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Two-Dimensional Boundary Value Problem of Heat Conduction in a Cone with Special Boundary Conditions

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Abstract—We consider the boundary value problem of heat conduction in a domain that is an inverted cone, while the boundary conditions contain a derivative with respect to the time variable. We prove a theorem on the solvability of a boundary value problem in weighted spaces of essentially bounded functions. The issues of solvability of the singular integral Volterra equation of the second kind, to which the original problem is reduced, are studied. Then we use the Carleman—Vekua regularization method to solve the resulting singular Volterra integral equation.

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1. INTRODUCTION

The need to solve boundary value problems for the equations of nonstationary transport in domains with boundaries that change with time is explained by the fact that they have a wide practical application. Problems of this kind describe electromagnetic, gas-dynamic, and thermophysical processes in low- and high-pressure gas-discharge plasmas. Mathematical modeling is indispensable for the development of plasma installations, as it provides the necessary information about the optimal sizes and values of the fundamental parameters of processes and devices [1, 2]. They also arise when studying the processes of melting electrical contacts, the effect of an electric arc on contacts [3]; when studying the problems of thermal shock in domains with a moving boundary [4], when solving a number of problems in hydromechanics [5]. Problems of this kind are of great practical value for studying thermal effects during crack propagation, when a constant temperature is set on the edges of a propagating crack (a domain with a moving boundary), which leads to the destruction of materials, mechanisms or aircraft; when studying the freezing of solutions, soils; in the study of the kinetic growth of crystals [6].

The peculiarity of the studying such problems is that when the size of the domain depends on time and the domain degenerates into a point at the initial moment of time, it is not possible to coordinate the solution of the equation with the motion of the domain boundaries. Finding analytical solutions for the indicated classes of heat conduction problems requires special methods or modifications of known approaches.

By the method of heat potentials, similar problems can be reduced to special Volterra type integral equations. It is important here that, if in the boundary value problem the variable domain does not degenerate into a point at the initial moment of time, then the equivalent integral equation is solved by the method of successive approximations. If the domain degenerates into a point at the initial moment of time, then the integral equation has a singularity, which is expressed in the fact that the integral from the kernel tends to unity as the upper limit of integration tends to the lower one, which means that the method of successive approximations is not applicable to it.

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2. STATEMENT OF THE BOUNDARY VALUE PROBLEM AND ITS TRANSFORMATION

In this paper, we study a two-dimensional boundary value problem in spatial variables in an inverted cone $Q = \left\{(x,y,t)|\ \sqrt{x^2+y^2} < t,\ 0 < t < 1\right\}$:

$$\frac{\partial u}{\partial t} - a^2 \triangle u = 0, \tag{1}$$

$$\frac{d\tilde{u}}{dt} + \frac{\partial u}{\partial \bar{n}}\bigg|_{\sqrt{x^2+y^2}=t} = g(x,y,t), \tag{2}$$

where $\tilde{u}(t) = u(x,y,t)|_{\sqrt{x^2+y^2}=t}$.

In [7], a one-dimensional version of a similar problem in weighted Holder classes is studied, where it is noted that the case of an nonhomogeneous boundary value problem turns out to be useful in the study of some problems with free boundaries.

Previously, we studied boundary value problems of heat conduction in one-dimensional, with respect to the spatial variable, domains degenerating into a point at the initial moment of time in Lebesgue classes [8–10].

Assuming that the axial symmetry condition is satisfied and passing in (1)–(2) to cylindrical coordinates, in the domain $G = \{(r,t)|\ 0 < r < t,\ 0 < t < 1\}$, we obtain the following boundary value problem:

$$\frac{\partial u}{\partial t} - a^2 \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = 0, \tag{3}$$

$$\left(2\frac{\partial u}{\partial r} + \frac{\partial u}{\partial t}\right)\bigg|_{r=t} = g(t), \quad u(r,t) \neq \infty \quad \text{as} \quad r \to 0. \tag{4}$$

Let's introduce a new function

$$w(r,t) = r\frac{\partial u}{\partial r}, \quad \left(\frac{\partial u}{\partial r} = \frac{1}{r}w(r,t)\right), \tag{5}$$

then the problem (3)–(4) is transformed into the following problem: in the domain $Q = \{(r,t)|0 < r < t, 0 < t < 1\}$ find a solution of the equation

$$\frac{\partial w}{\partial t} = a^2\frac{\partial^2 w}{\partial r^2} - a^2\frac{1}{r}\frac{\partial w}{\partial r}, \tag{6}$$

satisfying the boundary conditions

$$\begin{cases} \left\{\frac{1}{r}\frac{\partial w}{\partial r} + \frac{2}{a^2}\frac{1}{r}w\right\}\big|_{r=t} = \frac{g_1(t)}{a^2}, \\ w(r,t)|_{r=0} = g_2(t). \end{cases} \tag{7}$$

3. INTEGRAL REPRESENTATION OF THE SOLUTION OF THE PROBLEM (1)–(2) USING HEAT POTENTIALS

It is known [11, p. 76] that the function

$$G(r,\xi,t-\tau) = \frac{r}{2a(t-\tau)}\exp\left(-\frac{r^2+\xi^2}{4a^2(t-\tau)}\right) I_1\left(\frac{r\xi}{2a^2(t-\tau)}\right)$$

is the fundamental solution of the equation (6), ξ is a parameter. Hereinafter, $I_1(z)$ is the modified Bessel function of order 1. We will seek the solution of the problem (6)–(7) as the sum of heat potentials:

$$w(r,t) = \int\limits_0^t G(r,\xi,t-\tau)|_{\xi=\tau}\,\mu(\tau)d\tau + \int\limits_0^t \frac{\partial G(r,\xi,t-\tau)}{\partial \xi}\bigg|_{\xi=0}\nu(\tau)d\tau. \tag{8}$$

In the equality (8) the functions $\mu(t)$ and $\nu(t)$ are potential densities which are still unknown functions. We write the integral representation of the solution (8) in the following form:

$$w(r,t) = \int_0^t G(r,\xi,t-\tau)|_{\xi=\tau}\,\mu(\tau)d\tau + \int_0^t \left[\sum_{m=1}^{3} k_m(r,\xi,t-\tau)\right]\Bigg|_{\xi=0} \nu(\tau)d\tau, \tag{9}$$

where

$$k_1(r,\xi,t-\tau) = -\frac{1}{2a^2(t-\tau)}\frac{r}{\xi}\exp\left(-\frac{r^2+\xi^2}{4a^2(t-\tau)}\right)I_1\left(\frac{r\xi}{2a^2(t-\tau)}\right);$$

$$k_2(r,\xi,t-\tau) = \frac{r(r-\xi)}{[2a^2(t-\tau)]^2}\exp\left(-\frac{r^2+\xi^2}{4a^2(t-\tau)}\right)I_1\left(\frac{r\xi}{2a^2(t-\tau)}\right);$$

$$k_3(r,\xi,t-\tau) = \frac{r^2}{[2a^2(t-\tau)]^2}\exp\left(-\frac{r^2+\xi^2}{4a^2(t-\tau)}\right)\left[I_0\left(\frac{r\xi}{2a^2(t-\tau)}\right) - I_1\left(\frac{r\xi}{2a^2(t-\tau)}\right)\right].$$

Hence, we have

$$\left[\sum_{m=1}^{3} k_m(r,\xi,t-\tau)\right]\Bigg|_{\xi=0} = \lim_{\xi\to 0}\sum_{m=1}^{3} k_m(r,\xi,t-\tau) = \sum_{m=1}^{3} k_m(r,t-\tau).$$

In the last equality, we calculate separately the limits of each term:

$$\lim_{\xi\to 0} k_1(r,\xi,t-\tau) = -\frac{r^2}{8a^4(t-\tau)^2}\exp\left(-\frac{r^2}{4a^2(t-\tau)}\right); \quad \lim_{\xi\to 0} k_2(r,\xi,t-\tau) = 0;$$

$$\lim_{\xi\to 0} k_3(r,\xi,t-\tau) = \frac{r^2}{[2a^2(t-\tau)]^2}\exp\left(-\frac{r^2}{4a^2(t-\tau)}\right).$$

Thus, we finally have

$$\frac{\partial G(r,\xi,t-\tau)}{\partial\xi}\Bigg|_{\xi=0} = \frac{r^2}{8a^4(t-\tau)^2}\exp\left(-\frac{r^2}{4a^2(t-\tau)}\right). \tag{10}$$

It's obvious that

$$G(r,\xi,t-\tau)|_{\xi=\tau} = \frac{r}{2a^2(t-\tau)}\exp\left(-\frac{r^2+\tau^2}{4a^2(t-\tau)}\right)I_1\left(\frac{r\tau}{2a^2(t-\tau)}\right). \tag{11}$$

Substituting the obtained relations (10), (11) into the representation of the solution (9), we obtain the integral representation of the solution for the equation (6):

$$w(r,t) = \int_0^t \frac{r}{2a^2(t-\tau)}\exp\left(-\frac{r^2+\tau^2}{4a^2(t-\tau)}\right)I_1\left(\frac{r\tau}{2a^2(t-\tau)}\right)\mu(\tau)d\tau$$
$$+ \int_0^t \frac{r^2}{8a^4(t-\tau)^2}\exp\left(-\frac{r^2}{4a^2(t-\tau)}\right)\nu(\tau)d\tau. \tag{12}$$

4. REDUCTION OF THE BOUNDARY VALUE PROBLEM (6)–(7) TO A SINGULAR VOLTERRA TYPE INTEGRAL EQUATION

Now let us satisfy the boundary conditions (7). First we satisfy the boundary condition at $r = 0$, i.e. $w(r,t)|_{r=0} = g_2(t)$. We have

$$\lim_{r\to 0} w(r,t) = \lim_{r\to 0}\int_0^t \frac{r^2}{8a^4(t-\tau)^2}\exp\left(-\frac{r^2}{4a^2(t-\tau)}\right)\nu(\tau)d\tau = \frac{1}{2a^2}\nu(t) = g_2(t),$$

from here one of the sought-for densities $\nu(t)$ is directly determined $\nu(t) = 2a^2 g_2(t)$. Then the representation (12) can be rewritten as

$$w(r,t) = \int\limits_0^t \frac{r}{2a^2\,(t-\tau)} \exp\left(-\frac{r^2+\tau^2}{4a^2\,(t-\tau)}\right) I_1\left(\frac{r\tau}{2a^2\,(t-\tau)}\right)\mu(\tau)d\tau + \widetilde{g_2}(r,t), \tag{13}$$

where

$$\widetilde{g_2}(r,t) = \int\limits_0^t \frac{r^2}{4a^2(t-\tau)^2} \exp\left(-\frac{r^2}{4a^2(t-\tau)}\right) g_2(\tau)d\tau.$$

We introduce the following notations:

$$\mu_1(\tau) = \exp\left(\frac{\tau}{4a^2}\right)\mu(\tau), \quad \widetilde{I}_1\left(\frac{r\tau}{2a^2(t-\tau)}\right) = \exp\left(-\frac{r\tau}{2a^2(t-\tau)}\right) I_1\left(\frac{r\tau}{2a^2(t-\tau)}\right),$$

$$\widetilde{I_{01}}\left(\frac{t\tau}{2a^2(t-\tau)}\right) = \exp\left(-\frac{t\tau}{2a^2(t-\tau)}\right)\left[I_0\left(\frac{t\tau}{2a^2(t-\tau)}\right) - I_1\left(\frac{t\tau}{2a^2(t-\tau)}\right)\right]$$

and by satisfying the first boundary condition from (7), we obtain the following integral equation

$$\mu_1(t) + \int\limits_0^t \frac{t\tau}{2a^2(t-\tau)^2}\widetilde{I_{01}}\left(\frac{t\tau}{2a^2(t-\tau)}\right)\mu_1(\tau)d\tau$$

$$+\frac{3}{2}\int\limits_0^t \frac{t}{a^2(t-\tau)}\widetilde{I}_1\left(\frac{t\tau}{2a^2(t-\tau)}\right)\mu_1(\tau)d\tau = 2a^2\mathcal{F}_1(t), \tag{14}$$

where

$$\mathcal{F}_1(t) = \left[-\left.\frac{\partial \widetilde{g_2}(r,t)}{\partial r}\right|_{r=t} - \frac{2}{a^2}\,\widetilde{g_2}(r,t)|_{r=t} + \frac{1}{a^2}t\,g_1(t)\right]\exp\left(\frac{t}{4a^2}\right).$$

For convenience, we introduce a new function: $\mu_2(t) = t\mu_1(t)$. Then the equation (14) can be rewritten as follows

$$\mu_2(t) + \int\limits_0^t M(t,\tau)\mu_2(\tau)d\tau = 2a^2 t\mathcal{F}_1(t), \tag{15}$$

where

$$M(t,\tau) = \frac{t^2}{2a^2(t-\tau)^2}\widetilde{I_{01}}\left(\frac{t\tau}{2a^2(t-\tau)}\right) + \frac{3}{2a^2}\frac{t^2}{\tau(t-\tau)}\widetilde{I}_1\left(\frac{t\tau}{2a^2(t-\tau)}\right)$$
$$= M_1(t,\tau) + M_2(t,\tau). \tag{16}$$

Note the following property of the kernel (16), from which it follows that the method of successive approximations is not applicable to the integral equation (15).

Remark 1. We note that $\lim\limits_{t\to 0}\int\limits_0^t M(t,\tau)d\tau = 1$, and

$$\int\limits_0^t M_1(t,\tau)d\tau = 1, \quad \forall t > 0, \quad \int\limits_0^t M_2(t,\tau)d\tau = \frac{3}{2a^2}t.$$

Indeed [12, p. 43, 1.11.2(4)]:

$$\int_0^t \frac{1}{2a^2}\frac{t^2}{(t-\tau)^2}\exp\left(-\frac{t\tau}{2a^2(t-\tau)}\right)\left[I_0\left(\frac{t\tau}{2a^2(t-\tau)}\right)-I_1\left(\frac{t\tau}{2a^2(t-\tau)}\right)\right]d\tau$$

$$=\left\|\xi=\frac{t\tau}{2a^2(t-\tau)}\right\|=\int_0^\infty \exp\left(-\xi\right)\left[I_0\left(\xi\right)-I_1\left(\xi\right)\right]d\xi=1.$$

Let's prove the second equality [12, p. 272, 2.15.3(3)]:

$$\int_0^t \frac{1}{2a^2}\frac{3t^2}{\tau(t-\tau)}\exp\left(-\frac{t}{2a^2}\frac{\tau}{t-\tau}\right)I_1\left(\frac{t}{2a}\frac{\tau}{t-\tau}\right)d\tau=\left\|\frac{t}{2a^2}\frac{\tau}{t-\tau}=z\right\|$$

$$=\frac{3t}{2a^2}\int_0^\infty \frac{1}{z}\exp\left(-z\right)I_1(z)dz=\frac{3t}{2a^2}.$$

5. SOLUTION OF THE "TRUNCATED"-CHARACTERISTIC INTEGRAL EQUATION

We will seek a solution of the following "truncated" integral equation, which, by Remark 1, is characteristic for the equation (15)

$$\mu_2(t)+\int_0^t \frac{t^2}{2a^2(t-\tau)^2}\widetilde{I_{01}}\left(\frac{t\tau}{2a^2(t-\tau)}\right)\mu_2(\tau)d\tau=2a^2t\mathcal{F}_1(t). \quad (17)$$

If a solution of equation (17) is found, then the solution of equation (15) will be obtained by the Carleman–Vekua regularization method.

In the integral equation (17), we change the independent variables $t=\frac{1}{t_1}$, $\tau=\frac{1}{\tau_1}$ and by introducing the functions

$$\mu_2\left(\frac{1}{t_1}\right)=\mu_2\left(t_1\right),\quad 2a^2\frac{1}{t_1}\mathcal{F}_1\left(\frac{1}{t_1}\right)=\mathcal{F}_2(t_1)$$

we obtain an equation with a difference kernel

$$\mu_2(t_1)+\int_{t_1}^\infty M_{1-}\left(t_1-\tau_1\right)\mu_2(\tau_1)d\tau_1=\mathcal{F}_2(t_1), \quad (18)$$

where

$$M_{1-}\left(t_1-\tau_1\right)=\frac{1}{2a^2(\tau_1-t_1)^2}\widetilde{I_{01}}\left(\frac{1}{2a^2(\tau_1-t_1)}\right).$$

We apply the Laplace transform to both sides of the equation (18), and obtain [13, p. 425]:

$$\widehat{\mu}_2(p)=\frac{1}{2\cdot\frac{\sqrt{-p}}{a}I_0\left(\frac{\sqrt{-p}}{a}\right)K_1\left(\frac{\sqrt{-p}}{a}\right)}\cdot\widehat{\mathcal{F}}_2(p). \quad (19)$$

If we introduce the notation

$$\widehat{R}^*_-(-p)=\frac{1}{2\cdot\frac{\sqrt{-p}}{a}I_0\left(\frac{\sqrt{-p}}{a}\right)K_1\left(\frac{\sqrt{-p}}{a}\right)},\quad \mathrm{Re}p<0, \quad (20)$$

then the equality (19) can be represented as

$$\widehat{\mu}_2(p)=\frac{1}{2}\widehat{R}^*_-(-p)\,\widehat{\mathcal{F}}_2(p),\quad \mathrm{Re}p<0.$$

5.1. Finding the Original of the Resolvent

Let's find the original expression (20), representing it as follows: $\widehat{R}^*_-(-p) = 1 + \widehat{R}_-(-p)$, where

$$\widehat{R}_-(-p) = \frac{1 - 2\frac{\sqrt{-p}}{a} I_0\left(\frac{\sqrt{-p}}{a}\right) K_1\left(\frac{\sqrt{-p}}{a}\right)}{2\frac{\sqrt{-p}}{a} I_0\left(\frac{\sqrt{-p}}{a}\right) K_1\left(\frac{\sqrt{-p}}{a}\right)} \risingdotseq R_-(t_1).$$

Then the solution of the equation (18) has the form

$$\mu_2(t_1) = \mathcal{F}_2(t_1) + \int\limits_{t_1}^{\infty} R_-(t_1 - \tau_1)\mu_2(\tau_1)d\tau_1.$$

Next, we introduce the notation $\frac{\sqrt{-p}}{a} = s$ and find the original expression

$$\widehat{R^*}(s) = \frac{1 - 2sI_0(s)K_1(s)}{2sI_0(s)K_1(s)},$$

then we use the following properties [14, p. 191, 20.27]:

1^0. Let $\varphi(t) \risingdotseq \widehat{\varphi}(p)$, then $\varphi(\alpha t) \risingdotseq \frac{1}{\alpha}\widehat{\varphi}\left(\frac{p}{\alpha}\right)$, $\alpha > 0$.

2^0. Let $\widehat{\varphi}(p) \risingdotseq \varphi(t)$, then $\widehat{\varphi}(\sqrt{p}) \risingdotseq \frac{1}{2\sqrt{\pi}}\frac{1}{t^{\frac{3}{2}}}\int\limits_0^{\infty} \tau e^{-\frac{\tau^2}{4t}}\varphi(\tau)d\tau$.

Therefore

$$\widehat{R}^*(s) \risingdotseq R_-(t_1) = \sum_{k=-\infty}^{+\infty}{}' \frac{1}{2s_k\left[I_0(s)K_1(s)\right]'|_{s=s_k}} \exp{(s_k t_1)}. \tag{21}$$

The prime at the sign of the sum means the absence of the term at $k = 0$, here s_k are the roots of a function $I_0(s)$. In order to determine the roots of this function, we use the equality $I_0(s) = J_0(is)$, where $J_0(z)$ is the Bessel function — cylinder function of the first kind.

It is known that the function $J_0(z)$ has no roots other than real [15, p. 556]. Hence, the function $I_0(s)$ has roots $s_k = \pm i\alpha_k (k \neq 0)$, where α_k are real, in this case $is_{-k} = \alpha_{-k} = -is_k = -\alpha_k$ ($I_0(s)$ is even function). Thus, equality (21) takes the form

$$R_-(t_1) = \sum_{k=-\infty}^{+\infty}{}' A_k \exp{(s_k t_1)}, \tag{22}$$

where

$$A_k = \frac{1}{2s_k\left\{I_1(s)K_1(s) - I_0(s)\left[K_0(s) + \frac{1}{s}K_1(s)\right]\right\}\Big|_{s=s_k}} = \frac{1}{2s_kI_1(s_k)K_1(s_k)}.$$

From the equality (22) and the properties of the image 1^0 and 2^0, we have

$$\widehat{R}\left(\frac{\sqrt{s}}{a}\right) \risingdotseq R(t_1) = \sum_{k=-\infty}^{+\infty}{}' \frac{a^2A_k}{2\sqrt{\pi}t_1^{\frac{3}{2}}}E(t_1, s_k),$$

where

$$E(t_1, s_k) = \int\limits_0^{\infty} \tau \exp\left(-\frac{\tau^2}{4t_1} + s_k a^2 \tau\right)d\tau = 2t_1 + 2\sqrt{\pi}s_k a^2 t_1^{\frac{3}{2}} \exp{(s_k^2 a^4 t_1)}\mathrm{erfc}\left(-s_k a^2\sqrt{t_1}\right).$$

Thus, we obtain $R_-(t_1) = R_-^{(a)}(t_1) + R_-^{(b)}(t_1)$, where

$$R_-^{(a)}(t_1) = \frac{a^2}{\sqrt{\pi t_1}} \sum_{k=-\infty}^{+\infty}{}' A_k, \quad R_-^{(b)}(t_1) = a^4 \sum_{k=-\infty}^{+\infty}{}' A_k s_k \exp\left(s_k^2 a^4 t_1\right) \operatorname{erfc}\left(-s_k a^2 \sqrt{t_1}\right).$$

Using the properties of Bessel functions [12, p. 642], we obtain

$$R_-^{(a)}(t_1) = \frac{2a^2}{\pi^{\frac{3}{2}}\sqrt{t_1}} \sum_{k=1}^{\infty} B_k,$$

where

$$B_k = \frac{1}{\alpha_k\left[J_1^2(\alpha_k) + N_1^2(\alpha_k)\right]},$$

$N_1(z)$ is the Neumann function.

For the term $R_-^{(b)}(t_1)$ we will have

$$R_-^{(b)}(t_1) = a^4 \sum_{k=-\infty}^{+\infty}{}' A_k s_k \exp\left(-s_k^2 a^4 t_1\right) \operatorname{erfc}\left(-s_k a^2 \sqrt{t_1}\right) = \frac{2a^4}{\pi} \sum_{k=1}^{\infty} \frac{\exp\left(-\alpha_k^2 a^4 t_1\right)}{J_1^2(\alpha_k) + N_1^2(\alpha_k)}$$

$$- \frac{a^4}{\pi} \frac{2}{\sqrt{\pi}\alpha_k a^2 \sqrt{t_1}} \sum_{k=1}^{\infty} \frac{N_1(\alpha_k)}{J_1(\alpha_k)} \frac{1}{\alpha_2\left[J_1^2(\alpha_k) + N_1^2(\alpha_k)\right]} \int_0^{\infty} \exp\left(-\frac{\xi^2}{4\alpha_k^2 a^4}\frac{1}{t_1}\right) \sin\xi d\xi.$$

So, we get

$$R_-(t_1) = \sum_{i=1}^{3} R_-^{(i)}(t_1), \tag{23}$$

where

$$R_-^{(1)}(t_1) = \frac{2a^2}{\pi^{\frac{3}{2}}\sqrt{t_1}} \sum_{k=1}^{\infty} B_k, \quad R_-^{(2)}(t_1) = \frac{2a^4}{\pi} \sum_{k=1}^{\infty} \alpha_k B_k \exp\left(-\alpha_k^2 a^4 t_1\right),$$

$$R_-^{(3)}(t_1) = -\frac{2a^2}{\pi^{\frac{3}{2}}\sqrt{t_1}} \sum_{k=1}^{\infty} \frac{N_1(\alpha_k)}{J_1(\alpha_k)} B_k \int_0^{\infty} \exp\left(-\frac{\xi^2}{4\alpha_k^2 a^4 t_1}\right) \sin\xi d\xi.$$

5.2. Estimation of the Resolvent $R_-(t_1)$

Let us show that the following lemma holds for the resolvent (23).

Lemma 1. *The resolvent $R_-(t_1)$ satisfies the estimate $R_-(t_1) \le C/\sqrt{t_1}$.*

Proof. The resolvent (23) of the integral equation (18) is presented as a sum of three terms. We estimate each term separately. Using the equality

$$\alpha_k\left[J_1^2(\alpha_k) + N_1^2(\alpha_k)\right] = \alpha_k\left[J_1(\alpha_k) + iN_1(\alpha_k)\right] \cdot \left[J_1(\alpha_k) - iN_1(\alpha_k)\right] = \alpha_k H_1^{(1)}(\alpha_k) H_1^{(2)}(\alpha_k),$$

and also the well-known integral [12, p. 42]

$$\int_{x_0}^{\infty} \frac{dx}{x H_\nu^{(1)}(x) H_\nu^{(2)}(x)} = -\frac{\pi}{4i} \ln\left(\frac{H_\nu^{(2)}(x)}{H_\nu^{(1)}(x)}\right)\Bigg|_{x_0}^{\infty},$$

we have

$$R_-^{(1)}(t_1) \le \gamma \frac{a^2}{\sqrt{\pi}} \frac{1}{\sqrt{t_1}}.$$

Further, using the inequality [16, p. 928, 8, 479.1]

$$\frac{1}{x} \le \frac{\pi}{2}\left[J_1^2(x) + N_1^2(x)\right] < \frac{1}{\sqrt{x^2-1}}, \quad \frac{1}{2} \le \nu \le x; \quad \forall \alpha_k > 2.4,$$

we get that for $B_k (\forall k = 1, 2, 3, \dots)$ the inequality holds

$$B_k = \frac{1}{\alpha_k\left[J_1^2(\alpha_k) + N_1^2(\alpha_k)\right]} < \frac{\pi}{2}.$$

With this in mind, we estimate the second sum:

$$R_-^{(2)}(t_1) \le a^4 \sum_{k=1}^{\infty} \alpha_k \exp\left(-\alpha_k^2 a^4 t_1\right) \le a^4 \int\limits_{\alpha_1}^{\infty} \alpha_k \exp\left(-\alpha_k^2 a^4 t_1\right) d\alpha_k = \frac{1}{2t_1} \exp\left(-\alpha_1^2 a^4 t_1\right).$$

Now we estimate the third sum:

$$R_-^{(3)}(t_1) \le \frac{a^2}{\sqrt{\pi t_1}} \sum_{k=1}^{\infty} \frac{N_1(\alpha_k)}{J_1(\alpha_k)} \int\limits_0^{\infty} \exp\left(-\frac{\xi^2}{4\alpha_k^2 a^4 t_1}\right) \sin\xi d\xi.$$

Consider the integral

$$U(t_1, \alpha_k) = \int\limits_0^{\infty} \exp\left(-\frac{\xi^2}{4\alpha_k^2 a^4 t_1}\right) \sin\xi d\xi$$

$$= \sum_{n=0}^{\infty}\left[\int\limits_{2n\pi}^{(2n+1)\pi} \exp\left(-\frac{\xi^2}{4\alpha_k^2 a^4 t_1}\right) \sin\xi d\xi + \int\limits_{(2n+1)\pi}^{(2n+2)\pi} \exp\left(-\frac{\xi^2}{4\alpha_k^2 a^4 t_1}\right) \sin\xi d\xi\right]$$

$$\le \sum_{n=0}^{\infty}\left[\int\limits_{2n\pi}^{(2n+1)\pi} \exp\left(-\frac{(2n\pi)^2}{4\alpha_k^2 a^4 t_1}\right) \sin\xi d\xi + \int\limits_{(2n+1)\pi}^{(2n+2)\pi} \exp\left(-\frac{[(2n+2)\pi]^2}{4\alpha_k^2 a^4 t_1}\right) \sin\xi d\xi\right].$$

Therefore, we have

$$|U(t_1, \alpha_k)| \le 2\left\{\sum_{n=0}^{\infty}\left[\exp\left(-\frac{(2n)^2}{4\alpha_k^2 a^4 t_1}\right) - \exp\left(-\frac{(2n+2)^2}{4\alpha_k^2 a^4 t_1}\right)\right]\right\}$$

$$= 2\left\{\sum_{n=0}^{\infty} \exp\left(-\frac{(2n)^2}{4\alpha_k^2 a^4 t_1}\right)\left[1 - \exp\left(-\frac{2n+1}{4\alpha_k^2 a^4 t_1}\right)\right]\right\} \le 2\left\{\sum_{n=0}^{\infty} \frac{2n+1}{a^4 t_1 \alpha_k^2} \exp\left(-\frac{(2n)^2}{4a^4 t_1 \alpha_k^2}\right)\right\}$$

$$\le 2\int\limits_0^{\infty} \frac{2n+1}{a^4 t_1 \alpha_k^2} \exp\left(-\frac{(2n)^2}{4a^4 t_1 \alpha_k^2}\right) dn = 2\left[1 + \frac{\sqrt{\pi}}{2} \frac{1}{a^4 t_1 \alpha_k^2}\right].$$

Thus, for the third sum, we have the estimate

$$R_-^{(3)}(t_1) \le \frac{C_1}{\sqrt{t_1}} + \frac{C_2}{t_1}.$$

Taking into account $t_1 > 1$ $(0 < t < 1)$, we obtain $R_-(t_1) \le C \cdot \frac{1}{\sqrt{t_1}}$. Lemma is proved. □

Making the reverse transformation $t_1 = \frac{1}{t}$; $\tau_1 = \frac{1}{\tau}$, the solution of the characteristic equation (17) is written as follows:

$$\mu_2(t) = 2a^2 t \mathcal{F}_1(t_1) + 2a^2 \int\limits_0^t \widetilde{R}(t, \tau) \mathcal{F}_1(\tau_1) d\tau,$$

where the resolvent satisfies the estimate

$$\widetilde{R}(t,\tau) \le C\frac{\sqrt{t}}{\sqrt{\tau}\sqrt{t-\tau}}. \tag{24}$$

6. SOLUTION OF THE INTEGRAL EQUATION (14). THE CARLEMAN–VEKUA REGULARIZATION METHOD

Theorem 1. *Initial integral equation (14)*

$$\mu_1(t) + \int_0^t \frac{t\tau}{2a^2(t-\tau)^2}\widetilde{I_{01}}\left(\frac{t\tau}{2a^2(t-\tau)}\right)\mu_1(\tau)d\tau$$

$$+\int_0^t \frac{3}{2a^2}\frac{t}{t-\tau}\widetilde{I}_1\left(\frac{t\tau}{2a^2(t-\tau)}\right)\mu_1(\tau)d\tau = 2a^2\mathcal{F}_1(t),$$

$\forall t^{-\frac{1}{2}}\mathcal{F}_1(t) \in L_\infty(0,1)$, *has the unique solution in the class of functions* $t^{-\frac{1}{2}}\mu_1(t) \in L_\infty(0,1)$, *which can be found by the method of successive approximations.*

Proof. We write the integral equation (14) as $(t\mu_1(t) = \mu_2(t))$:

$$\mu_2(t) + \int_0^t \frac{t^2}{2a^2(t-\tau)^2}\exp\left(-\frac{t\tau}{2a^2(t-\tau)}\right)\widetilde{I_{01}}\left(\frac{t\tau}{2a^2(t-\tau)}\right)\mu_2(\tau)d\tau$$

$$= -\int_0^t \frac{3}{2a^2}\frac{t^2}{\tau(t-\tau)}\exp\left(-\frac{t\tau}{2a^2(t-\tau)}\right)I_1\left(\frac{t\tau}{2a^2(t-\tau)}\right)\mu_2(\tau)d\tau + 2a^2t\mathcal{F}_1(t). \tag{25}$$

Assuming the right-hand side of the equation (25) to be temporarily known, we write down its solution

$$\mu_3(t) + \int_0^t \widetilde{M}(t,\tau)\mu_3(\tau)d\tau = 2a^2\frac{1}{t^{\frac{5}{2}}}\int_0^t \widetilde{R}(t,\tau)\mathcal{F}_1(\tau)d\tau + 2a^2\frac{1}{t^{\frac{1}{2}}}\mathcal{F}_1(t), \tag{26}$$

where

$$\widetilde{M}(t,\tau) = \widetilde{M_1}(t,\tau) + \widetilde{M_2}(t,\tau), \tag{27}$$

$$\widetilde{M_1}(t,\tau) = \frac{3}{2a^2}\frac{\sqrt{t}\sqrt{\tau}}{t-\tau}\exp\left(-\frac{t\tau}{2a^2(t-\tau)}\right)I_1\left(\frac{t\tau}{2a^2(t-\tau)}\right)\mu_3(\tau),$$

$$\widetilde{M_2}(t,\tau) = \frac{3}{2a^2}\frac{1}{t^{\frac{3}{2}}}\int_\tau^t \widetilde{R}(t,\tau_1)\frac{\tau_1\sqrt{\tau}}{\tau_1-\tau}\exp\left(-\frac{\tau_1\tau}{2a^2(\tau_1-\tau)}\right)I_1\left(\frac{\tau_1\tau}{2a^2(\tau_1-\tau)}\right)\mu_3(\tau_1)d\tau_1,$$

$$\mu_3(t) = t^{-\frac{3}{2}}\mu_2(t) = t^{-\frac{1}{2}}\mu_1(t).$$

To prove the theorem, it suffices to show that the kernel $\widetilde{M}(t,\tau)$ of the integral equation (26) has a weak singularity. We estimate the kernel (27) of the integral equation (26), taking into account the estimate of the resolvent (24). Using inequality $\exp(-z)I_1(z) \le \dfrac{C}{\sqrt{z}}$ for the first term in the kernel $\widetilde{M_1}(t,\tau)$, we obtain the estimate

$$\widetilde{M_1}(t,\tau) \le D_1\frac{1}{\sqrt{t-\tau}}.$$

Now we estimate $\widetilde{M_2}(t,\tau)$:

$$\left|\widetilde{M_2}(t,\tau)\right| \le \frac{3C}{2a^2}t^{-\frac{3}{2}}\int\limits_{\tau}^{t}\frac{\sqrt{t}}{\sqrt{\tau_1}\sqrt{t-\tau_1}}\frac{\tau_1\sqrt{\tau}}{\tau_1-\tau}\exp\left(-\frac{\tau_1\tau}{2a^2(\tau_1-\tau)}\right)I_1\left(\frac{\tau_1\tau}{2a^2(\tau_1-\tau)}\right)d\tau_1$$

$$= \left\|\frac{\tau_1-\tau}{t-\tau}=z\right\| = \frac{3C}{2a^2}\frac{\sqrt{\tau}}{t}\int\limits_0^1\frac{\sqrt{z+\frac{\tau}{t-\tau}}}{z\sqrt{1-z}}\exp\left(-\frac{\tau\left(z+\frac{\tau}{t-\tau}\right)}{2a^2z}\right)I_1\left(\frac{\tau\left(z+\frac{\tau}{t-\tau}\right)}{2a^2z}\right)dz.$$

To estimate the last integral, we represent it as the sum of two integrals $\widetilde{M_2}^{(1)}(t,\tau)$ and $\widetilde{M_2}^{(2)}(t,\tau)$ taken respectively over the intervals $\left(0,\frac{1}{N}\right)$ and $\left(\frac{1}{N},1\right)$, where N is a large enough fixed number. Thus,

$$\left|\widetilde{M_2}(t,\tau)\right| \le \widetilde{M_2}^{(1)}(t,\tau)+\widetilde{M_2}^{(2)}(t,\tau).$$

Now we estimate separately each integral:

$$\widetilde{M_2}^{(1)}(t,\tau) = \frac{3C}{2a^2}\frac{\sqrt{\tau}}{t}\int\limits_0^{1/N}\frac{\sqrt{z+\frac{\tau}{t-\tau}}}{z\sqrt{1-z}}\exp\left(-\frac{\tau\left(z+\frac{\tau}{t-\tau}\right)}{2a^2z}\right)I_1\left(\frac{\tau\left(z+\frac{\tau}{t-\tau}\right)}{2a^2z}\right)dz$$

$$\le \frac{3C}{2a^2}\frac{\sqrt{\tau}}{t}\int\limits_0^{1/N}\frac{\sqrt{\frac{1}{N}+\frac{\tau}{t-\tau}}}{\sqrt{1-\frac{1}{N}}}\frac{1}{z}\exp\left(-\frac{\tau\left(z+\frac{\tau}{t-\tau}\right)}{2a^2z}\right)I_1\left(\frac{\tau\left(z+\frac{\tau}{t-\tau}\right)}{2a^2z}\right)dz$$

$$\le \frac{3C}{2a^2}\sqrt{\frac{N}{N-1}}\frac{\sqrt{\tau}}{\sqrt{t(t-\tau)}}\int\limits_0^{1/N}\frac{1}{z}\exp\left(-\frac{\tau}{2a^2}-\frac{\tau^2}{2a^2(t-\tau)}\frac{1}{z}\right)I_1\left(\frac{\tau}{2a^2}+\frac{\tau^2}{2a^2(t-\tau)}\frac{1}{z}\right)dz.$$

In the last integral, we make a change of variables: $\frac{\tau^2}{2a^2(t-\tau)}\frac{1}{z}=\theta$, then

$$\widetilde{M_2}^{(1)}(t,\tau) \le \frac{3C}{2a^2}\sqrt{\frac{N}{N-1}}\frac{\sqrt{\tau}}{\sqrt{t(t-\tau)}}\int\limits_{\frac{N\tau^2}{a^2(t-\tau)}}^{\infty}\frac{1}{\theta}\exp\left(-\frac{\tau}{2a^2}-\theta\right)I_1\left(\frac{\tau}{2a^2}+\theta\right)d\theta.$$

It is known that the function $-y=e^{-x}I_1(x)$ is bounded $\forall x\ge 0$ $(0\le e^{-x}I_1(x)<\frac{1}{4})$ and decreases monotonically $\forall x>\frac{3}{2}$.

Consider two possible options:

1. If τ,t,N such that $\frac{\tau}{2a^2}+\theta>\frac{3}{2}$ for $\theta>\frac{N\tau^2}{a^2(t-\tau)}$, then $e^{-\frac{\tau}{2a^2}-\theta}I_1\left(\frac{\tau}{2a^2}+\theta\right)<e^{-\theta}I_1(\theta)$. In this case we have

$$\widetilde{M_2}^{(1)}(t,\tau) \le \frac{3C}{2a^2}\sqrt{\frac{N}{N-1}}\frac{\sqrt{\tau}}{\sqrt{t(t-\tau)}}\int\limits_{\frac{N\tau^2}{a^2(t-\tau)}}^{\infty}\frac{1}{\theta}\exp(-\theta)I_1(\theta)\,d\theta$$

$$\le \frac{3C}{2a^2}\sqrt{\frac{N}{N-1}}\frac{\sqrt{\tau}}{\sqrt{t(t-\tau)}}\int\limits_0^{\infty}\frac{1}{\theta}\exp(-\theta)I_1(\theta)\,d\theta = \frac{3C}{2a^2}\sqrt{\frac{N}{N-1}}\frac{\sqrt{\tau}}{\sqrt{t(t-\tau)}}$$

$$\le \frac{D_2}{a^2}\frac{\sqrt{\tau}}{\sqrt{t(t-\tau)}} \le \frac{D_2}{a^2}\frac{1}{\sqrt{t-\tau}}.$$

2. If τ, t, N such that $0 < \frac{\tau}{2a^2} + \theta < \frac{3}{2}$ for $\theta > \frac{N\tau^2}{a^2(t-\tau)}$, then integral

$$\int\limits_{\frac{N\tau^2}{a^2(t-\tau)}}^{\infty} \frac{1}{\theta}\exp\left(-\frac{\tau}{2a^2}-\theta\right)I_1\left(\frac{\tau}{2a^2}+\theta\right)d\theta$$

can be written as

$$\int\limits_{\frac{N\tau^2}{a^2(t-\tau)}}^{3/2} \frac{1}{\theta}\exp\left(-\frac{\tau}{2a^2}-\theta\right)I_1\left(\frac{\tau}{2a^2}+\theta\right)d\theta + \int\limits_{3/2}^{\infty} \frac{1}{\theta}\exp\left(-\frac{\tau}{2a^2}-\theta\right)I_1\left(\frac{\tau}{2a^2}+\theta\right)d\theta.$$

For the second integral, we again have

$$\int\limits_{3/2}^{\infty} \frac{1}{\theta}\exp\left(-\frac{\tau}{2a^2}-\theta\right)I_1\left(\frac{\tau}{2a^2}+\theta\right)d\theta < \int\limits_{0}^{\infty} \frac{1}{\theta}\exp\left(-\theta\right)I_1\left(\theta\right)d\theta = 1.$$

Let's estimate the first integral

$$\int\limits_{\frac{N\tau^2}{a^2(t-\tau)}}^{3/2} \frac{1}{\theta}\exp\left(-\frac{\tau}{2a^2}-\theta\right)I_1\left(\frac{\tau}{2a^2}+\theta\right)d\theta \leq \frac{1}{4}\int\limits_{\frac{N\tau^2}{a^2(t-\tau)}}^{3/2}\frac{d\theta}{\theta} = \frac{1}{4}\ln\frac{3a^2(t-\tau)}{2N\tau^2}$$

$$= \frac{1}{4}\left[\left(\ln 3a^2(t-\tau)\right) + \ln\left(2N\tau^2\right)\right] = \frac{1}{4}\left[\frac{(t-\tau)^{\varepsilon}\ln 3a^2(t-\tau)}{(t-\tau)^{\varepsilon}} + \frac{\tau^{\varepsilon}\ln\left(\tau\sqrt{2N}\right)^2}{\tau^{\varepsilon}}\right]$$

$$\leq \frac{D_3}{(t-\tau)^{\varepsilon}} + \frac{D_4}{\tau^{\varepsilon}}, \quad 0 < \varepsilon < \frac{1}{2}.$$

Therefore, $\widetilde{M_2}^{(1)}(t,\tau)$ satisfies the estimate

$$\widetilde{M_2}^{(1)}(t,\tau) \leq D_5\left[\frac{1}{(t-\tau)^{\frac{1}{2}+\varepsilon}} + \frac{1}{\tau^{\varepsilon}\sqrt{t-\tau}}\right], \quad 0 < \varepsilon < \frac{1}{2}.$$

Now we estimate the integral

$$\widetilde{M_2}^{(2)}(t,\tau) = \frac{3C}{2a^2}\frac{\sqrt{\tau}}{t}\int\limits_{1/N}^{1}\frac{\sqrt{z+\frac{\tau}{t-\tau}}}{z\sqrt{1-z}}\exp\left(-\frac{\tau\left(z+\frac{\tau}{t-\tau}\right)}{2a^2z}\right)I_1\left(\frac{\tau\left(z+\frac{\tau}{t-\tau}\right)}{2a^2z}\right)dz$$

$$\leq \frac{3C}{2a^2}\frac{1}{4}\frac{\sqrt{\tau}}{t}\int\limits_{1/N}^{1}\frac{\sqrt{1+\frac{\tau}{t-\tau}}}{\frac{1}{N}\sqrt{1-z}}dz = \frac{3CN}{2a^2}\frac{1}{4}\frac{\sqrt{\tau}}{t}\sqrt{\frac{t}{t-\tau}} \leq D_6\frac{1}{\sqrt{t-\tau}}.$$

Thus, we show that the kernel $\widetilde{M}(t,\tau)$ of the integral equation (26) has a weak singularity and satisfies the estimate

$$\widetilde{M}(t,\tau) \leq D\left[\frac{1}{(t-\tau)^{\frac{1}{2}+\varepsilon}} + \frac{1}{\tau^{\varepsilon}\sqrt{t-\tau}}\right], \quad 0 < \varepsilon < \frac{1}{2}.$$

This means that the original integral equation (14) in the class of functions has a unique solution, which can be found by the method of successive approximations. This completes the proof of the Theorem. □

7. SOLUTION OF THE BOUNDARY VALUE PROBLEM

Theorem 2. *If the conditions $\sqrt{t}g_1(t) \in L_\infty(0,1)$ and $g_2(t) \in L_\infty(0,1)$ are satisfied, then the boundary value problem (6)−(7) has a unique solution $w(r,t) \in L_\infty(G)$.*

Proof. We estimate each term in the integral representation (13) of the solution of the boundary value problem (6)–(7):

$$w(r,t) \le \int\limits_0^t \frac{r}{2a^2(t-\tau)} \exp\left(-\frac{r^2+\tau^2}{4a^2(t-\tau)}\right) I_1\left(\frac{r\tau}{2a^2(t-\tau)}\right) d\tau = \left\| \frac{\tau}{t-\tau} = z \right\|$$

$$\le \frac{r}{2a^2t} \exp\left(-\frac{r^2}{4a^2t}\right) \int\limits_0^\infty \frac{1}{z} \exp\left(-\frac{(r-t)^2}{4a^2t} z\right) \exp\left(-\frac{r}{2a^2} z\right) I_1\left(\frac{r}{2a^2} z\right) dz$$

$$\le \frac{r}{2a^2t} \exp\left(-\frac{r^2}{4a^2t}\right) \int\limits_0^\infty \frac{1}{z} \exp\left(-\frac{r}{2a^2} z\right) I_1\left(\frac{r}{2a^2} z\right) dz$$

$$= \frac{r}{2a^2t} \exp\left(-\frac{r^2}{4a^2t}\right) \le \frac{1}{2a^2} \exp\left(-\frac{r^2}{4a^2t}\right) \le \frac{1}{2a^2}.$$

Now we estimate the second term

$$\widetilde{g_2}(r,t) \le \int\limits_0^t \frac{r^2}{4a^2(t-\tau)^2} \exp\left(-\frac{r^2}{4a^2(t-\tau)}\right) d\tau = \int\limits_{\frac{r^2}{4a^2t}}^\infty \exp(-z) dz = \exp\left(-\frac{r^2}{4a^2t}\right) \le 1.$$

These estimates imply the validity of the Theorem 2. □

From Theorem 2 and equality (5) we obtain the main result.

Theorem 3. *If $\sqrt{t}g(t) \in L_\infty(0,1)$, then boundary value problem (3)−(4) has a unique solution $u(r,t) \in L_\infty(G)$.*

CONCLUSION

In weighted spaces of essentially bounded functions, the issues of the solvability of a two-dimensional boundary value problem for the heat equation in a domain degenerating into a point at the initial moment of time are studied. In addition to this feature, in the problem under consideration, the boundary conditions contain a derivative with respect to the time variable. The integral equation to which the problem under consideration is reduced is a special Volterra integral equation of the second kind. The study of such integral equations is of independent interest, since for such kind of integral equations problems arise that are absent for the classical Volterra equation of the second kind [8−10].

The obtained results can be useful in the problems of modeling thermophysical processes in gas-discharge plasma of low and high pressure, as well as in studying the processes of melting of electrical contacts, the effect of an electric arc on contacts.

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