\alpha+\varepsilon+\frac{\mathbf{1}}{\mathbf{p}'},\alpha+\frac{\mathbf{1}}{(\mathbf{p}+\delta)'}\right)$.

In order to show that $f_\beta^{(3)}\in L_{\mathbf{q},\tau}(\mathbb{T}^n)$ let's use Theorem B. We have

$$\left\|f_\beta^{(3)}\right\|_{L_{\mathbf{q},\tau}(\mathbb{T}^n)} \leq C_4\left\|f_\beta^{(3)}\right\|_{B_{\mathbf{p}}^{(\frac{\mathbf{1}}{\mathbf{p}}-\frac{\mathbf{1}}{\mathbf{q}})\tau}(\mathbb{T}^n)} =$$

$$= C_4\left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty}\left(2^{\left(\frac{\mathbf{1}}{\mathbf{p}}-\frac{\mathbf{1}}{\mathbf{q}},\mathbf{s}\right)}\left\|\Delta_{\mathbf{s}}(f_\beta^{(3)})\right\|_{L_{\mathbf{p}}(\mathbb{T}^n)}\right)^{\tau}\right)^{1/\tau} \leq$$

$$\leq C_5\left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty}\left(2^{\left(\frac{\mathbf{1}}{\mathbf{p}}-\frac{\mathbf{1}}{\mathbf{q}},\mathbf{s}\right)}2^{\left(-\beta+\frac{\mathbf{1}}{\mathbf{p}'},\mathbf{s}\right)}\right)^{\tau}\right)^{1/\tau} =$$

$$= C_5\left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty}\left(2^{\left(\frac{\mathbf{1}}{\mathbf{p}}+\frac{\mathbf{1}}{\mathbf{p}'}-\beta-\frac{\mathbf{1}}{\mathbf{q}},\mathbf{s}\right)}\right)^{\tau}\right)^{1/\tau} =$$

$$= C_5 \left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty} \left(2^{(\mathbf{1}-\frac{\mathbf{1}}{\mathbf{q}}-\beta,\mathbf{s})} \right)^{\tau} \right)^{1/\tau} = C_5 \left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty} \left(2^{(\frac{\mathbf{1}}{\mathbf{q}'}-\beta,\mathbf{s})} \right)^{\tau} \right)^{1/\tau} < +\infty,$$

as $\frac{\mathbf{1}}{\mathbf{q}'} - \beta < \mathbf{0}$, i.e. $f_{\beta}^{(3)} \in L_{\mathbf{q},\tau}(\mathbb{T}^n)$.

Let's show that $f_{\beta} \notin B_{\mathbf{p}}^{\alpha\mathbf{q}}(\mathbb{T}^n)$.

Let us estimate the norm of this function from below

$$\left\| f_{\beta}^{(3)} \right\|_{B_{\mathbf{p}}^{(\alpha+\varepsilon)\tau}(\mathbb{T}^n)} = \left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty} \left(2^{(\alpha+\varepsilon,\mathbf{s})} \left\| \Delta_{\mathbf{s}}(f_{\beta}^{(3)}) \right\|_{L_{\mathbf{p}}(\mathbb{T}^n)} \right)^{\tau} \right)^{\mathbf{1}/\tau} \geq$$

$$\geq C_6 \left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty} \left(2^{(\alpha+\varepsilon,\mathbf{s})} 2^{\left(\frac{\mathbf{1}}{\mathbf{p}'}-\beta,\mathbf{s}\right)} \right)^{\tau} \right)^{\mathbf{1}/\tau} =$$

$$= C_6 \left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty} \left(2^{\left(\alpha+\varepsilon+\frac{\mathbf{1}}{\mathbf{p}'}-\beta,\mathbf{s}\right)} \right)^{\tau} \right)^{\mathbf{1}/\tau} = +\infty,$$

as $\alpha + \varepsilon - \beta + \frac{\mathbf{1}}{\mathbf{p}'} > \mathbf{0}$, i.e. $f_{\beta}^{(3)} \notin B_{\mathbf{p}}^{(\alpha+\varepsilon)\tau}(\mathbb{T}^n)$.

Further, we will show that $f_{\beta}^{(3)} \notin B_{(\mathbf{p_0}+\delta)}^{\alpha_{\mathbf{0}}\ \tau}(\mathbb{T}^n)$.

Considering that $\alpha_{\mathbf{0}} + \frac{\mathbf{1}}{(\mathbf{p_0}+\delta)'} - \beta > 0$, we have

$$\left\| f_{\beta}^{(3)} \right\|_{B_{(\mathbf{p_0}+\delta)}^{\alpha_{\mathbf{0}}\ \tau}(\mathbb{T}^n)} = \left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty} \left(2^{(\alpha_{\mathbf{0}},\mathbf{s})} \left\| \Delta_{\mathbf{s}}(f_{\beta}^{(3)}) \right\|_{L_{(\mathbf{p_0}+\delta)}(\mathbb{T}^n)} \right)^{\tau} \right)^{\mathbf{1}/\tau} \geq$$

$$\geq C_7 \left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty} \left(2^{(\alpha_{\mathbf{0}},\mathbf{s})} 2^{\left(\frac{\mathbf{1}}{(\mathbf{p_0}+\delta)'}-\beta,\mathbf{s}\right)} \right)^{\tau} \right)^{\mathbf{1}/\tau} =$$

$$= C_7 \left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty} \left(2^{\left(\alpha_{\mathbf{1}}+\frac{\mathbf{1}}{(\mathbf{p_0}+\delta)'}-\beta,\mathbf{s}\right)} \right)^{\tau} \right)^{\mathbf{1}/\tau} = +\infty,$$

as $f_{\beta}^{(3)} \notin B_{(\mathbf{p_0}+\delta)}^{\alpha_{\mathbf{0}}\ \tau}(\mathbb{T}^n)$.

Thus, we have shown that $f_{\beta}^{(3)} \notin B_{\mathbf{p_0}}^{(\alpha_{\mathbf{0}}+\varepsilon)\tau}(\mathbb{T}^n) \cup B_{(\mathbf{p_0}+\delta)}^{\alpha_{\mathbf{0}}\ \tau}(\mathbb{T}^n)$.

The proof is complete.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Үстем аралас туындысы және аралас метрикасы бар анизотропты Никольский-Бесов кеңістіктері және анизотропты Лоренц кеңістіктері үшін ену теоремаларының жетілдірілмейтіндігі туралы

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Дифференциалданатын функциялар кеңістіктерінің енгізу теориясы әртүрлі метрикалардағы функциялардың дифференциалдық (тегістіліктік) қасиеттерінің маңызды байланыстары мен қатынастарын зерттейді. Математикалық физиканың шектік есептер теориясында, жуықтау теориясында және математиканың басқа да салаларында кеңінен колданысқа ие. Мақалада үстем аралас тегістілігі және аралас метрикасы бар Никольский-Бесовтың кеңістіктері үшін және Лоренцтің анизотропты кеңістіктері үшін енгізу теоремалары берілген. Ұсынылған жұмыста жоғарыда көрсетілген теоремалардағы параметрлердің жетілдірілмейтіндігі көрсетілді. Осыны көрсетуге біз сол жақтағы енулердегі кеңістіктер үшін шекті функцияларды құрамыз және олар оң жақтағы енулерде "сәл ғана жіңішкертілген" кеңістіктерде жатпайтындығы көрсетілген.

Кілт сөздер: Лоренцтің анизотропты кеңістіктері, Никольский-Бесов типтес кеңістіктер, үстем аралас туынды, аралас метрика, ену теоремалары.

О неулучшаемости теорем вложения для анизотропных пространств Никольского-Бесова с доминирующей смешанной производной и смешанной метрикой и анизотропных пространств Лоренца

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Теория вложения пространств дифференцируемых функций многих переменных изучает важные связи и соотношения между дифференциальными (гладкостными) и метрическими свойствами функций и имеет широкое применение в различных разделах чистой математики и ее приложениях. Ранее нами были получены предельные теоремы вложения разных метрик для пространств Никольского–Бесова с доминирующей смешанной гладкостью и со смешанной метрикой и для анизотропных пространств Лоренца. В данной работе мы показали, что условия на параметры пространств в отмеченных выше теоремах являются неулучшаемыми. Для этого мы построили крайние функции, входящие в пространства из левых сторон вложений и не входящие в «немного зауженные» пространства, чем пространства, стоящие в правых частях вложений.

Ключевые слова: анизотропные пространства Лоренца, пространства Никольского–Бесова, доминирующая смешанная производная, смешанная метрика, теоремы вложения.

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