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## Characterizing the Ordered AG-Groupoids Through the Properties of Their Different Classes of Ideals

In this article, we have presented some important characterizations of the ordered non-associative semigroups in relation to their ideals. We have initially characterized the ordered AG-groupoid through the properties of the their ideals, then we characterized the two important classes of these AG-groupoids, namely the regular and intraregular non-associative AG-groupoids. Our aim is also to encourage the research and the maturity of the associative algebraic structures by studying a class of non-associative and non-commutative algebraic structures called the ordered AG-groupoid.

*Keywords:* Ordered AG-groupoids, left (right, interior, quasi-, bi-, generalized bi-) ideals, regular (intra-regular) ordered AG-groupoids.

### Introduction

In 1972, a generalization of commutative semigroups has been established by Kazim et. al [1]. In ternary commutative law:  $abc = cba$ , they introduced the braces on the left side of this law and explored a new pseudo associative law, that is  $(ab)c = (cb)a$ . They have called the left invertive law of this law. A groupoid  $S$  is said to be a left almost semigroup (abbreviated as LA-semigroup) if it satisfies the left invertive law :  $(ab)c = (cb)a$ . This structure is also known as Abel-Grassmann's groupoid (abbreviated as AG-groupoid) in [2]. An AG-groupoid is a midway structure between an abelian semigroup and a groupoid. Mushtaq et. al [3], investigated the concept of ideals in AG-groupoids.

In [4] (resp. [5]), a groupoid  $S$  is said to be medial (resp. paramedial) if  $(ab)(cd) = (ac)(bd)$  (resp.  $(ab)(cd) = (db)(ca)$ ). In [1], an AG-groupoid is medial, but in general an AG-groupoid needs not to be paramedial. Every AG-groupoid with left identity is paramedial by Protic et. al [2] and also satisfies  $a(bc) = b(ac)$ ,  $(ab)(cd) = (dc)(ba)$ .

In [6, 7], if  $(S, \cdot, \leq)$  is an ordered semigroup and  $\emptyset \neq A \subseteq S$ , we define a subset of  $S$  as follows :  $[A] = \{s \in S : s \leq a \text{ for some } a \in A\}$ . A non-empty subset  $A$  of  $S$  is called a subsemigroup of  $S$  if  $A^2 \subseteq A$ .

A non-empty subset  $A$  of  $S$  is called a left (resp. right) ideal of  $S$  if following hold (1)  $SA \subseteq A$  (resp.  $AS \subseteq A$ ). (2) If  $a \in A$  and  $b \in S$  such that  $b \leq a$  implies  $b \in A$ . Equivalent definition:  $A$  is called a left(resp. right) ideal of  $S$  if  $[A] \subseteq A$  and  $SA \subseteq A$  (resp.  $AS \subseteq A$ ).

A non-empty subset  $A$  of  $S$  is called an interior (resp. quasi-) ideal of  $S$  if (1)  $SAS \subseteq A$  (resp.  $(AS] \cap (SA] \subseteq A$ ). (2) If  $a \in A$  and  $b \in S$  such that  $b \leq a$  implies  $b \in A$ .

A subsemigroup (A non-empty subset)  $A$  of  $S$  is called a bi- (generalized bi-) ideal of  $S$  if (1)  $ASA \subseteq A$ . (2) If  $a \in A$  and  $b \in S$  such that  $b \leq a$  implies  $b \in A$ . Every bi-ideal of  $S$  is a generalized bi-ideal of  $S$ .

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In [7, 8], an ordered semigroup is said to be regular if for every  $a \in S$ , there exists an element  $x \in S$  such that  $a \leq axa$ . Equivalent definitions are as follows: (1)  $A \subseteq (ASA)$  for every  $A \subseteq S$ . (2)  $a \in (aSa)$  for every  $a \in S$ .

In [9, 10], an ordered semigroup  $S$  is intra-regular if for every  $a \in S$  there exist elements  $x, y \in S$  such that  $a \leq xa^2y$ . Equivalent definitions are as follows: (1)  $A \subseteq (SA^2S)$  for every  $A \subseteq S$ . (2)  $a \in (Sa^2S)$  for every  $a \in S$ .

We will define left (right, interior, quasi-, bi-, generalized bi-) ideals in ordered AG-groupoids. We will establish a study by discussing the different properties of such ideals. We will also characterize regular (resp. intra-regular, both regular and intra-regular) ordered AG-groupoids by the properties of left (right, quasi-, bi-, generalized bi-) ideals.

### Ideals in Ordered AG-groupoids

An ordered AG-groupoid  $S$ , is a partially ordered set, at the same time an AG-groupoid such that  $a \leq b$ , implies  $ac \leq bc$  and  $ca \leq cb$  for all  $a, b, c \in S$ . Two conditions are equivalent to the one condition  $(ca)d \leq (cb)d$  for all  $a, b, c, d \in S$ .

*Example 1.* Consider a set  $S = \{e, f, a, b, c\}$  with the following multiplication “ $\cdot$ ” and order relation “ $\leq$ ”

$\cdot$	$e$	$f$	$a$	$b$	$c$
$e$	$e$	$f$	$a$	$b$	$c$
$f$	$f$	$f$	$f$	$b$	$e$
$a$	$a$	$f$	$c$	$b$	$c$
$b$	$c$	$c$	$c$	$f$	$b$
$c$	$b$	$b$	$b$	$c$	$f$

$$\leq = \{(e, e), (e, a), (e, b), (e, c), (f, f), (f, b), (f, c), (a, a), (a, c), (b, b), (b, c), (c, c)\}.$$

Then  $(S, \cdot, \leq)$  is an ordered AG-groupoid with left identity  $e$ .

For  $\emptyset \neq A \subseteq S$ , we define a subset  $[A] = \{s \in S : s \leq a \text{ for some } a \in A\}$  of  $S$  and obviously  $A \subseteq [A]$ . For  $\emptyset \neq A, B \subseteq S$ , then  $([A]) = [A], [A][B] \subseteq [AB], ([A][B]) = [AB]$ , if  $A \subseteq B$ , then  $[A] \subseteq [B], (A \cap B) \neq [A] \cap [B]$ , in general.

For  $\emptyset \neq A \subseteq S$ . Then  $A$  is called an ordered AG-subgroupoid of  $S$  if  $A^2 \subseteq A$ .  $A$  is called a left (resp. right) ideal of  $S$  if the following hold (1)  $SA \subseteq A$  (resp.  $AS \subseteq A$ ). (2) If  $a \in A$  and  $b \in S$  such that  $b \leq a$  implies  $b \in A$ .  $A$  is called an ideal of  $S$  if  $A$  is both a left and a right ideal of  $S$ .

We denote by  $L(a), R(a), I(a)$  the left ideal, the right ideal and the ideal of  $S$ , respectively, generated by  $a$ . we have  $L(a) = \{s \in S : s \leq a \text{ or } s \leq xa \text{ for some } x \in S\} = (a \cup Sa], R(a) = (a \cup aS], I(a) = (a \cup Sa \cup aS \cup (Sa)S]$ .

A non-empty subset  $A$  of an ordered AG-groupoid  $S$  is called an interior (resp. quasi-) ideal of  $S$  if (1)  $(SA)S \subseteq A$  (resp.  $(AS) \cap (SA) \subseteq A$ ). (2) If  $a \in A$  and  $b \in S$  such that  $b \leq a$  implies  $b \in A$ .

An AG-subgroupoid  $A$  of  $S$  is called a bi-ideal of  $S$  if (1)  $(AS)A \subseteq A$ . (2) If  $a \in A$  and  $b \in S$  such that  $b \leq a$  implies  $b \in A$ . A non-empty subset  $A$  of  $S$  is called generalized bi-ideal of  $S$  if (1)  $(AS)A \subseteq A$ . (2) If  $a \in A$  and  $b \in S$  such that  $b \leq a$  implies  $b \in A$ .

Now we give the imperative properties of such ideals of an ordered AG-groupoid  $S$ , which will be play a vital rule in the later sections. Specifically we show:

(1) Let  $S$  be an ordered AG-groupoid with left identity  $e$ . Then every right ideal of  $S$  is a ideal of  $S$ .

(2) Let  $S$  be an ordered AG-groupoid with left identity  $e$ , such that  $(xe)S = xS$  for all  $x \in S$ . Then every quasi-ideal of  $S$  is a bi-ideal of  $S$ .

*Lemma 1.* Let  $S$  be an ordered AG-groupoid with left identity  $e$ . Then  $SS = S$  and  $eS = S = Se$ .

*Proof:* Since  $SS \subseteq S$  and  $x = ex \in SS$ , i.e.,  $S \subseteq SS$ , thus  $SS = S$ . Obviously,  $eS = S$  and  $Se = (SS)e = (eS)S = SS = S$ .

*Lemma 2.* Let  $S$  be an ordered AG-groupoid with left identity  $e$  and  $a \in S$ . Then  $Sa$  is a smallest left ideal of  $S$  containing  $a$ .

Proof: Let  $x \in Sa$  and  $s \in S$ , this implies that  $x = s_1a$ , where  $s_1 \in S$ . Now

$$\begin{aligned} sx &= s(s_1a) = (es)(s_1a) = ((s_1a)s)e = ((s_1a)(es))e \\ &= ((s_1e)(as))e = (e(as))(s_1e) = (as)(s_1e) = ((s_1e)s)a \in Sa. \end{aligned}$$

Thus  $sx \in Sa$  and  $(Sa] \subseteq Sa$ . Since  $a = ea \in Sa$ , hence  $Sa$  is a left ideal of  $S$  containing  $a$ . Let  $I$  be another left ideal of  $S$  containing  $a$ . Since  $sa \in I$ , because  $I$  is a left ideal of  $S$ . But  $sa \in Sa$ , this means that  $Sa \subseteq I$ . Therefore  $Sa$  is a smallest left ideal of  $S$  containing  $a$ .

*Lemma 3.* Let  $S$  be an ordered AG-groupoid with left identity  $e$  and  $a \in S$ . Then  $aS$  is a left ideal of  $S$ .

Proof: Straight forward.

*Proposition 1.* Let  $S$  be an ordered AG-groupoid with left identity  $e$  and  $a \in S$ . Then  $aS \cup Sa$  is a smallest right ideal of  $S$  containing  $a$ .

Proof: Let  $x \in aS \cup Sa$ . We have to show that  $(aS \cup Sa)S \subseteq aS \cup Sa$ . Now

$$\begin{aligned} (aS \cup Sa)S &= (aS)S \cup (Sa)S = (SS)a \cup (Sa)(eS) \\ &\subseteq Sa \cup (Se)(aS) = Sa \cup S(aS) \\ &= Sa \cup a(SS) \subseteq Sa \cup aS = aS \cup Sa. \end{aligned}$$

Thus  $(aS \cup Sa)S \subseteq aS \cup Sa$  and  $(aS \cup Sa] \subseteq aS \cup Sa$ . Therefore  $aS \cup Sa$  is a right ideal of  $S$ . Since  $a \in Sa$ , i.e.,  $a \in aS \cup Sa$ . Let  $I$  be another right ideal of  $S$  containing  $a$ . Now  $aS \in IS \subseteq I$  and  $Sa = (SS)a = (aS)S \in (IS)S \subseteq IS \subseteq I$ , i.e.,  $aS \cup Sa \subseteq I$ . Hence  $aS \cup Sa$  is a smallest right ideal of  $S$  containing  $a$ .

*Lemma 4.* Let  $S$  be an ordered AG-groupoid with left identity  $e$ . Then every right ideal of  $S$  is an ideal of  $S$ .

Proof: Let  $R$  be a right ideal of  $S$  and  $r \in R, s \in S$ . Now  $sr = (es)r = (rs)e \in (RS)S \subseteq RS \subseteq R$ . Thus  $SR \subseteq R$  and  $(R] \subseteq R$ . Hence  $R$  is an ideal of  $S$ .

*Lemma 5.* Let  $S$  be an ordered AG-groupoid with left identity  $e$  such that  $(xe)S = xS$  for all  $x \in S$ . Then  $(AS)S \subseteq AS$  and  $(AS]S \subseteq (AS]$ .

Proof: Since

$$\begin{aligned} (AS)S &= (AS)(eS) = (Ae)(SS) \subseteq (Ae)S = AS. \\ \text{and } (AS]S &= (AS](S] \subseteq ((AS)S] \subseteq (AS]. \end{aligned}$$

*Remark 1.* Let  $S$  be an ordered AG-groupoid with left identity  $e$  such that  $(xe)S = xS$  for all  $x \in S$ , then  $(AS]$  is an ideal of  $S$ .

Let  $S$  be an ordered AG-groupoid with left identity  $e$  such that  $(xe)S = xS$  for all  $x \in S$  and  $A, B \subseteq S$ . Then  $(AS)(BS) \subseteq (AB)S$  and  $(AS](BS] \subseteq ((AB)S]$ . Similarly  $(SA)(SB) \subseteq S(AB)$  and  $(SA](SB] \subseteq (S(AB))]$ .

In general for  $A_1, A_2, \dots, A_n \subseteq S$ , then  $(A_1S)(A_2S)\dots(A_nS) \subseteq (A_1A_2, \dots, A_n)S$  and  $(A_1S](A_2S] \dots (A_nS] \subseteq ((A_1A_2, \dots, A_n)S]$ .

Similarly,  $(SA_1)(SA_2)\dots(SA_n) \subseteq S(A_1A_2, \dots, A_n)$  and  $(SA_1](SA_2] \dots (SA_n] \subseteq (S(A_1A_2, \dots, A_n))]$ .

*Lemma 6.* Let  $S$  be an ordered AG-groupoid.  $A$  is a right ideal of  $S$  and  $B$  is a right ideal of  $A$ , then  $(B] = B$ .

Proof: Since  $(B] = \{s \in S \mid s \leq b \text{ for some } b \in B\}$  and  $s \in (B]$ , this implies that there exists an element  $s \in S$  such that  $s \leq b$  for some  $b \in B \subseteq A$ . Thus  $S \ni s \leq b \in A$ . Now  $A \ni s \leq b \in B$  and  $B$  is a right ideal of  $A$ , i.e.,  $s \in B$ , so  $(B] \subseteq B$ . Since  $B \subseteq (B]$ , thus  $(B] = B$ .

*Proposition 2.* Let  $S$  be an ordered AG-groupoid with left identity  $e$  such that  $(xe)S = xS$  for all  $x \in S$ .  $A$  is a right ideal of  $S$  and  $B$  is a right ideal of  $A$  such that  $(B^2] = B$ . Then  $B$  is an ideal of  $S$ .

Proof: We have to show that  $B$  is a right ideal of  $S$ . Now

$$\begin{aligned} BS &= (B^2]S = (B^2](S] \subseteq (B^2S] = ((BB)S] \\ &= ((SB)B] \subseteq ((SB)A] = ((SB)(eA)] \\ &= ((Se)(BA)] = (B((Se)A)] = (B((Ae)S)] \\ &= (B(AS)] \subseteq (BA] \subseteq (B] = B \text{ by the Lemma 6.} \end{aligned}$$

Thus  $BS \subseteq B$  and  $(B] \subseteq B$ , i.e.,  $B$  is a right ideal of  $S$ . Hence  $B$  is an ideal of  $S$  by the Lemma 4.

*Lemma 7.* Let  $S$  be an ordered AG-groupoid.  $A$  is a left ideal of  $S$  and  $B$  is a left ideal of  $A$ , then  $(B] = B$ .

Proof: Same as Lemma 6.

*Proposition 3.* Let  $S$  be an ordered AG-groupoid with left identity  $e$ .  $A$  is a left ideal of  $S$  and  $B$  is a left ideal of  $A$  such that  $(B^2] = B$ . Then  $B$  is left ideal of  $S$ .

Proof: We have to show that  $B$  is a left ideal of  $S$ . Now

$$\begin{aligned} SB &= S(B^2] = (S](B^2] \subseteq (SB^2] = (S(BB)] \\ &= ((Se)(BB)] = ((SB)(eB)] \\ &\subseteq ((SA)(eB)] \subseteq (AB] \subseteq (B] = B, \text{ by the Lemma 7.} \end{aligned}$$

Thus  $SB \subseteq B$  and  $(B] \subseteq B$ . Hence  $B$  is a left ideal of  $S$ .

*Lemma 8.* Every two-sided ideal of  $S$  is an interior ideal of  $S$ .

Proof: Straight forward.

*Proposition 4.* Let  $S$  be an ordered AG-groupoid with left identity  $e$ . Then any non-empty subset  $I$  of  $S$  is an ideal of  $S$  if and only if  $I$  is an interior ideal of  $S$ .

Proof: Suppose that  $I$  is an interior ideal of  $S$ . Let  $i \in I$  and  $s \in S$ . Now  $is = (ei)s \in (SI)S \subseteq I$ , this implies that  $IS \subseteq I$  and  $(I] \subseteq I$ , i.e.,  $I$  is a right ideal of  $S$ . Hence  $I$  is an ideal of  $S$  by the Lemma 4. Converse is true by the Lemma 8.

*Lemma 9.* Every right (two-sided) ideal of  $S$  is a bi-ideal of  $S$ .

Proof: Straight forward.

*Lemma 10.* Every bi-ideal of  $S$  is a generalized bi-ideal of  $S$ .

Proof: Obvious.

*Lemma 11.* Every left (right, two-sided) ideal of  $S$  is a quasi-ideal of  $S$ .

Proof: Let  $I$  be a right ideal of  $S$ . Now  $(IS] \cap (SI] \subseteq (IS] \subseteq (I] \subseteq I$  and  $(I] \subseteq I$ . Thus  $I$  is a quasi-ideal of  $S$ .

*Proposition 5.* Every quasi-ideal of  $S$  is an ordered AG-subgroupoid of  $S$ .

Proof: Suppose that  $I$  is a quasi-ideal of  $S$ . Now  $II \subseteq IS \subseteq (I](S] \subseteq (IS]$  and  $II \subseteq SI \subseteq (S](I] \subseteq (SI]$ , i.e.,  $I^2 = II \subseteq (IS] \cap (SI] \subseteq I$ . Hence  $I$  is an AG-subgroupoid of  $S$ .

*Proposition 6.* Let  $R$  be a right ideal and  $L$  be a left ideal of an ordered AG-groupoid  $S$ , respectively. Then  $R \cap L$  is a quasi-ideal of  $S$ .

Proof: Since  $((R \cap L)S] \cap (S(R \cap L)] \subseteq (RS] \cap (SL] \subseteq (R] \cap (L] \subseteq R \cap L$  and  $(R \cap L] = R \cap L$ . Thus  $R \cap L$  is a quasi-ideal of  $S$ .

*Lemma 12.* Let  $S$  be an ordered AG-groupoid with left identity  $e$  such that  $(xe)S = xS$  for all  $x \in S$ . Then every quasi-ideal of  $S$  is a bi-ideal of  $S$ .

Proof: Let  $Q$  be a quasi-ideal of  $S$ . Now  $(QS)Q \subseteq (SS)Q \subseteq SQ \subseteq (SQ]$  and  $(QS)Q \subseteq (QS)S = (QS)(eS) = (Qe)(SS) = (Qe)S = QS \subseteq (QS]$ , thus  $(QS)Q \subseteq (QS] \cap (SQ] \subseteq Q$ . Therefore  $(QS)Q \subseteq Q$  and  $(Q] \subseteq Q$ . Hence  $Q$  is a bi-ideal of  $S$ .

## Regular Ordered AG-groupoids

An ordered AG-groupoid  $S$  is called regular if for every  $a \in S$ , there exists an element  $x \in S$  such that  $a \leq (ax)a$ . Equivalent definitions are as follows:

- (1)  $A \subseteq ((AS)A]$  for every  $A \subseteq S$ .
- (2)  $a \in ((aS)a]$  for every  $a \in S$ .

An ideal  $I$  of an ordered AG-groupoid  $S$  is called idempotent if  $(I^2] = I$ .

In this section, we characterize regular ordered AG-groupoids by the properties of (left, right, quasi-, bi-, generalized bi-) ideals.

*Lemma 13.* Every right ideal of a regular ordered AG-groupoid  $S$

Proof: Let  $R$  be a right ideal of  $S$ . Let  $r \in R$  and  $a \in S$ , this implies that there exists an element  $x \in S$  such that  $a \leq (ax)a$ . Now  $ar \leq ((ax)a)r = (ra)(ax) \in RS \subseteq R$ , thus  $SR \subseteq R$  and  $(R] = R$ . Hence  $R$  is an ideal of  $S$ .

*Lemma 14.* Every ideal of a regular ordered AG-groupoid  $S$  is an idempotent.

Proof: Suppose that  $I$  is an ideal of  $S$  and  $(I^2] = (II] \subseteq (I] = I$ . Let  $a \in I$ , this means that there exists an element  $x \in S$  such that  $a \leq (ax)a$ . Now  $a \leq (ax)a \in (IS)I \subseteq II = I^2$ , i.e.,  $I \subseteq (I^2]$ . Therefore  $(I^2] = I$ .

*Remark 2.* Every right ideal of a regular ordered AG-groupoid  $S$  is an idempotent.

*Proposition 7.* Let  $S$  be a regular ordered AG-groupoid. Then any non-empty subset  $I$  of  $S$  is an ideal of  $S$  if and only if  $I$  is an interior ideal of  $S$ .

Proof: Assume that  $I$  is an interior ideal of  $S$ . Let  $a \in I$  and  $s \in S$ , then there exists an element  $x \in S$ , such that  $a \leq (ax)a$ . Now  $as \leq ((ax)a)s = (sa)(ax) \in (SI)S \subseteq I$ . Thus  $IS \subseteq I$  and  $(I] = I$ , i.e.,  $I$  is a right ideal of  $S$ . Hence  $I$  is an ideal of  $S$  by the Lemma 4. Converse is true by the Lemma 13.

*Proposition 8.* Let  $S$  be a regular ordered AG-groupoid with left identity  $e$ . Then  $(IS] \cap (SI] = I$ , for every right ideal  $I$  of  $S$ .

Proof: Let  $I$  be an ideal of  $S$ . This implies that  $(IS] \cap (SI] \subseteq I$ , because every ideal of  $S$  is a quasi-ideal of  $S$ . Let  $a \in I$ , this means that there exists an element  $x \in S$  such that  $a \leq (ax)a$ . Now  $a \leq (ax)a \in (IS)I \subseteq II \subseteq IS$ , i.e.,  $I \subseteq (IS]$ . Now  $a \leq (ax)a = (ax)(ea) = (ae)(xa) \in II \subseteq SI$ , i.e.,  $I \subseteq (SI]$ . Thus  $I \subseteq (IS] \cap (SI]$ . Hence  $(IS] \cap (SI] = I$ .

*Lemma 15.* Let  $S$  be a regular ordered AG-groupoid. Then  $(RL] = R \cap L$ , for every right ideal  $R$  and every left ideal  $L$  of  $S$ .

Proof: Since  $(RL] \subseteq (RS] \subseteq (R] = R$  and  $(RL] \subseteq (SL] \subseteq (L] = L$ , i.e.,  $(RL] \subseteq R \cap L$ . Let  $a \in R \cap L$ , this implies that there exists an element  $x \in S$  such that  $a \leq (ax)a$ . Now  $a \leq (ax)a \in (RS)L \subseteq RL$ , i.e.,  $R \cap L \subseteq (RL]$ . Therefore  $(RL] = R \cap L$ .

*Theorem 1.* Let  $S$  be an ordered AG-groupoid with left identity  $e$  such that  $(xe)S = xS$  for all  $x \in S$ . Then the following conditions are equivalent.

- (1)  $S$  is a regular.
- (2)  $R \cap L = (RL]$  for every right ideal  $R$  and every left ideal  $L$  of  $S$ .
- (3)  $Q = ((QS)Q]$  for every quasi-ideal  $Q$  of  $S$ .

Proof: Suppose that (1) holds. Let  $Q$  be a quasi-ideal of  $S$  and  $a \in Q$ , this implies that there exists an element  $x \in S$  such that  $a \leq (ax)a$ . Now  $a \leq (ax)a \in (QS)Q$ , i.e.,  $Q \subseteq ((QS)Q] \subseteq (Q] = Q$ , because every quasi-ideal of  $S$  is a bi-ideal of  $S$ . Hence  $Q = ((QS)Q]$ , i.e., (1)  $\Rightarrow$  (3). Assume that (3) holds, let  $R$  be a right ideal and  $L$  be a left ideal of  $S$ . Then  $R$  and  $L$  be quasi-ideals of  $S$  by the Lemma 11, so  $R \cap L$  be a quasi-ideal of  $S$ . Now  $R \cap L = (((R \cap L)S)(R \cap L)] \subseteq ((RS)L] \subseteq (RL]$ . Since  $(RL] \subseteq R \cap L$ , so  $(RL] = R \cap L$ , i.e., (3)  $\Rightarrow$  (2). Suppose that (2) is true, let  $a \in S$ , then  $Sa$  is a left ideal of  $S$  containing  $a$  by the Lemma 2 and  $aS \cup Sa$  is a right ideal of  $S$  containing  $a$  by the

Proposition 1. By (2),

$$\begin{aligned} (aS \cup Sa) \cap Sa &= ((aS \cup Sa)(Sa)] = ((aS)(Sa) \cup (Sa)(Sa)]. \\ (Sa)(Sa) &= ((Se)a)(Sa) = ((ae)S)(Sa) = (aS)(Sa). \end{aligned}$$

Thus

$$\begin{aligned} (aS \cup Sa) \cap Sa &= ((aS)(Sa) \cup (Sa)(Sa)] \\ &= ((aS)(Sa) \cup (aS)(Sa)] = ((aS)(Sa)]. \end{aligned}$$

Since  $a \in (aS \cup Sa) \cap Sa$ , Implies  $a \in ((aS)(Sa)]$ . Then  $a \leq (ax)(ya) = ((ya)x)a = (((ey)a)x)a = (((ay)e)x)a = ((xe)(ay))a = (a((xe)y))a \in (aS)a$  for any  $x, y \in S$ , i.e.,  $a \in ((aS)a]$ . Hence  $a$  is regular, so  $S$  is a regular, i.e., (2)  $\Rightarrow$  (1).

*Theorem 2.* Let  $S$  be an ordered AG-groupoid with left identity  $e$  such that  $(xe)S = xS$  for all  $x \in S$ . Then the following conditions are equivalent.

- (1)  $S$  is a regular.
- (2)  $Q = ((QS)Q]$  for every quasi-ideal  $Q$  of  $S$ .
- (3)  $B = ((BS)B]$  for every bi-ideal  $B$  of  $S$ .
- (4)  $G = ((GS)G]$  for every generalized bi-ideal  $G$  of  $S$ .

Proof: (1)  $\Rightarrow$  (4), is obvious. (4)  $\Rightarrow$  (3), since every bi-ideal of  $S$  is a generalized bi-ideal of  $S$  by the Lemma 10. (3)  $\Rightarrow$  (2), since every quasi-ideal of  $S$  is bi-ideal of  $S$  by the Lemma 12. (2)  $\Rightarrow$  (1), by the Theorem 1.

*Theorem 3.* Let  $S$  be an ordered AG-groupoid with left identity  $e$  such that  $(xe)S = xS$  for all  $x \in S$ . Then the following conditions are equivalent.

- (1)  $S$  is a regular.
- (2)  $Q \cap I = ((QI)Q]$  for every quasi-ideal  $Q$  and every ideal  $I$  of  $S$ .
- (3)  $B \cap I = ((BI)B]$  for every bi-ideal  $B$  and every ideal  $I$  of  $S$ .
- (4)  $G \cap I = ((GI)G]$  for every generalized bi-ideal  $G$  and every ideal  $I$  of  $S$ .

Proof: Suppose that (1) is true. Let  $G$  be a generalized bi-ideal and  $I$  be an ideal of  $S$ . Now  $((GI)G] \subseteq ((SI)S] \subseteq (I] = I$  and  $((GI)G] \subseteq ((GS)G] \subseteq (G] = G$ , thus  $((GI)G] \subseteq G \cap I$ . Let  $a \in G \cap I$ , this means that there exists an element  $x \in S$  such that  $a \leq (ax)a$ . Now  $a \leq (ax)a = (((ax)a)x)a = ((xa)(ax))a = (a((xa)x))a \in (GI)G$ , thus  $G \cap I \subseteq ((GI)G]$ . Hence  $G \cap I = ((GI)G]$ , i.e., (1)  $\Rightarrow$  (4). (4)  $\Rightarrow$  (3), since every bi-ideal of  $S$  is a generalized bi-ideal of  $S$  by the Lemma 10. (3)  $\Rightarrow$  (2), since every quasi-ideal of  $S$  is a bi-ideal of  $S$  by the Lemma 12. Assume that (2) is true. Now  $Q \cap S = ((QS)Q]$ , i.e.,  $Q = ((QS)Q]$ , where  $Q$  is a quasi-ideal of  $S$ . Hence  $S$  is a regular by the Theorem 1.

*Theorem 4.* Let  $S$  be an ordered AG-groupoid with left identity  $e$  such that  $(xe)S = xS$  for all  $x \in S$ . Then the following conditions are equivalent.

- (1)  $S$  is a regular.
- (2)  $R \cap Q \subseteq (RQ]$  for every quasi-ideal  $Q$  and every right ideal  $R$  of  $S$ .
- (3)  $R \cap B \subseteq (RB]$  for every bi-ideal  $B$  and every right ideal  $R$  of  $S$ .
- (4)  $R \cap G \subseteq (RG]$  for every generalized bi-ideal  $G$  and every right ideal  $R$  of  $S$ .

Proof: (1)  $\Rightarrow$  (4), is obvious. (4)  $\Rightarrow$  (3), since every bi-ideal of  $S$  is a generalized bi-ideal of  $S$ . (3)  $\Rightarrow$  (2), since every quasi-ideal of  $S$  is a bi-ideal of  $S$  by the Lemma 12. . Suppose that (2) is true. Now  $R \cap Q = Q \cap R \subseteq (RQ]$ , where  $Q$  is a left ideal and  $R$  is right ideal of  $S$ , because every left ideal of  $S$  is a quasi-ideal of  $S$ . Since  $(RQ] \subseteq R \cap Q$ , thus  $R \cap Q = (RQ]$ . Hence  $S$  is a regular, by the Theorem 1.

*Intra-regular Ordered AG-groupoids*

An ordered AG-groupoid  $S$  is called intra-regular if for every  $a \in S$ , there exist elements  $x, y \in S$  such that  $a \leq (xa^2)y$ . Equivalent definitions are as follows:

- (1)  $A \subseteq ((SA^2)S]$  for every  $A \subseteq S$ .
- (2)  $a \in ((Sa^2)S]$  for every  $a \in S$ .

In this section, we characterize intra-regular ordered AG-groupoids by the properties of (left, right, quasi-, bi-, generalized bi-) ideals.

*Lemma 16.* Every left (right) ideal of an intra-regular ordered AG-groupoid  $S$  is an ideal of  $S$ .

Proof: Let  $R$  be a right ideal of  $S$ . Let  $r \in R$  and  $a \in S$ , this implies that there exist elements  $x, y \in S$  such that  $a \leq (xa^2)y$ . Now  $ar \leq ((xa^2)y)r = (ry)(xa^2) \in RS \subseteq R$ . Thus  $SR \subseteq R$  and  $(R] \subseteq R$ . Hence  $R$  is an ideal of  $S$ .

*Lemma 17.* Every ideal of an intra-regular ordered AG-groupoid  $S$  with left identity  $e$ , is an idempotent.

Proof: Suppose that  $I$  is an ideal of  $S$  and  $(I^2] = (II] \subseteq (I] = I$ . Let  $a \in I$ , this means that there exist elements  $x, y \in S$  such that  $a \leq (xa^2)y$ . Now

$$\begin{aligned} a &\leq (xa^2)y = (x(aa))y = (a(xa))y \\ &= (a(xa))(ey) = (ae)((xa)y) = (xa)((ae)y) \in II. \end{aligned}$$

Thus  $a \in (II] = (I^2]$ . Therefore  $(I^2] = I$ .

*Proposition 9.* Let  $S$  be an intra-regular ordered AG-groupoid with left identity  $e$ . Then any non-empty subset  $I$  of  $S$  is an ideal of  $S$  if and only if  $I$  is an interior ideal of  $S$ .

Proof: Assume that  $I$  is an interior ideal of  $S$ . Let  $i \in I$  and  $a \in S$ , then there exist elements  $x, y \in S$  such that  $x \leq (yx^2)z$ . Now

$$\begin{aligned} ia &\leq i((xa^2)y) = i((x(aa))y) \\ &= i((a(xa))y) = i((a(xa))(ey)) \\ &= i((ae)((xa)y)) = i((xa)((ae)y)) \\ &= (xa)(i((ae)y)) = (xi)(a((ae)y)) \in (SI)S \subseteq I. \end{aligned}$$

Thus  $IS \subseteq I$  and  $(I] \subseteq I$ , i.e.,  $I$  is a right ideal of  $S$ . So  $I$  is an ideal of  $S$  by the Lemma 16. Converse is obvious.

*Lemma 18.* Let  $S$  be an intra-regular ordered AG-groupoid with left identity  $e$ . Then  $L \cap R \subseteq (LR]$  for every left ideal  $L$  and every right ideal  $R$  of  $S$ .

Proof: Let  $a \in L \cap R$ , where  $L$  is a left ideal and  $R$  is a right ideal of  $S$ , respectively, this implies that there exist elements  $x, y \in S$  such that  $a \leq (xa^2)y$ . Now

$$\begin{aligned} a &\leq (xa^2)y = (x(aa))y = (a(xa))y = (a(xa))(ey) \\ &= (ae)((xa)y) = (xa)((ae)y) \in LR. \\ &\Rightarrow L \cap R \subseteq (LR]. \end{aligned}$$

*Theorem 5.* Let  $S$  be an ordered AG-groupoid with left identity  $e$  such that  $(xe)S = xS$  for all  $x \in S$ . Then the following conditions are equivalent.

- (1)  $S$  is an intra-regular.
- (2)  $L \cap R \subseteq (LR]$  for every left ideal  $L$  and every right ideal  $R$  of  $S$ .

Proof: Since (1)  $\Rightarrow$  (2) holds by the Lemma 18. Suppose that (2) holds and  $a \in S$ , then  $Sa$  is a left ideal of  $S$  containing  $a$  and  $aS \cup Sa$  is a right ideal of  $S$  containing  $a$ . By our supposition

$$\begin{aligned} Sa \cap (aS \cup Sa) &\subseteq ((Sa)(aS \cup Sa)] = ((Sa)(aS) \cup (Sa)(Sa)]. \\ (Sa)(aS) &= (Sa)((ea)S) = (Sa)((Sa)e) = (Sa)((Sa)(ee)) \\ &= (Sa)((Se)(ae)) = (Sa)(S(ae)) = (Sa)(Sa). \end{aligned}$$

Thus

$$\begin{aligned}
 (aS \cup Sa) \cap Sa &\subseteq ((Sa)(aS) \cup (Sa)(Sa)) \\
 &= ((Sa)(Sa) \cup (Sa)(Sa)) \\
 &= ((Sa)(Sa)) = (S^2a^2) = (Sa^2) \\
 &= (S(a^2e)) = ((SS)(a^2e)) = ((eS)(a^2S)) = (S(a^2S)) \\
 &= (a^2(SS)) = ((ea^2)(SS)) = ((Sa^2)(Se)) = ((Sa^2)S].
 \end{aligned}$$

Since  $a \in (aS \cup Sa) \cap Sa$ , implies  $a \in ((Sa^2)S]$ , thus  $a$  is an intra regula. Hence  $S$  is an intra-regular, i.e., (2)  $\Rightarrow$  (1).

*Theorem 6.* Let  $S$  be an ordered AG-groupoid with left identity  $e$  such that  $(xe)S = xS$  for all  $x \in S$ . Then the following conditions are equivalent.

- (1)  $S$  is an intra-regular.
- (2)  $Q \cap I = ((QI)Q]$  for every quasi-ideal  $Q$  and every ideal  $I$  of  $S$ .
- (3)  $B \cap I = ((BI)B]$  for every bi-ideal  $B$  and every ideal  $I$  of  $S$ .
- (4)  $G \cap I = ((GI)G]$  for every generalized bi-ideal  $G$  and every ideal  $I$  of  $S$ .

*Proof:* Suppose that (1) holds. Let  $a \in G \cap I$ , where  $G$  is a generalized bi-ideal and  $I$  is an ideal of  $S$ , this implies that there exist elements  $x, y \in S$  such that  $a \leq (xa^2)y$ . Now

$$\begin{aligned}
 a &\leq (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a. \\
 y(xa) &\leq y(x((xa^2)y)) = y((xa^2)(xy)) = (xa^2)(y(xy)) \\
 &= (xa^2)(xy^2) = (x(aa))m, \text{ say } xy^2 = m \\
 &= (a(xa))m = (m(xa))a. \\
 m(xa) &\leq m(x((xa^2)y)) = m((xa^2)(xy)) = (xa^2)(m(xy)) \\
 &= (x(aa))n, \text{ say } m(xy) = n \\
 &= (a(xa))n = (n(xa))a \\
 &= va, \text{ say } n(xa) = v. \\
 \Rightarrow y(xa) &= (m(xa))a = (va)a = (va)(ea) = (ve)(aa) = a((ve)a).
 \end{aligned}$$

Thus  $a \leq (xa^2)y = (y(xa))a = (a((ve)a))a \in (GI)G$ . This means that  $a \in ((GI)G]$ , i.e.,  $G \cap I \subseteq ((GI)G]$ . Now  $((GI)G] \subseteq ((SI)S] \subseteq (I) = I$  and  $((GI)G] \subseteq ((GS)G] \subseteq (G] = G$ , thus  $((GI)G] \subseteq G \cap I$ . Hence  $G \cap I = ((GI)G]$ , i.e., (1)  $\Rightarrow$  (4). (4)  $\Rightarrow$  (3), every bi-ideal of  $S$  is a generalized bi-ideal of  $S$  by the Lemma 10. (3)  $\Rightarrow$  (2), every quasi-ideal of  $S$  is a bi-ideal of  $S$  by the Lemma 12. Assume that (2) is true and let  $R$  be a right ideal and  $I$  be a two-sided ideal of  $S$ . Now  $I \cap R = ((RI)R] \subseteq ((SI)R] \subseteq (IR]$ , since every right ideal of  $S$  is a quasi-ideal of  $S$ . Therefore  $S$  is an intra-regular by the Theorem 5, i.e., (2)  $\Rightarrow$  (1).

*Theorem 7.* Let  $S$  be an ordered AG-groupoid with left identity  $e$  such that  $(xe)S = xS$  for all  $x \in S$ . Then the following conditions are equivalent.

- (1)  $S$  is an intra-regular.
- (2)  $L \cap Q \subseteq (LQ]$  for every quasi-ideal  $Q$  and every left ideal  $L$  of  $S$ .
- (3)  $L \cap B \subseteq (LB]$  for every bi-ideal  $B$  and every left ideal  $L$  of  $S$ .
- (4)  $L \cap G \subseteq (LG]$  for every generalized bi-ideal  $G$  and every left ideal  $L$  of  $S$ .

*Proof:* Suppose that (1) holds. Let  $a \in L \cap G$ , where  $L$  is a left ideal and  $G$  is a generalized bi-ideal of  $S$ , this means that there exist elements  $x, y \in S$  such that  $a \leq (xa^2)y$ . Now  $a \leq (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a \in LG$ , i.e.,  $a \in (LG]$ . Thus  $L \cap G \subseteq (LG]$ , i.e., (1)  $\Rightarrow$  (4). (4)  $\Rightarrow$  (3), every bi-ideal of  $S$  is a generalized bi-ideal of  $S$ . (3)  $\Rightarrow$  (2), every quasi-ideal of  $S$  is a bi-ideal of  $S$ . Assume that (2) is true and let  $R$  be a right ideal of  $S$  and  $L$  be a left ideal of  $S$ . Now  $L \cap R \subseteq (LR]$ , where  $R$  is a quasi-ideal of  $S$ . Hence  $S$  is an intra-regular by the Theorem 5, i.e., (2)  $\Rightarrow$  (1).

*Theorem 8.* Let  $S$  be an ordered AG-groupoid with left identity  $e$  such that  $(xe)S = xS$  for all  $x \in S$ . Then the following conditions are equivalent.

- (1)  $S$  is an intra-regular.
- (2)  $L \cap Q \cap R \subseteq ((LQ)R]$  for every quasi-ideal  $Q$ , every right ideal  $R$  and every left ideal  $L$  of  $S$ .
- (3)  $L \cap B \cap R \subseteq ((LB)R]$  for every bi-ideal  $B$ , every right ideal  $R$  and every left ideal  $L$  of  $S$ .
- (4)  $L \cap G \cap R \subseteq ((LG)R]$  for every generalized bi-ideal  $G$ , every right ideal  $R$  and every left ideal  $L$  of  $S$ .

*Proof:* Suppose that (1) holds. Let  $a \in L \cap G \cap R$ , where  $L$  is a left ideal,  $G$  is a generalized bi-ideal and  $R$  is a right ideal of  $S$ , this implies that there exist elements  $x, y \in S$  such that  $a \leq (xa^2)y$ . Now

$$\begin{aligned} a &\leq (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a. \\ y(xa) &\leq y(x((xa^2)y)) = y((xa^2)(xy)) = (xa^2)(y(xy)) \\ &= (xa^2)(xy^2) = (x(aa))m, \text{ say } xy^2 = m \\ &= (a(xa))m = (m(xa))a. \end{aligned}$$

Thus  $a \leq (xa^2)y = (y(xa))a = ((m(xa))a)a \in (LG)R$ , i.e.,  $a \in ((LG)R]$ . Hence  $L \cap G \cap R \subseteq ((LG)R]$ , i.e., (1)  $\Rightarrow$  (4). (4)  $\Rightarrow$  (3), every bi-ideal of  $S$  is a generalized bi-ideal of  $S$ . (3)  $\Rightarrow$  (2), every quasi-ideal of  $S$  is a bi-ideal of  $S$ . Assume that (2) is true. Now

$$\begin{aligned} L \cap S \cap R &\subseteq ((LS)R] = (((eL)S)R] = (((SL)e)R] = (((SL)(ee))R] \\ &= (((Se)(Le))R] \subseteq ((S(Le))R] \subseteq ((SL)R] \subseteq (LR]. \\ &\Rightarrow L \cap R \subseteq (LR]. \end{aligned}$$

Hence  $S$  is an intra-regular by the Theorem 5, i.e., (2)  $\Rightarrow$  (1).

### Regular and Intra-regular Ordered AG-groupoids

In this section, we characterize both regular and intra-regular ordered AG-groupoids by the properties of (left, right, quasi-, bi-, generalized bi-) ideals.

*Theorem 9.* Let  $S$  be an ordered AG-groupoid with left identity  $e$  such that  $(xe)S = xS$  for all  $x \in S$ . Then the following conditions are equivalent.

- (1)  $S$  is a regular and an intra-regular.
- (2)  $(B^2] = B$  for every bi-ideal  $B$  of  $S$ .
- (3)  $B_1 \cap B_2 = (B_1B_2] \cap (B_2B_1]$  for all bi-ideals  $B_1, B_2$  of  $S$ .

*Proof:* Suppose that (1) holds and  $B$  be a bi-ideal of  $S$ . Since  $(B^2] = (BB] \subseteq (B] = B$ . Let  $a \in B$ , this implies that there exists an element  $x \in S$  such that  $a \leq (ax)a$ , also there exist elements  $y, z \in S$  such that  $a \leq (ya^2)z$ . Now

$$\begin{aligned} a &\leq (ax)a \leq (ax)((ya^2)z) = (((ya^2)z)x)a. \\ ((ya^2)z)x &= (xz)(ya^2) = m(ya^2), \text{ say } m = xz \\ &= m(y(aa)) = m(a(ya)) = a(m(ya)) \\ &\leq ((ax)a)(m(ya)) = ((ax)m)(a(ya)) \\ &= ((m(x)a)(a(ya)) = (na)(a(ya)), \text{ say } n = mx \\ &= ((en)a)(a(ya)) = ((an)e)(a(ya)) \\ &= ((an)a)(e(ya)) = ((an)a)(ya) = (sa)(ya), \text{ say } s = an \\ &= (aa)(ys) = (aa)t, \text{ say } t = ys \\ &\leq (((ax)a)a)t = ((aa)(ax))t = (t(ax))(aa) \\ &= (a(tx))(aa) = (aw)(aa), \text{ say } w = tx. \end{aligned}$$

Thus  $a \leq ((ya^2)z)x a \leq ((aw)(aa))a \in ((BS)B)B \subseteq B^2$ , i.e.,  $a \in (B^2]$ . So  $B \subseteq (B^2]$ , i.e.,  $(B^2] = B$ . Hence (1)  $\Rightarrow$  (3). Assume that (2) is true. Let  $B_1, B_2$  be bi-ideals of  $S$ , then  $B_1 \cap B_2$  be also a bi-ideal of  $S$ . Now  $B_1 \cap B_2 = ((B_1 \cap B_2)(B_1 \cap B_2)] \subseteq (B_1 B_2]$  and  $B_1 \cap B_2 = ((B_1 \cap B_2)(B_1 \cap B_2)] \subseteq (B_2 B_1]$ , thus  $B_1 \cap B_2 \subseteq (B_1 B_2] \cap (B_2 B_1]$ . First of all we have to show that  $(B_1 B_2]$  is a bi-ideal of  $S$ . It is enough to show that  $((B_1 B_2]S)(B_1 B_2] \subseteq (B_1 B_2]$ . Now

$$\begin{aligned} ((B_1 B_2]S)(B_1 B_2] &= ((B_1 B_2](S))(B_1 B_2] \\ &\subseteq ((B_1 B_2]S)(B_1 B_2] \\ &\subseteq (((B_1 B_2)S)(B_1 B_2]) \\ &= (((B_1 B_2)(SS))(B_1 B_2]) \\ &= (((B_1 S)(B_2 S))(B_1 B_2]) \\ &= (((B_1 S)B_1)((B_2 S)B_2]) \subseteq (B_1 B_2] \\ &\Rightarrow (((B_1 B_2)S)(B_1 B_2]) \subseteq (B_1 B_2]. \end{aligned}$$

Thus  $(B_1 B_2]$  is a bi-ideal of  $S$ , similarly  $(B_2 B_1]$  is also a bi-ideal of  $S$ . Then  $(B_1 B_2] \cap (B_2 B_1]$  is also a bi-ideal of  $S$ . Now

$$\begin{aligned} (B_1 B_2] \cap (B_2 B_1] &= (((B_1 B_2] \cap (B_2 B_1])(B_1 B_2] \cap (B_2 B_1])) \\ &\subseteq ((B_1 B_2](B_2 B_1]) \subseteq (((B_1 B_2)(B_2 B_1])) \\ &= ((B_1 B_2)(B_2 B_1]) \subseteq ((B_1 S)(SB_1]) \\ &= (((SB_1)S)B_1] = (((Se)B_1)S)B_1] \\ &= (((B_1 e)S)S)B_1] = (((B_1 S)S)B_1]) \\ &= (((SS)B_1)B_1] = ((SB_1)B_1] = (((Se)B_1)B_1]) \\ &= (((B_1 e)S)B_1] = ((B_1 S)B_1] \subseteq (B_1] \\ &\Rightarrow (B_1 B_2] \cap (B_2 B_1] \subseteq (B_1] = B_1. \end{aligned}$$

Similarly, we have  $(B_1 B_2] \cap (B_2 B_1] \subseteq (B_2] = B_2$ , thus  $(B_1 B_2] \cap (B_2 B_1] \subseteq B_1 \cap B_2$ . Therefore  $B_1 \cap B_2 = (B_1 B_2] \cap (B_2 B_1]$ , i.e., (2)  $\Rightarrow$  (3). Suppose that (3) holds, let  $R$  be right ideal of  $S$  and  $I$  be an ideal of  $S$ . Then  $R$  and  $I$  be bi-ideals of  $S$ , because every right ideal and two sided ideal of  $S$  is bi-ideal of  $S$  by the Lemma 9. Now  $R \cap I = (RI] \cap (IR]$ , this implies that  $R \cap I \subseteq (RI] \cap (IR]$ . Thus  $R \cap I \subseteq (RI]$  and  $R \cap I \subseteq (IR]$ , where  $I$  is also a left ideal of  $S$ . Since  $(RI] \subseteq R \cap I$ , i.e.,  $(RI] = R \cap I$ , thus  $S$  is a regular by the Theorem 1. Also,  $R \cap I \subseteq (IR]$ , thus  $S$  is an intra-regular by the Theorem 5. Hence (3)  $\Rightarrow$  (1).

*Theorem 10.* Let  $S$  be an ordered AG-groupoid with left identity  $e$  such that  $(xe)S = xS$  for all  $x \in S$ . Then the following conditions are equivalent.

- (1)  $S$  is regular and intra-regular.
- (2) Every quasi-ideal of  $S$  is an idempotent.

*Proof:* Suppose that (1) holds. Let  $Q$  be a quasi-ideal of  $S$  and  $(Q^2] = (QQ] \subseteq (Q] = Q$ , i.e.,  $(Q^2] \subseteq Q$ . Let  $a \in Q$ , this implies that there exists an element  $x \in S$  such that  $a \leq (ax)a$ , also there exist elements  $y, z \in S$  such that  $a \leq (ya^2)z$ . Now

$$\begin{aligned}
 a &\leq (ax)a \leq (ax)((ya^2)z) = (((ya^2)z)x)a. \\
 ((ya^2)z)x &= (xz)(ya^2) = m(ya^2), \text{ say } m = xz \\
 &= m(y(aa)) = m(a(ya)) = a(m(ya)) \\
 &\leq ((ax)a)(m(ya)) = ((ax)m)(a(ya)) \\
 &= ((mx)a)(a(ya)) = (qa)(a(ya)), \text{ say } q = mx \\
 &= ((eq)a)(a(ya)) = ((aq)e)(a(ya)) \\
 &= ((aq)a)(e(ya)) = ((aq)a)(ya) = (sa)(ya), \text{ say } s = aq \\
 &= (aa)(ys) = (aa)t, \text{ say } t = ys \\
 &\leq (((ax)a)a)t = ((aa)(ax))t = (t(ax))(aa) \\
 &= (a(tx))(aa) = (aw)(aa), \text{ say } w = tx
 \end{aligned}$$

Thus  $a \leq (((ya^2)z)x)a \leq ((aw)(aa))a \in ((QS)Q)Q \subseteq QQ \subseteq Q^2$ , i.e.,  $a \in (Q^2]$ , because every quasi-ideal of  $S$  is a bi-ideal of  $S$  by the Lemma 12. Thus  $Q \subseteq (Q^2]$ , i.e.,  $(Q^2] = Q$ . Hence (1)  $\Rightarrow$  (2). Assume that (2) is true. Let  $a \in S$ , then  $Sa$  is a left ideal of  $S$  containing  $a$ , so  $Sa$  is a quasi-ideal of  $S$ , because every left ideal of  $S$  is a quasi-ideal of  $S$ . Now  $Sa = ((Sa)^2] = ((Sa)(Sa)]$ , i.e.,  $a \in ((Sa)(Sa)]$ . Thus  $S$  is an intra-regular by the Theorem 5. Now  $Sa = ((Sa)(Sa)] = (((Se)a)(Sa)] = (((ae)S)(Sa)] = ((aS)(Sa)]$ , i.e.,  $a \in ((aS)(Sa)]$ . Thus  $S$  is a regular by the Theorem 1. Therefore (2)  $\Rightarrow$  (1).

*Theorem 11.* Let  $S$  be an ordered AG-groupoid with left identity  $e$  such that  $(xe)S = xS$  for all  $x \in S$ . Then the following conditions are equivalent.

- (1)  $S$  is regular and intra-regular.
- (2) Every quasi-ideal of  $S$  is an idempotent.
- (3) Every bi-ideal of  $S$  is an idempotent.

Proof: (1)  $\Rightarrow$  (3), by the Theorem 9. (3)  $\Rightarrow$  (2), every quasi-ideal of  $S$  is a bi-ideal of  $S$ , by the Lemma 12. (2)  $\Rightarrow$  (1), by the Theorem 10.

*Theorem 12.* Let  $S$  be an ordered AG-groupoid with left identity  $e$  such that  $(xe)S = xS$  for all  $x \in S$ . Then the following conditions are equivalent.

- (1)  $S$  is regular and intra-regular.
- (2)  $Q_1 \cap Q_2 \subseteq (Q_1Q_2]$  for all quasi-ideals  $Q_1, Q_2$  of  $S$ .
- (3)  $Q \cap B \subseteq (QB]$  for every quasi-ideal  $Q$  and every bi-ideal  $B$  of  $S$ .
- (4)  $B \cap Q \subseteq (BQ]$  for every bi-ideal  $B$  and every quasi-ideal  $Q$  of  $S$ .
- (5)  $B_1 \cap B_2 \subseteq (B_1B_2]$  for all bi-ideals  $B_1, B_2$  of  $S$ .

Proof: Suppose that (1) holds. Let  $B_1, B_2$  be bi-ideals of  $S$ , then  $B_1 \cap B_2$  be also a bi-ideal of  $S$ . Since every bi-ideal of  $S$  is an idempotent by the Theorem 9, then  $B_1 \cap B_2 = ((B_1 \cap B_2)^2] = ((B_1 \cap B_2)(B_1 \cap B_2)] \subseteq (B_1B_2]$ . Hence (1)  $\Rightarrow$  (5). Since (5)  $\Rightarrow$  (4)  $\Rightarrow$  (2) and (5)  $\Rightarrow$  (3)  $\Rightarrow$  (2), because every quasi-ideal of  $S$  is a bi-ideal of  $S$  by the Lemma 12. Assume that (2) is true. Now  $R \cap L \subseteq (RL]$ , where  $R$  is a right ideal and  $L$  is a left ideal of  $S$ . Since  $(RL] \subseteq R \cap L$ , i.e.,  $R \cap L = (RL]$ , thus  $S$  is regular. Again by (2)  $L \cap R \subseteq (LR]$ , thus  $S$  is an intra-regular. Therefore (2)  $\Rightarrow$  (1).

### Conclusion

In this article, we have characterized the non-associative ordered semigroups in terms of their one-sided ideals, ideals, interior ideals, bi-ideals and quais ideals. We have also characterized the intraregular and regular ordered AG-groupoids through the properties of their ideals.

References

- 1 Kazim, M. A. & Naseerudin, M. (1972). On almost semigroups. *Alig. Bull. Math.*, 2, 1-7.
- 2 PV, P. (1995). AG-test and some general properties of Abel-Grassmann's groupoids. *Pure Mathematics and Applications*, 6, 4, 371-383.
- 3 Mushtaq, Q. & Yusuf, S. M., (1978). On LA-semigroups. *Alig. Bull. Math.*, 8, 65-70.
- 4 Jezek, J. & Kepka, T. (1983). Medial groupoids. *Rozpravy CSAV Rada Mat. a Prir. Ved*, 93, 2, 93.
- 5 Cho, J. R., Jezek, J. & Kepka, T. (1999). Paramedial groupoids. *Czechoslovak Mathematical Journal*, 49, 2, 277-290.
- 6 Kehayopulu, N. (1990). Remark on ordered semigroups. *Math. Japon.*, 35, 6, 1061-1063.
- 7 Kehayopulu, N. (1990). On left regular ordered semigroups. *Math. Japon.*, 35, 6, 1051-1060.
- 8 Kehayopulu, N. (1993, December). On intra-regular ordered semigroups. *In Semigroup Forum, Springer-Verlag*, 46, 1, 271-278.
- 9 Kehayopulu, N. (1997). On regular ordered semigroups. *Mathematica japonicae*, 45, 3, 549-553.
- 10 Kehayopulu, N. (1998). On completely regular ordered semigroups. *Sci. Math*, 1, 1, 27-32.

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### Реттелген АG-группоидтардың әртүрлі идеалды кластарының қасиеттері бойынша сипаттамасы

Мақалада ассоциативті емес жартылай группалардың идеалдарына қатысты кейбір маңызды сипаттамалар ұсынылған. Біріншіден, біз реттелген АG-группоидты оның идеалының қасиеттері тұрғысынан сипаттадық, содан кейін осы АG-группоидтардың екі маңызды класына, яғни регулярлық және ішкі регулярлық емес ассоциативті емес АG-группоидтарға сипаттама бердік. Біздің мақсатымыз – реттелген АG-группоид деп аталатын ассоциативті емес және коммутативті емес алгебралық құрылымдар класын зерттеу арқылы ассоциативті алгебралық құрылымдарды зерттеу мен дамытуды ынталандыру.

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### Характеризация упорядоченных АG-группоидов через свойства их различных классов идеалов

В статье представлены некоторые важные характеристики упорядоченных неассоциативных полугрупп относительно их идеалов. Сначала были охарактеризован упорядоченный АG-группоид через свойства его идеалов, затем два важных класса этих АG-группоидов, а именно, регулярные и внутриврегулярные неассоциативные АG-группоиды. Цель настоящей работы – стимулирование исследования и развитие ассоциативных алгебраических структур путем изучения класса неассоциативных и некоммутативных алгебраических структур, называемых упорядоченным АG-группоидом.

*Ключевые слова:* упорядоченные АG-группоиды, левые (правые, внутренние, квази-, би-, обобщенные би-) идеалы, регулярные (внутриврегулярные) упорядоченные АG-группоиды.