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Best approximation by «angle» and the absolute Cesàro summability of double Fourier series

This article is devoted to the topic of absolute summation of series or Cesaro summation. The relevance of this article lies in the fact that a type of absolute summation with vector index which has not been previously studied is considered. In this article, a sufficient condition for the vector index absolute summation method was obtained in terms of the best approximation by «angle» of the functions from Lebesgue space. The theorem that gives a sufficient condition proves the conditions that are sufficient in different cases, which may depend on the parameters. From this proved theorem, a sufficient condition on the term mixed smoothness modulus of the function from Lebesgue space, which is easily obtained by a well-known inequality, is also presented.

Keywords: trigonometric series, Fourier series, Lebesgue space, best approximation by «angle», absolute summability of the series.

Introduction and preliminaries

Let $I_2 = \{(x_1, x_2) \in \mathbf{R}^2 : 0 \leq x_j < 2\pi\}$.

We denote by $L_q(I_2)$ the space of all measurable by Lebesque, 2π-periodic on each variable functions $f(x_1, x_2)$, such that

$$\|f\|_q = \left(\int_0^{2\pi}\int_0^{2\pi} |f(x_1, x_2)|^q dx_1 dx_2\right)^{\frac{1}{q}} < +\infty, 1 \leq q < +\infty.$$

Let $Y_{n_1 n_2}(f)_q$ is two-dimensional best approximation by «angle» of function $f \in L_q(I_2)$. By definition [1–3],

$$Y_{n_1 n_2}(f)_q = \inf_{T_{n_1,\infty}, T_{\infty,n_2}} \|f - T_{n_1,\infty} - T_{\infty,n_2}\|_q \,,$$

where the function $T_{n_1,\infty} \in L_q(I_2)$ is a trigonometric polynomial of degree at most n_1 in x_1, and the function $T_{\infty,n_2} \in L_q(I_2)$ is a trigonometric polynomial of degree at most n_2 in x_2.

Let $r \in \mathbf{N}$, $h_1, h_2 \in \mathbf{R}$. For a function $f \in L_q(I_2)$, the difference of order $r \in \mathbf{N}$ with respect to the variable x_1 and the difference of order $r \in \mathbf{N}$ with respect to the variable x_2 are defined as follows [1–3]:

$$\Delta^r_{h_1,x_1} f(x_1, x_2) = \sum_{\nu_1=0}^{r} (-1)^{r-\nu_1} \cdot C_r^{\nu_1} \cdot f(x_1 + h_1\nu_1, x_2),$$

and, respectively

$$\Delta^r_{h,x_2} f(x_1, x_2) = \sum_{\nu_2=0}^{r} (-1)^{r-\nu_2} \cdot C_r^{\nu_2} \cdot f(x_1, x_2 + h_2\nu_2).$$

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Denote by $\Omega_r(f;t_1,t_2)_q$ the mixed module of smoothness of an order r of function $f \in L_q(I_2)$ [1–3]:

$$\Omega_r(f;t_1,t_2)_q = \sup_{\substack{|h_j|\le t_j\\ j=1,2}} \|\Delta^r_{h_2,x_2}\left(\Delta^r_{h_1,x_1}(f)\right)\|_q.$$

Consider a double trigonometric series

$$\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}(a_{n_1,n_2}\cos n_1x_1\cos n_2x_2 + b_{n_1,n_2}\sin n_1x_1\cos n_2x_2 +$$

$$+c_{n_1,n_2}\cos n_1x_1\sin n_2x_2 + d_{n_1,n_2}\sin n_1x_1\sin n_2x_2) \equiv \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} B_{n_1,n_2}(x_1,x_2). \tag{1}$$

Let's write it down

$$A_n^{(\beta)} = \frac{(\beta+1)(\beta+2)\dots(\beta+n)}{n!},$$

where β is a real number, n is a natural number.

The sum

$$\sigma_{n_1,n_2}^{(\beta_1,\beta_2)}(x_1,x_2) = \sum_{k_1=1}^{n_1}\sum_{k_2=1}^{n_2}\prod_{j=1}^{2} A_{n_j-k_j}^{(\beta_j-1)}\left(A_{n_j}^{(\beta_j)}\right)^{-1} B_{k_1,k_2}(x_1,x_2)$$

called $(C;\beta_1,\beta_2)$ mean of series (1).

The series (1) called $|C;\beta_1,\beta_2|_{\lambda_1,\lambda_2}$-summable (or absolute summable with vector index), $\lambda_j \ge 1$, $j=1,2$, at the point $(x_1,x_2)\in I_2$, if the following series converges:

$$\sum_{n_2=1}^{\infty} n_2^{\lambda_2-1}\left[\sum_{n_1=1}^{\infty} n_1^{\lambda_1-1}\left|\sigma_{n_1,n_2}^{(\beta_1,\beta_2)}(x_1,x_2) - \sigma_{n_1-1,n_2}^{(\beta_1,\beta_2)}(x_1,x_2) -\right.\right.$$

$$\left.\left. -\sigma_{n_1,n_2-1}^{(\beta_1,\beta_2)}(x_1,x_2) + \sigma_{n_1-1,n_2-1}^{(\beta_1,\beta_2)}(x_1,x_2)\right|^{\lambda_1}\right]^{\frac{\lambda_2}{\lambda_1}}. \tag{2}$$

Let

$$\tau_{n_1,n_2}^{(\beta_1,\beta_2)}(x_1,x_2) = \left(\prod_{j=1}^{2} A_{n_j}^{(\beta_j)}\right)^{-1} \sum_{k_1=1}^{n_1}\sum_{k_2=1}^{n_2}\left(\prod_{j=1}^{2} k_j A_{n_j-k_j}^{(\beta_j-1)}\right) B_{k_1,k_2}(x_1,x_2).$$

Then the convergence of the series (2) is equivalent to the convergence of the following series

$$\sum_{n_2=1}^{\infty} n_2^{-1}\left[\sum_{n_1=1}^{\infty} n_1^{-1}\left|\tau_{n_1,n_2}^{(\beta_1,\beta_2)}(x_1,x_2)\right|^{\lambda_1}\right]^{\frac{\lambda_2}{\lambda_1}}.$$

In case $\lambda_2=\lambda_1=\lambda$ we will write $|C;\beta_1,\beta_2|_{\lambda}$ (absolute summability with scalar index) instead of $|C;\beta_1,\beta_2|_{\lambda_1,\lambda_2}$.

Issues related to the absolute Cesàro summability of series began to be studied intensively in the twentieth century. Among the many scientists we can mention the work of F.T. Wang [4], I.E. Zhak and M.F. Timan [5], K. Tandori [6], L. Leindler [7], M.F. Timan [8], Yu.A. Ponomarenko and M.F. Timan [9],

I. Szalay [10, 11], G. Sunouchi [12], who studied the conditions of the absolute Cesàro summability of trigonometric and orthogonal series. In recent years, various generalizations of absolute Cesàro summability have been defined, and the former classical results have been proved for these generalizations. For example, we can cite the works of H. Bor [13–15], Yu. Dansheng and Zhou Guanzhen [16], S. Sonker and A. Munjal [17, 18], E. Savaş [19, 20], B. Rhoades and E. Savaş [21]. In addition, L. Leindler [22] and H. Bor [23] gave a new application of power increasing sequence by applying absolute Cesàro summability for an infinity series. Problems of absolute Cesàro summability of multiple trigonometric Fourier series of functions from different spaces studied in works [24–29]. Almost all of this work is devoted to the topic of absolute Cesàro summability with scalar index. The absolute Cesàro summability with vector index was first defined in [24]. In the article [24] the condition $\beta_1 = \beta_2$ is considered and only sufficient conditions are obtained. Feature of our work is that under sufficient conditions $\beta_1 \neq \beta_2$.

Main results

Now we prove the main results.

We denote $\rho_{k_1k_2} = \sqrt{a^2_{k_1k_2} + b^2_{k_1k_2} + c^2_{k_1k_2} + d^2_{k_1k_2}}$.

Theorem 1. Let $1 < q \leq 2$, $1 \leq \lambda_2 \leq \lambda_1 \leq q$, $\frac{1}{q} + \frac{1}{q'} = 1$. Then for $|C;\beta_1,\beta_2|_{\lambda_1,\lambda_2}$-summability almost everywhere on I_2 Fourier series of function $f(x_1,x_2) \in L_q(I_2)$ is sufficiently,

1) in case of $\frac{1}{q'} < \beta_1 < +\infty$, $\frac{1}{q'} = \beta_2$, for the condition to be met:

$$\sum_{n_2=2}^{\infty} (\ln n_2)^{\frac{\lambda_2}{q}} n_2^{\lambda_2\left(\frac{2}{q}-1\right)-1} \left[\sum_{n_1=1}^{\infty} n_1^{\lambda_1\left(\frac{2}{q}-1\right)-1} \cdot Y^{\lambda_1}_{n_1n_2}(f)_q\right]^{\frac{\lambda_2}{\lambda_1}} < +\infty;$$

2) in case of $\frac{1}{q'} < \beta_1 < +\infty$, $-1 < \beta_2 < \frac{1}{q'}$, for the condition to be met:

$$\sum_{n_2=1}^{\infty} n_2^{\left(\frac{1}{q}-\beta_2\right)\lambda_2-1} \left[\sum_{n_1=1}^{\infty} n_1^{\left(\frac{2}{q}-1\right)\lambda_1-1} \cdot Y^{\lambda_1}_{n_1n_2}(f)_q\right]^{\frac{\lambda_2}{\lambda_1}} < +\infty;$$

3) in case of $-1 < \beta_1 < \frac{1}{q'}$, $\frac{1}{q'} = \beta_2$ for the condition to be met:

$$\sum_{n_2=2}^{\infty} n_2^{\lambda_2\left(\frac{2}{q}-1\right)-1} (\ln n_2)^{\frac{\lambda_2}{q}} \left[\sum_{n_1=1}^{\infty} n_1^{\left(\frac{1}{q}-\beta_1\right)\lambda_1-1} \cdot Y^{\lambda_1}_{n_1n_2}(f)_q\right]^{\frac{\lambda_2}{\lambda_1}} < +\infty.$$

Proof of item 1). It was proved in work [29] that in case of $\frac{1}{q'} < \beta_1 < +\infty$, $\frac{1}{q'} = \beta_2$, if the next series

$$\sum_{n_2=0}^{\infty} \left[\sum_{n_1=0}^{\infty} \left(\sum_{k_1=2^{n_1}}^{2^{n_1+1}-1} \sum_{k_2=2^{n_2}}^{2^{n_2+1}-1} \rho^q_{k_1k_2} \ln k_2\right)^{\frac{\lambda_1}{q}}\right]^{\frac{\lambda_2}{\lambda_1}}$$

converges, then series (1) is $|C;\beta_1,\beta_2|_{\lambda_1,\lambda_2}$-summable almost everywhere on I_2.

By simple calculations, we get

$$S = \sum_{n_2=0}^{\infty} \left[\sum_{n_1=0}^{\infty} \left(\sum_{k_1=2^{n_1}}^{2^{n_1+1}-1} \sum_{k_2=2^{n_2}}^{2^{n_2+1}-1} \rho^q_{k_1k_2} \ln k_2\right)^{\frac{\lambda_1}{q}}\right]^{\frac{\lambda_2}{\lambda_1}} =$$

$$= \sum_{n_2=0}^{\infty}\left[\sum_{n_1=0}^{\infty}\left(\sum_{k_1=2^{n_1}}^{2^{n_1+1}-1}\sum_{k_2=2^{n_2}}^{2^{n_2+1}-1}\rho_{k_1k_2}^{q}(k_1k_2)^{q-2}\ln k_2(k_1k_2)^{2-q}\right)^{\frac{\lambda_1}{q}}\right]^{\frac{\lambda_2}{\lambda_1}} \leq$$

$$\leq C\sum_{n_2=0}^{\infty}\left[\sum_{n_1=0}^{\infty}\left(2^{n_1(2-q)}2^{n_2(2-q)}n_2\sum_{k_1=2^{n_1}}^{2^{n_1+1}-1}\sum_{k_2=2^{n_2}}^{2^{n_2+1}-1}\rho_{k_1k_2}^{q}(k_1k_2)^{q-2}\right)^{\frac{\lambda_1}{q}}\right]^{\frac{\lambda_2}{\lambda_1}} \leq$$

$$\leq C\sum_{n_2=1}^{\infty}2^{n_2\lambda_2\left(\frac{2}{q}-1\right)}n_2^{\frac{\lambda_2}{q}}\left[\sum_{n_1=0}^{\infty}2^{n_1\lambda_1\left(\frac{2}{q}-1\right)}\left(\sum_{k_1=2^{n_1}}^{2^{n_1+1}-1}\sum_{k_2=2^{n_2}}^{2^{n_2+1}-1}\rho_{k_1k_2}^{q}(k_1k_2)^{q-2}\right)^{\frac{\lambda_1}{q}}\right]^{\frac{\lambda_2}{\lambda_1}}.$$

Hence, using the Hardy-Littlewood theorem [30], we obtain

$$S \leq C\sum_{n_2=1}^{\infty}2^{n_2\lambda_2\left(\frac{2}{q}-1\right)}n_2^{\frac{\lambda_2}{q}}\left[\sum_{n_1=0}^{\infty}2^{n_1\lambda_1\left(\frac{2}{q}-1\right)}\left\|\sum_{k_1=2^{n_1}}^{2^{n_1+1}-1}\sum_{k_2=2^{n_2}}^{2^{n_2+1}-1}B_{k_1k_2}(\cdot,\cdot)\right\|_q^{\lambda_1}\right]^{\frac{\lambda_2}{\lambda_1}}. \quad (3)$$

Now, using inequality [1]

$$\left\|\sum_{k_1=2^{n_1}}^{2^{n_1+1}-1}\sum_{k_2=2^{n_2}}^{2^{n_2+1}-1}B_{k_1k_2}(\cdot,\cdot)\right\|_q \leq C\cdot Y_{2^{n_1}-1,2^{n_2}-1}(f)_q, \quad (4)$$

due to the monotonicity of the best approximation by an «angle» from (3), we have

$$S \leq C\sum_{n_2=1}^{\infty}2^{n_2\lambda_2\left(\frac{2}{q}-1\right)}n_2^{\frac{\lambda_2}{q}}\left[\sum_{n_1=0}^{\infty}2^{n_1\lambda_1\left(\frac{2}{q}-1\right)}Y_{2^{n_1}-1,2^{n_2}-1}^{\lambda_1}(f)_q\right]^{\frac{\lambda_2}{\lambda_1}} \leq$$

$$\leq C\sum_{n_2=2}^{\infty}(\ln n_2)^{\frac{\lambda_2}{q}}\,n_2^{\lambda_2\left(\frac{2}{q}-1\right)-1}\left[\sum_{n_1=1}^{\infty}n_1^{\lambda_1\left(\frac{2}{q}-1\right)-1}Y_{n_1n_2}^{\lambda_1}(f)_q\right]^{\frac{\lambda_2}{\lambda_1}}.$$

Proof of item 2). In case of $\frac{1}{q'} < \beta_1 < +\infty, \quad -1 < \beta_2 < \frac{1}{q'}$, by Theorem 2 in [29], the convergence of series

$$\sum_{n_2=0}^{\infty}\left[\sum_{n_1=0}^{\infty}\left(\sum_{k_1=2^{n_1}}^{2^{n_1+1}-1}\sum_{k_2=2^{n_2}}^{2^{n_2+1}-1}\rho_{k_1k_2}^{q}k_2^{q(1-\beta_2)-1}\right)^{\frac{\lambda_1}{q}}\right]^{\frac{\lambda_2}{\lambda_1}}$$

implies the $|C;\beta_1,\beta_2|_{\lambda_1,\lambda_2}$-summability of series (1) almost everywhere on I_2.

Carrying out simple calculations, using the Hardy-Littlewood theorem [30] and inequality (4), we obtain

$$\sum_{n_2=0}^{\infty}\left[\sum_{n_1=0}^{\infty}\left(\sum_{k_1=2^{n_1}}^{2^{n_1+1}-1}\sum_{k_2=2^{n_2}}^{2^{n_2+1}-1}\rho_{k_1k_2}^{q}k_2^{q(1-\beta_2)-1}\right)^{\frac{\lambda_1}{q}}\right]^{\frac{\lambda_2}{\lambda_1}} =$$

$$= \sum_{n_2=0}^{\infty}\left[\sum_{n_1=0}^{\infty}\left(\sum_{k_1=2^{n_1}}^{2^{n_1+1}-1}\sum_{k_2=2^{n_2}}^{2^{n_2+1}-1}\rho_{k_1k_2}^{q}(k_1k_2)^{q-2}k_2^{q(1-\beta_2)-1}(k_1k_2)^{2-q}\right)^{\frac{\lambda_1}{q}}\right]^{\frac{\lambda_2}{\lambda_1}} \leq$$

$$\leq C\sum_{n_2=0}^{\infty}2^{n_2\lambda_2\left(\frac{1}{q}-\beta_2\right)}\left[\sum_{n_1=0}^{\infty}2^{n_1\lambda_1\left(\frac{2}{q}-1\right)}\left(\sum_{k_1=2^{n_1}}^{2^{n_1+1}-1}\sum_{k_2=2^{n_2}}^{2^{n_2+1}-1}\rho_{k_1k_2}^{q}(k_1k_2)^{q-2}\right)^{\frac{\lambda_1}{q}}\right]^{\frac{\lambda_2}{\lambda_1}} \leq$$

$$\leq C\sum_{n_2=0}^{\infty}2^{n_2\lambda_2\left(\frac{1}{q}-\beta_2\right)}\left[\sum_{n_1=0}^{\infty}2^{n_1\lambda_1\left(\frac{2}{q}-1\right)}\left\|\sum_{k_1=2^{n_1}}^{2^{n_1+1}-1}\sum_{k_2=2^{n_2}}^{2^{n_2+1}-1}B_{k_1k_2}(\cdot,\cdot)\right\|_q^{\lambda_1}\right]^{\frac{\lambda_2}{\lambda_1}} \leq$$

$$\leq C\sum_{n_2=0}^{\infty}2^{n_2\lambda_2\left(\frac{1}{q}-\beta_2\right)}\left[\sum_{n_1=0}^{\infty}2^{n_1\lambda_1\left(\frac{2}{q}-1\right)}\cdot Y^{\lambda_1}_{2^{n_1}-1,2^{n_2}-1}(f)_q\right]^{\frac{\lambda_2}{\lambda_1}} \leq$$

$$\leq C\sum_{n_2=1}^{\infty}n_2^{\left(\frac{1}{q}-\beta_2\right)\lambda_2-1}\left[\sum_{n_1=1}^{\infty}n_1^{\left(\frac{2}{q}-1\right)\lambda_1-1}\cdot Y^{\lambda_1}_{n_1n_2}(f)_q\right]^{\frac{\lambda_2}{\lambda_1}}.$$

Proof of item 3). Let $-1 < \beta_1 < \frac{1}{q'}$, $\frac{1}{q'} = \beta_2$. Then by Theorem 2 in [29] the convergence of series

$$\sum_{n_2=0}^{\infty}\left[\sum_{n_1=0}^{\infty}\left(\sum_{k_1=2^{n_1}}^{2^{n_1+1}-1}\sum_{k_2=2^{n_2}}^{2^{n_2+1}-1}\rho_{k_1k_2}^{q}k_1^{q(1-\beta_1)-1}\cdot\ln k_2\right)^{\frac{\lambda_1}{q}}\right]^{\frac{\lambda_2}{\lambda_1}}$$

implies the $|C;\beta_1,\beta_2|_{\lambda_1,\lambda_2}$-summability of series (1) almost everywhere on I_2.

In a similar way to the proof of the previous points, using the Hardy-Littlewood theorem [30] and inequality (4), we get

$$\sum_{n_2=0}^{\infty}\left[\sum_{n_1=0}^{\infty}\left(\sum_{k_1=2^{n_1}}^{2^{n_1+1}-1}\sum_{k_2=2^{n_2}}^{2^{n_2+1}-1}\rho_{k_1k_2}^{q}k_1^{q(1-\beta_1)-1}\cdot\ln k_2\right)^{\frac{\lambda_1}{q}}\right]^{\frac{\lambda_2}{\lambda_1}} =$$

$$= \sum_{n_2=0}^{\infty}\left[\sum_{n_1=0}^{\infty}\left(\sum_{k_1=2^{n_1}}^{2^{n_1+1}-1}\sum_{k_2=2^{n_2}}^{2^{n_2+1}-1}\rho_{k_1k_2}^{q}(k_1k_2)^{q-2}k_1^{q(1-\beta_1)-1}(k_1k_2)^{2-q}\cdot\ln k_2\right)^{\frac{\lambda_1}{q}}\right]^{\frac{\lambda_2}{\lambda_1}} \leq$$

$$\leq C\sum_{n_2=1}^{\infty}2^{n_2\lambda_2\left(\frac{2}{q}-1\right)}n_2^{\frac{\lambda_2}{q}}\left[\sum_{n_1=0}^{\infty}2^{n_1\lambda_1\left(\frac{1}{q}-\beta_1\right)}\left(\sum_{k_1=2^{n_1}}^{2^{n_1+1}-1}\sum_{k_2=2^{n_2}}^{2^{n_2+1}-1}\rho_{k_1k_2}^{q}(k_1k_2)^{q-2}\right)^{\frac{\lambda_1}{q}}\right]^{\frac{\lambda_2}{\lambda_1}} \leq$$

$$\leq C\sum_{n_2=1}^{\infty}2^{n_2\lambda_2\left(\frac{2}{q}-1\right)}n_2^{\frac{\lambda_2}{q}}\left[\sum_{n_1=0}^{\infty}2^{n_1\lambda_1\left(\frac{1}{q}-\beta_1\right)}\left\|\sum_{k_1=2^{n_1}}^{2^{n_1+1}-1}\sum_{k_2=2^{n_2}}^{2^{n_2+1}-1}B_{k_1k_2}(\cdot,\cdot)\right\|_q^{\lambda_1}\right]^{\frac{\lambda_2}{\lambda_1}} \leq$$

$$\leq C \sum_{n_2=2}^{\infty} n_2^{\lambda_2\left(\frac{2}{q}-1\right)-1} (\ln n_2)^{\frac{\lambda_2}{q}} \left[\sum_{n_1=1}^{\infty} n_1^{\left(\frac{1}{q}-\beta_1\right)\lambda_1-1} \cdot Y_{n_1 n_2}^{\lambda_1}(f)_q\right]^{\frac{\lambda_2}{\lambda_1}}.$$

Thus, the theorem is fully proved.

Using the following inequality [1]:

$$Y_{n_1 n_2}(f)_q \leq C \cdot \Omega_r(f; \frac{1}{n_1+1}, \frac{1}{n_2+1})_q$$

we can formulate another result.

Theorem 2. Let $1 < q \leq 2$, $1 \leq \lambda_2 \leq \lambda_1 \leq q$, $\frac{1}{q} + \frac{1}{q'} = 1$ and r is a natural number. Then for $|C; \beta_1, \beta_2|_{\lambda_1, \lambda_2}$-summability almost everywhere on I_2 Fourier series of function $f(x_1, x_2) \in L_q(I_2)$ is sufficiently,

1) in case of $\frac{1}{q'} < \beta_1 < +\infty$, $\frac{1}{q'} = \beta_2$, for the condition to be met:

$$\sum_{n_2=2}^{\infty} (\ln n_2)^{\frac{\lambda_2}{q}} n_2^{\lambda_2\left(\frac{2}{q}-1\right)-1} \left[\sum_{n_1=1}^{\infty} n_1^{\lambda_1\left(\frac{2}{q}-1\right)-1} \cdot \Omega_r^{\lambda_1}(f; \frac{1}{n_1}, \frac{1}{n_2})_q\right]^{\frac{\lambda_2}{\lambda_1}} < +\infty;$$

2) in case of $\frac{1}{q'} < \beta_1 < +\infty$, $-1 < \beta_2 < \frac{1}{q'}$, for the condition to be met:

$$\sum_{n_2=1}^{\infty} n_2^{\left(\frac{1}{q}-\beta_2\right)\lambda_2-1} \left[\sum_{n_1=1}^{\infty} n_1^{\left(\frac{2}{q}-1\right)\lambda_1-1} \cdot \Omega_r^{\lambda_1}(f; \frac{1}{n_1}, \frac{1}{n_2})_q\right]^{\frac{\lambda_2}{\lambda_1}} < +\infty;$$

3) in case of $-1 < \beta_1 < \frac{1}{q'}$, $\frac{1}{q'} = \beta_2$ for the condition to be met:

$$\sum_{n_2=2}^{\infty} n_2^{\lambda_2\left(\frac{2}{q}-1\right)-1} (\ln n_2)^{\frac{\lambda_2}{q}} \left[\sum_{n_1=1}^{\infty} n_1^{\left(\frac{1}{q}-\beta_1\right)\lambda_1-1} \cdot \Omega_r^{\lambda_1}(f; \frac{1}{n_1}, \frac{1}{n_2})_q\right]^{\frac{\lambda_2}{\lambda_1}} < +\infty.$$

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«Бұрышпен» ең жақсы жуықтау және екі еселі Фурье қатарының Чезаро бойынша абсолютті қосындылануы

Мақала қатарлардың абсолютті қосындылануы немесе Чезаро бойынша қосындылану тақырыбына арналған. Бұл жұмыстың өзектілігі мынада: бұрын көп зерттелмеген векторлық индекстің абсолютті қосындылану түрі қарастырылатындығында. Авторлар векторлық индекстің абсолютті қосындылану тәсілі үшін Лебег кеңістігі функциясының «бұрышпен» ең жақсы жуықтауы терминіндегі жеткілікті шартты алған. Жеткілікті шартты беретін теорема параметрлерге байланысты әртүрлі жағдайларда жеткілікті шарттарды дәлелдейді. Осы дәлелденген теоремадан белгілі теңсіздіктің көмегімен Лебег кеңістігі функциясының аралас тегістік модулі терминіндегі жеткілікті шарт алынады.

Кілт сөздер: тригонометриялық қатар, Фурье қатары, Лебег кеңістігі, «бұрышпен» ең жақсы жуықтау, қатардың абсолютті қосындылануы.

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Наилучшее приближение «углом» и абсолютная суммируемость по Чезаро двойных рядов Фурье

Статья посвящена теме абсолютного суммирования рядов, или суммирования Чезаро. Актуальность данной работы заключается в том, что рассматривается не изученный ранее вид абсолютного суммирования с векторным индексом. Авторами получено достаточное условие для метода абсолютного суммирования с векторным индексом в терминах наилучшего приближения «углом» функций из пространства Лебега. Теорема, дающая достаточное условие, доказывает достаточные условия в различных случаях, которые могут зависеть от параметров. Из этой доказанной теоремы выводится также достаточное условие в термине смешанного модуля гладкости функции из пространства Лебега, которое легко получается с помощью известного неравенства.

Ключевые слова: тригонометрический ряд, ряд Фурье, пространство Лебега, наилучшее приближение «углом», абсолютная суммируемость ряда.

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