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Forward and inverse problems for a mixed-type equation with the Caputo fractional derivative and Dezin-type non-local condition

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This paper investigates a mixed-type partial differential equation involving the Caputo fractional derivative of order $\rho \in (0,1)$ for $t > 0$, and a classical parabolic equation for $t < 0$. The problem is studied in an arbitrary N-dimensional domain Ω with smooth boundary, subject to Dezin-type non-local boundary and gluing conditions. For the forward problem, existence and uniqueness of the classical solution are established under suitable assumptions on the data, employing the Fourier method. The influence of the parameter λ in the non-local boundary condition on solvability is analyzed. Additionally, an inverse problem is considered, where the source term is separable as $F(x,t) = f(x)g(t)$, with known $g(t)$ and unknown spatial function $f(x)$. Under certain conditions on $g(t)$, the uniqueness and existence of the solution are proven. This work extends previous results on mixed-type equations, highlighting the role of fractional derivatives and non-local conditions in both forward and inverse settings. The findings contribute to the theory of mixed-type and fractional differential equations, with potential applications in subdiffusion and related processes.

Keywords: mixed type equation, the Caputo derivative, forward problem, inverse problem, Fourier method, Dezin-type non-local condition, existence and uniqueness, gluing conditions.

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Introduction and formulation of problems

Numerous researchers have investigated boundary value problems for differential equations of mixed type. These problems first attracted attention through the work of S. Chaplygin, who applied mixed-type partial differential equations to model gas dynamics. Later, A. Bitsadze [1] demonstrated the ill-posedness of the Dirichlet problem for the equation $u_{xx} + \mathrm{sgn}(y)u_{yy} = 0$.

Let $0 < \rho < 1$. The Caputo fractional derivative of order ρ of a function f is given by [2, p. 336]

$$D_t^\rho f(t) = \frac{1}{\Gamma(1-\rho)} \int_0^t \frac{f'(\tau)}{(t-\tau)^{1-\rho}} d\tau, \quad t > 0,$$

provided the right-hand side exists. Here $\Gamma(\cdot)$ denotes the well-known gamma function.

Let Ω be an arbitrary N-dimensional domain with a sufficiently smooth boundary $\partial\Omega$. Consider the following mixed-type equation:

$$\begin{cases} D_t^\rho u - \Delta u = F(x,t), & x \in \Omega, \quad 0 < t \le \beta, \\ u_t + \Delta u = F(x,t), & x \in \Omega, \quad -\alpha < t < 0, \end{cases} \tag{1}$$

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where $F(x,t)$ is a continuous function and $\alpha > 0$, $\beta > 0$ are given real numbers and Δ is the Laplace operator.

The Dezin problem. Find a function $u(x,t)$ satisfying equation (1) and the boundary condition

$$u(x,t)|_{\partial\Omega} = 0, \quad t \in [\alpha, \beta], \tag{2}$$

and the gluing condition

$$\lim_{t\to+0} u(x,t) = \lim_{t\to-0} u(x,t), \quad x \in \Omega, \tag{3}$$

and also the non-local condition

$$u(x,-\alpha) = \lambda u(x,0), \quad x \in \Omega, \tag{4}$$

where $\lambda = const$, $\lambda \neq 0$.

This problem is called the Dezin problem due to condition (4). Note that if $\lambda = 0$ then we arrive at the backward problem for the subdiffusion equation.

Definition 1. A function $u(t,x) \in AC([0,\beta]; C(\overline{\Omega}))$ with the properties

1. $u(x,t) \in C(\overline{\Omega} \times [-\alpha, 0])$,
2. $\Delta u(x,t) \in C(\overline{\Omega} \times (-\alpha, 0) \cup (0, \beta])$,
3. $D_t^{\rho} u(x,t) \in C(\overline{\Omega} \times (0, \beta])$,
4. $u_t(x,t) \in C(\overline{\Omega} \times (-\alpha, 0))$,

and satisfying conditions (1)–(4) is called a (classical) solution of the problem (1)–(4).

In equation (1), the derivatives of the function $u(x,t)$ are considered in the open domain. The condition of continuity for these derivatives in the closed domain $\overline{\Omega}$ is imposed to facilitate a straightforward proof of the solution's uniqueness. The requirement of absolute continuity of the solution for $t \geq 0$ is necessary to exclude singular functions from consideration, as their inclusion would violate the uniqueness of the solution. Notably, the solution derived via the Fourier method inherently satisfies these continuity and absolute continuity requirements.

Inverse problem. Let $F(x,t) = f(x)g(t)$, and let the function $g(t)$ be known. Find functions $f(x)$ and $u(x,t)$, such that $f(x) \in C(\overline{\Omega})$ and the function $u(x,t)$ satisfies conditions (1)–(4) and conditions of Definition 1, also an additional condition

$$u(x,t_0) = \varphi_0(x), \quad x \in \Omega, \tag{5}$$

here $\varphi_0(x)$ is a given sufficiently smooth function and t_0 is a given point in $(0,\beta)$.

In 1963 A.A. Dezin [3] (see the condition (Γ_1)) studied solvable extensions of mixed-type differential equations. He formulated a boundary value problem characterized by 2π-periodicity and non-local conditions, where the value of the unknown function within a rectangular domain is related to the value of its derivative on the boundary. This formulation involves the Lavrentiev–Bitsadze operator and reflects a significant development in the theory of mixed-type equations.

In works [4–7] non-local boundary value problems of Dezin's type for mixed-type differential equations have been investigated. Let us dwell in more detail on these works.

In [4], the following degenerating mixed type equation is considered:

$$Lu \equiv K(t)u_{xx} + u_{tt} - bK(t)u = F(x,t), \tag{6}$$

in the rectangular domain $D = \{(x,t) : 0 < x < l,\ -\alpha < t < \beta\}$, where $K(t) = (\operatorname{sgn} t)|t|^m$, and $m, b, l > 0$ are given real constants. The study addresses an inhomogeneous Dezin-type non-local boundary condition of the form $u_t(x,-\alpha) - \lambda u(x,0) = \psi(x)$. In [5], a similar problem is examined under the assumptions $m = b = 0$, $\alpha = l$, $\psi(x) = 0$, and $F(x,t) = f(x,t)H(t)$ ($H(t)$ is the Heaviside function), with $\lambda \geq 0$. It is also shown that in the case $\lambda < 0$, the homogeneous problem admits a nontrivial solution. In [6], equation (6) is investigated under the same conditions as in [4], except for the homogeneous case where $F(x,t) \equiv 0$. It should be emphasized that all the abovementioned works focus on forward problems. In the work [8], the forward and inverse problems for equation (1) were studied. In solving the forward problem, instead of the non-local condition (4), the gluing condition $D_t^\rho u(x,+0) = u_t(x,-0)$ was used. The inverse problem of determining the unknown function $f(x)$ was investigated for the case where $g(t) \equiv 1$. In [9], the inverse problem is also considered, where the equation involves for $t > 0$ a Caputo fractional derivative of order ρ, and for $t < 0$ the equation is of hyperbolic type. Furthermore, in [10–12], similar inverse problems are studied for the subdiffusion and mixed-type equations.

In this paper, we consider the forward problem (1)–(4) and the inverse problem (1)–(5) of determining the right-hand side.

1 Preliminaries

Let us denote by $\{v_k\}$ the complete orthonormal eigenfunctions in $L_2(\Omega)$ and by λ_k (where the values λ_k are a sequence of non-negative integers that do not decrease with increasing index k: $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq ...$) the set of positive eigenvalues of the following spectral problem

$$\begin{cases} -\Delta v(x) = \lambda v(x), \ x \in \Omega, \\ v(x)\,|_{\partial\Omega} = 0. \end{cases} \tag{7}$$

Let σ be an arbitrary real number. In the space $L_2(\Omega)$, we introduce the operator $\hat{A}^\sigma$, which operates according to the rule

$$\hat{A}^\sigma g(x) = \sum_{k=1}^{\infty} \lambda_k^\sigma g_k v_k(x).$$

Here $g_k = (g, v_k)$ are the Fourier coefficients of an element $g \in L_2(\Omega)$. Obviously, this operator $\hat{A}^\sigma$ with the domain $D\left(\hat{A}^\sigma\right) = \left\{g \in L_2(\Omega) : \sum_{k=1}^{\infty} \lambda_k^{2\sigma} |g_k|^2 < \infty\right\}$ is selfadjoint. If we denote the operator by A in $L_2(\Omega)$ acting according to the rule $Ag(x) = -\Delta g(x)$ and with the domain of definition $D(A) = \left\{g \in C^2(\overline{\Omega}) : g(x) = 0, \ x \in \partial\Omega\right\}$, then the operator $\hat{A} = \hat{A}^1$ is the selfadjoint extension in $L_2(\Omega)$ of the operator A [13, p. 139].

Our reasoning will largely rely on the methodology developed in the monograph [14].

Lemma 1. [14, p. 453] Let $\sigma > \frac{N}{4}$. Then the following estimate $||\hat{A}^{-\sigma} g||_{C(\Omega)} \leq C||g||_{L_2(\Omega)}$ holds.

In order to prove the existence of a solution to the forward and inverse problems, it is necessary to study the convergence of the following series:

$$\sum_{k=1}^{\infty} \lambda_k^\tau |h_k|^2, \quad \tau > \frac{N}{2}, \tag{8}$$

here h_k are the Fourier coefficients of the function $h(x) \in L_2(\Omega)$. In the case of integer τ, in the paper by Il'in [13] we obtain the conditions for convergence of such series in terms of function $h(x)$ belonging to the classical Sobolev space. In order to formulate this condition, let us introduce the class $\hat{W}_2^1(\Omega)$ as a closure in the $W_2^1(\Omega)$ norm of the set of functions from $C_0^\infty(\Omega)$ that vanish on the boundary of the domain Ω. Il'in's lemma states that if the function $h(x)$ satisfies the following conditions (we can take $\tau = \frac{N}{2} + 1$, if N is even and $\tau = \frac{N+1}{2}$, if N is odd):

$$h(x) \in W_2^{\left[\frac{N}{2}\right]+1}(\Omega), \quad h(x), \, \Delta h(x), ..., \Delta^{\left[\frac{N}{4}\right]} h(x) \in \hat{W}_2^1(\Omega), \tag{9}$$

then the series (8) converges. Similarly, if in (8) τ is replaced by $\tau+1$, then the convergence conditions are

$$h(x) \in W_2^{\left[\frac{N}{2}\right]+2}(\Omega), \quad h(x),\ \Delta h(x), ...,\ \Delta^{\left[\frac{N}{4}\right]} h(x) \in \hat{W}_2^1(\Omega). \tag{10}$$

Next we recall some properties of the Mittag-Leffler function.

Let μ be an arbitrary complex number. The function defined by the following infinite series

$$E_{\rho,\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\rho k + \mu)}$$

is called a Mittag-Leffler function with two parameters [2, p. 56]. If the parameter $\mu = 1$, then we have the classical Mittag-Leffler function: $E_\rho(z) = E_{\rho,1}(z)$.

Lemma 2. [2, p. 61, Eq. (4.4.5)] For any $t \ge 0$ one has

$$0 < E_{\rho,\mu}(-t) \le \frac{C_0}{1+t},$$

where the constant C_0 does not depend on t and μ.

Lemma 3. [2, p. 47]) The classical Mittag-Leffler function of the negative argument $E_\rho(-t)$ is a monotonically decreasing function for all $0 < \rho < 1$ and

$$0 < E_\rho(-t) < 1, \quad E_\rho(0) = 1.$$

Lemma 4. [2, p. 61, Eq. (4.4.5)] Let $\rho > 0$, $\mu > 0$ and $\lambda \in C$. Then for all positive t one has

$$\int_0^t \eta^{\rho-1} E_{\rho,\rho}(\lambda \eta^\rho) d\eta = t^\rho E_{\rho,\rho+1}(\lambda t^\rho).$$

Lemma 5. [2, p. 57, Eq. (4.2.3)] For all $\alpha > 0$, $\mu \in \mathbb{C}$, the following recurrence relation holds:

$$E_{\rho,\mu}(-t) = \frac{1}{\Gamma(\mu)} - tE_{\rho,\mu+\rho}(-t).$$

Lemma 6. [15] Let $\lambda > 0$, $0 < \varepsilon < \rho$. Then, for all $t > 0$, the following coarser estimate holds:

$$\left|t^{\rho-1} E_{\rho,\rho}\left(-\lambda t^\rho\right)\right| \le C \lambda^{\varepsilon-1} t^{\varepsilon\rho-1},$$

where $C > 0$ is a constant independent of λ and t.

2 *Constructing the solution of the forward problem (1)–(4)*

We seek the unknown function $u(x,t)$, which is a solution to the problem (1)–(4), in the form

$$u(x,t) = \sum_{k=1}^{\infty} T_k(t) v_k(x).$$

It is easy to see that the unknown coefficients $T_k(t)$ have the form [2, p. 174]

$$T_k(t) = \begin{cases} a_k E_{\rho,1}(-\lambda_k t^\rho) + \int_0^t s^{\rho-1} E_{\rho,\rho}\left(-\lambda_k s^\rho\right) F_k(t-s)\, ds, & t > 0, \\ b_k e^{\lambda_k t} - \int_t^0 F_k(s) e^{\lambda_k (t-s)} ds, & t < 0, \end{cases}$$

where a_k, b_k are arbitrary constants, and $F_k(t)$ are the Fourier coefficients of the function $F(x,t)$.

By the gluing condition (3) one has $a_k = b_k$. The non-local condition (4) implies:

$$a_k \delta_k = F_k^*, \tag{11}$$

where $F_k^* = \int\limits_{-\alpha}^{0} F_k(s) e^{\lambda_k(-\alpha-s)} ds$ and $\delta_k := \delta_k(\lambda) = e^{-\lambda_k \alpha} - \lambda,\ k \geq 1$.

If for some k, we have $\delta_k = 0$, then equation (11) has a solution only if the free term is zero, i.e., $F_k^* = 0$. In this case, the coefficients a_k remain arbitrary, and problem (1)–(4) does not have a unique solution.

Thus, if $\delta_k \neq 0$ for all k, then the unknown coefficients a_k are uniquely determined, and problem (1)–(4) has a unique solution. Indeed let $u \equiv u_1 - u_2$. We have the following problem for $u(x,t)$:

$$\begin{cases} D_t^\rho u(x,t) - \Delta u(x,t) = 0, & 0 < t < \beta,\ \ x \in \Omega, \\ u_t(x,t) + \Delta u(x,t) = 0, & -\alpha < t < 0,\ \ x \in \Omega, \end{cases} \tag{12}$$

and the conditions (2), (3) and (4).

Assume that $u(x,t)$ satisfies all the conditions of the homogeneous problem, and let v_k be an arbitrary eigenfunction of the spectral problem (7) corresponding to the eigenvalue λ_k. Let

$$T_k(t) = \int\limits_\Omega u(x,t) v_k(x)\, dx, \quad k = 1, 2, \ldots$$

Differentiating under the integral sign with respect to t, which is allowed by the definition of the solution, and using equation (12), we obtain

$$D_t^\rho T_k(t) = \int\limits_\Omega D_t^\rho u(x,t) v_k(x)\, dx = \int\limits_\Omega \Delta u(x,t) v_k(x)\, dx, \quad t > 0,$$

$$\frac{dT_k(t)}{dt} = \int\limits_\Omega \frac{\partial u(x,t)}{\partial t} v_k(x)\, dx = -\int\limits_\Omega \Delta u(x,t) v_k(x)\, dx, \quad t < 0.$$

Integrating by parts and using condition (2), we get:

$$D_t^\alpha T_k(t) = -\lambda_k T_k(t), \quad t > 0, \quad \frac{dT_k(t)}{dt} = \lambda_k T_k(t), \quad t < 0.$$

The solutions to these equations are given by [2, p. 175]:

$$T_k(t) = a_k E_{\rho,1}(-\lambda_k t^\rho), \quad t > 0, \quad T_k(t) = b_k e^{\lambda_k t}, \quad k = 1, 2, \ldots, \quad t < 0. \tag{13}$$

Gluing condition (3) translates into: $T_k(+0) = T_k(-0)$. Using this condition, we find $a_k = b_k$. Applying the non-local condition (4) to get: $a_k \delta_k = 0$. Since $\delta_k \neq 0$ for all $k \in \mathbb{N}$, $a_k = b_k = 0$. Therefore, from (13), we can see that the right-hand sides must be identically zero, which implies that $u(x,t)$ is orthogonal to the complete system $\{v_k(x)\}$. As a result, we conclude that $u(x,t) \equiv 0$ in $\overline{\Omega}$.

Thus, we arrive at the criterion for the uniqueness of the solution to the forward problem (1)–(4):

Theorem 1. If there is a solution to the forward problem (1)–(4), then this solution is unique if and only if the condition $\delta_k \neq 0$ is satisfied for all $k \in \mathbb{N}$.

So we obtain a formal solution to problem (1)–(4) represented in the form

$$u(x,t)=\begin{cases}\sum\limits_{k=1}^{\infty}\left(\frac{F_k^*}{\delta_k}E_{\rho,1}(-\lambda_k t^{\rho})+\int\limits_0^t s^{\rho-1}E_{\rho,\rho}\left(-\lambda_k s^{\rho}\right)F_k(t-s)\,ds\right)v_k(x), & 0\le t\le\beta,\\ \sum\limits_{k=1}^{\infty}\left(\frac{F_k^*}{\delta_k}e^{\lambda_k t}-\int\limits_t^0 F_k(s)e^{\lambda_k(t-s)}ds\right)v_k(x), & -\alpha\le t\le 0.\end{cases}\tag{14}$$

To show that these series satisfy the conditions of Definition 1, we need to estimate the denominator δ_k from below.

3 Lower estimates for the denominator of the solution to the forward problem (1)–(4)

In this section, we investigate the conditions under which δ_k may be equal to zero, and for those cases where $\delta_k\neq 0$, we derive lower bounds for δ_k. It is not hard to see that the following lemma is true:

Lemma 7. Let $\lambda\notin[0,1)$. Then there exists a constant $\delta_0>0$ such that, for all $k\in\mathbb{N}$, the following estimate holds:

$$|\delta_k|>\delta_0,\quad \delta_0=\begin{cases}|\lambda|, & \lambda<0,\\ \lambda-e^{-\lambda_1\alpha}, & \lambda\ge 1.\end{cases}$$

Proof. We consider two separate cases based on the value of the parameter λ.
Case 1. $\lambda<0$. In this case, since $e^{-\lambda_k\alpha}>0$, we have:

$$|\delta_k|=|e^{-\lambda_k\alpha}-\lambda|=|\lambda|+e^{-\lambda_k\alpha}\ge|\lambda|=\delta_0>0.$$

Case 2. $\lambda\ge 1$. In this case, we observe that $e^{-\lambda_k\alpha}\in(0,1)$ for all $k\in\mathbb{N}$, and therefore:

$$|\delta_k|=\left|e^{-\lambda_k\alpha}-\lambda\right|=\lambda-e^{-\lambda_k\alpha}\ge\lambda-e^{-\lambda_1\alpha}=\delta_0>0.$$

This completes the proof. □

Theorem 2. Let $\lambda\notin[0,1)$. Let the function $F(x,t)$ be continuous for all $t\in[-\alpha,\beta]$ and satisfy condition (10) uniformly with respect to t. Then there exists a unique solution of the forward problem (1)–(4), determined by the series (14).

Proof. Now we will show the existence of a solution. The formal solution of problem (1)–(4) has the form

$$u(x,t)=\begin{cases}\sum\limits_{k=1}^{\infty}\left(\frac{F_k^*}{\delta_k}E_{\rho,1}(-\lambda_k t^{\rho})+\int\limits_0^t s^{\rho-1}E_{\rho,\rho}\left(-\lambda_k s^{\rho}\right)F_k(t-s)\,ds\right)v_k(x), & t>0,\\ \sum\limits_{k=1}^{\infty}\left(\frac{F_k^*}{\delta_k}e^{\lambda_k t}-\int\limits_t^0 F_k(s)e^{\lambda_k(t-s)}ds\right)v_k(x), & t<0.\end{cases}\tag{15}$$

Let us now show that the sum of series (15) is indeed a solution to the forward problem. Consider the case for $t>0$, and in the case $t<0$ the absolute convergence of the solution (15) is proved in a similar way. This series is the sum of two series. We denote the first sum by $-\Delta S_1(x,t)$, and the second by $-\Delta S_2(x,t)$. Let the partial sums of the first and second terms have the following forms, respectively:

$$-\Delta S_1^j(x,t)=\sum_{k=1}^{j}\frac{\lambda_k\left(\int\limits_{-\alpha}^{0}F_k(s)e^{\lambda_k(-\alpha-s)}ds\right)E_{\rho,1}(-\lambda_k t^{\rho})}{\delta_k}v_k(x), \tag{16}$$

$$-\Delta S_2^j(x,t)=\sum_{k=1}^{j}\lambda_k\left(\int\limits_{0}^{t}\eta^{\rho-1}E_{\rho,\rho}\left(-\lambda_k\eta^{\rho}\right)F_k(t-s)\,ds\right)v_k(x). \tag{17}$$

In what follows, the symbol C will denote a positive constant, not necessarily the same one.

Let $\sigma>\frac{N}{4}$. Since $\hat{A}^{-\sigma}v_k(x)=\lambda_k^{-\sigma}v_k(x)$, we have by (16)

$$-\Delta S_1^j(x,t)=\hat{A}^{-\sigma}\sum_{k=1}^{j}\frac{\lambda_k^{\sigma+1}\left(\int\limits_{-\alpha}^{0}F_k(s)e^{\lambda_k(-\alpha-s)}ds\right)E_{\rho,1}(-\lambda_k t^{\rho})}{\delta_k}v_k(x).$$

By virtue of Lemma 1 we obtain

$$\left\|-\Delta S_1^j(x,t)\right\|^2_{C(\Omega)}\le C\left\|\sum_{k=1}^{j}\frac{\lambda_k^{\sigma+1}\left(\int\limits_{-\alpha}^{0}F_k(s)e^{\lambda_k(-\alpha-s)}ds\right)E_{\rho,1}(-\lambda_k t^{\rho})}{\delta_k}v_k(x)\right\|^2_{L_2(\Omega)}.$$

Since the system $\{v_k\}$ is orthonormal, by applying Parseval's equality and using Lemma 2 we have

$$\left\|-\Delta S_1^j(x,t)\right\|^2_{C(\Omega)}\le Ct^{-2\rho}\sum_{k=1}^{j}\lambda_k^{2\sigma}\left|\int\limits_{-\alpha}^{0}F_k(s)e^{\lambda_k(-\alpha-s)}ds\right|^2.$$

Applying the Cauchy-Schwarz inequality

$$\left\|-\Delta S_1^j(x,t)\right\|^2_{C(\Omega)}\le\frac{Ct^{-2\rho}}{\lambda_1^2}\int\limits_{-\alpha}^{0}\sum_{k=1}^{j}\lambda_k^{2\sigma}\left|F_k(s)\right|^2ds,\quad \tau=2\sigma>\frac{N}{2}.$$

This means that we have the series, similar to the series (8). Thus, if the function $F(x,t)$ satisfies the conditions (10) with $\tau>\frac{N}{2}$, then the series $|-\Delta S_1(x,t)|^2_{C(\overline{\Omega})}\le C$ will converge if $t>0$.

For the series (17) by virtue of Lemma 6 we get

$$\left\|-\Delta S_2^j(x,t)\right\|^2_{C(\Omega)}\le C\sum_{k=1}^{j}\left|\int_0^t s^{\varepsilon\rho-1}\lambda_k^{\sigma+\varepsilon}F_k(t-s)ds\right|^2.$$

Further, we will apply the generalized Minkowski inequality. Then

$$\left\|-\Delta S_2^j(x,t)\right\|^2_{C(\Omega)}\le C\left[\int_0^t s^{\rho\varepsilon-1}\left(\sum_{k=1}^{j}\left|\lambda_k^{2(\sigma+\varepsilon)}||F_k(t-s)\right|^2\right)^{\frac{1}{2}}ds\right]^2,\quad \tau=2\sigma+2\varepsilon>\frac{N}{2}. \tag{18}$$

Here we again get a series similar to (8). In this case, $\tau=2\sigma+2\varepsilon$. Since ε is an arbitrarily small number, the series (18) converges under the same conditions (10) for the function $F(x,t)$.

Consequently, $|-\Delta S_1(x,t)|^2_{C(\overline{\Omega})} \le C$, $|-\Delta S_2(x,t)|^2_{C(\Omega)} \le C$, $t > 0$. Thus $\Delta u(x,t) \in C(\overline{\Omega} \times (0,\beta])$, in particular $u(x,t) \in C(\overline{\Omega} \times [0,\beta])$. Using completely similar reasoning, it can be shown that sum (15) for $t < 0$ has the same properties as sum (15) for $t > 0$. Hence, $\Delta u(x,t) \in C(\overline{\Omega} \times (-\alpha, 0))$, in particular $u(x,t) \in C(\overline{\Omega} \times [-\alpha, 0])$.

From equation (1), we have $D_t^\rho u(x,t) \in C(\overline{\Omega} \times (0,\beta])$, $u_t(x,t) \in C(\overline{\Omega} \times (-\alpha, 0))$. That $u(x,t)$ is absolutely continuous in a closed region follows from the fact that every function $T_k(t)v_k(x)$ is such. Theorem 2 is proved. □

Lemma 8. Let $0 < \lambda < 1$. Then there exists a number $k_0 \in \mathbb{N}$, such that for all $k > k_0$, the following estimate holds:

$$|\delta_k| \ge \frac{\lambda}{2}.$$

If $0 < \lambda < 1$, then obviously, there is a unique $\lambda_0 > 0$ such that $e^{-\lambda_0 \alpha} = \lambda$. If $\lambda_k \ne \lambda_0$ for all $k \in \mathbb{N}$ then the formal solution of problem (1)–(4) has the form (14).

If $\lambda_k = \lambda_0$ for $k = k_0,\ k_0+1,\ \ldots,\ k_0+p_0-1$, where p_0 is the multiplicity of the eigenvalue λ_{k_0}, then for the solvability of problem (1)–(4) it is necessary and sufficient that the following equality holds (see (11)):

$$F_k^* = (F^*, v_k) = 0, \quad k \in K_0, \quad K_0 = \{k_0, k_0+1, \ldots, k_0+p_0-1\}. \tag{19}$$

In this case, the solution of problem (1)–(4) can be written as follows:

$$u(x,t) = \begin{cases} \sum\limits_{k \notin K_0} T_k(t) v_k(x) + \sum\limits_{k \in K_0} a_k E_{\rho,1}(\lambda_k t^\rho) v_k(x), & t > 0, \\ \sum\limits_{k \notin K_0} T_k(t) v_k(x) + \sum\limits_{k \in K_0} a_k e^{\lambda_k t} v_k(x), & t < 0, \end{cases} \tag{20}$$

here, a_k are arbitrary constants.

Thus, we obtain the following statement:

Theorem 3. Let $0 < \lambda < 1$ and let the function $F(x,t)$ be continuous for all $t \in [-\alpha, \beta]$ and satisfy condition (10) uniformly with respect to t.

1) If $\lambda_k \ne \lambda_0$, for all $k \ge 1$, then there exists a unique solution of the problem (1)–(4) and it can be represented in the form (14).

2) If $\lambda_k = \lambda_0$, for some k and the orthogonality condition (19) holds for indices $k \in K_0$, then the problem (1)–(4) has a solution, which is expressed in the form (20) with arbitrary coefficients a_k.

Proof. We have considered the proof of the first part of the theorem above in Theorem 2. Now, we need to show the convergence of the series (20). If $k \in K_0$, then in the solution (20) additional series are formed as

$$u_0(x,t) = \begin{cases} \sum\limits_{k \in K_0} a_k E_{\rho,1}\left(-\lambda_k t^\rho\right) v_k(x), & t > 0, \\ \sum\limits_{k \in K_0} a_k e^{\lambda_k t} v_k(x), & t < 0. \end{cases}$$

Since K_0 has a finite number of elements, these series consist of finite sum of smooth functions. Therefore, these series satisfy all conditions of Definition 1. □

4 Existence and uniqueness of the solution of the inverse problem (1)–(5)

We study the inverse problem for the equation (1) with the right-hand side of the form $F(x,t) = f(x)g(t)$, where $g(t)$ is a given function and $f(x)$ is an unknown function. Furthermore, since we use the solution of the forward problem when solving the inverse problem, in all subsequent sections we assume that $\delta_k \neq 0$ for all k. According to the additional condition (5), it is sufficient to construct the solution of the inverse problem (1)–(5) only for $t > 0$. Using the representation (14), we obtain the following solution to the inverse problem (1)–(5):

$$u(x,t) = \sum_{k=1}^{\infty} \left(a_{1k} E_{\rho,1}(-\lambda_k t^\rho) + f_k \int_0^t s^{\rho-1} E_{\rho,\rho}\left(-\lambda_k s^\rho\right) g(t-s)\, ds \right) v_k(x), \quad 0 \le t \le \beta, \tag{21}$$

where

$$a_{1k} = \frac{f_k \int\limits_{-\alpha}^{0} g(s) e^{\lambda_k(-\alpha-s)} ds}{\delta_k}.$$

Substituting the function (21) into the condition (5), we obtain the equation

$$\sum_{k=1}^{\infty} T_k(t_0) v_k(x) = \varphi_0(x) = \sum_{k=1}^{\infty} \varphi_{0k} v_k(x), \tag{22}$$

where

$$T_k(t_0) = \frac{f_k \int\limits_{-\alpha}^{0} g(s) e^{\lambda_k(-\alpha-s)} ds}{\delta_k} E_{\rho,1}(-\lambda_k t_0^\rho) + f_k \int_0^{t_0} \eta^{\rho-1} E_{\rho,\rho}\left(-\lambda_k \eta^\rho\right) g(s)\, ds,$$

and

$$\varphi_{0k} = \int_\Omega \varphi_0(x) v_k(x)\, dx, \quad k = 1, 2, \ldots,$$

the numbers f_k are so far unknown and have to be determined.

From the relation (22), we have

$$f_k \Delta_k(t_0) = \delta_k \varphi_{0k} = (e^{-\lambda_k \alpha} - \lambda) \varphi_{0k}, \tag{23}$$

here

$$\Delta_k(t_0) = E_{\rho,1}(-\lambda_k t_0^\rho) \int_{-\alpha}^{0} g(s) e^{\lambda_k(-\alpha-s)} ds + (e^{-\lambda_k \alpha} - \lambda) \int_0^{t_0} s^{\rho-1} E_{\rho,\rho}\left(-\lambda_k s^\rho\right) g(t_0 - s)\, ds.$$

Let us introduce the following notation:

$$I_k(\alpha) = \int_{-\alpha}^{0} g(s) e^{\lambda_k(-\alpha-s)} ds, \quad I_{k,\rho}(t_0) = \int_0^{t_0} s^{\rho-1} E_{\rho,\rho}\left(-\lambda_k s^\rho\right) g(t_0 - s)\, ds.$$

Again, as we noted above, if $\Delta_k(t_0) \neq 0$ for all k, then the coefficients f_k are found uniquely, otherwise, i.e. if $\Delta_k(t_0) = 0$ for some k, according to the equation (23), the coefficients f_k are chosen arbitrarily. Therefore, we have the following uniqueness criterion for the inverse problem (1)–(5):

Theorem 4. The uniqueness of the solution to the inverse problem (1)–(5) is guaranteed if and only if $\Delta_k(t_0) \neq 0$ for all $k \geq 1$.

The uniqueness of the solution of the inverse problem follows from the completeness of the eigenfunctions (see the proof of Theorem 1).

5 Lower estimates for the denominator of the solution to the inverse problem (1)–(5)

We now provide a lower estimate for $\Delta_k(t_0)$. Let $g \in C[-\alpha, \beta]$ and $g(t) \neq 0$, we define

$$m = \min_{t\in[-\alpha,t_0]} |g(t)| > 0, \quad M = \max_{t\in[-\alpha,t_0]} |g(t)| > 0.$$

Lemma 9. Let $\lambda < 0$, $g(t) \in C[-\alpha, \beta]$ and $g(t) \neq 0$, $t \in [-\alpha, \beta]$. Then, there is a constant $C > 0$, depending on t_0 and α, such that for all k:

$$\Delta_k(t_0) \geq \frac{C}{\lambda_k}.$$

Proof. It is sufficient to consider the case $g(t) > 0$, $t \in [-\alpha, \beta]$. If $t_0 \in (0, \beta]$, then (see Lemma 4)

$$I_{k,\rho}(t_0) \geq m \int\limits_0^{t_0} s^{\rho-1} E_{\rho,\rho}(-\lambda_k s^\rho) ds = m t_0^\rho E_{\rho,\rho+1}(-\lambda_k t_0^\rho).$$

Taking into account (see Lemma 5), we obtain

$$I_{k,\rho}(t_0) \geq \frac{1}{\lambda_k}(1 - E_\rho(-\lambda_k t_0^\rho))m \geq \frac{1}{\lambda_k}(1 - E_\rho(-\lambda_1 t_0^\rho))m \geq \frac{C_{t_0}}{\lambda_k},\ C_{t_0} > 0,$$

$$I_k(\alpha) \geq m \int\limits_{-\alpha}^{0} e^{\lambda_k(-\alpha-s)} ds = m\frac{1 - e^{-\lambda_k \alpha}}{\lambda_k} \geq \frac{C_\alpha}{\lambda_k},\ C_\alpha > 0.$$

Therefore,

$$\Delta_k(t_0) \geq E_{\rho,1}(-\lambda_k t_0^\rho)\frac{C_\alpha}{\lambda_k} + (e^{-\lambda_k \alpha} - \lambda)\frac{C_{t_0}}{\lambda_k} \geq (e^{-\lambda_k \alpha} - \lambda)\frac{C_{t_0}}{\lambda_k},$$

which implies the desired assertion because $\lambda < 0$. Lemma 9 is proved. □

Lemma 10. Let $\lambda \geq 1$, $g(t) \in C[-\alpha, \beta]$ and $g(t) \neq 0$, $t \in [-\alpha, \beta]$.

If the number t_0 satisfies the following condition

$$t_0^\rho > \frac{C_0}{\lambda_1}\left(1 + \frac{M}{m}\right), \tag{24}$$

where C_0 is the number in Lemma 2 then, there is a constant $C > 0$ depending on t_0, ρ and α, such that for all k:

$$|\Delta_k(t_0)| \geq \frac{C}{\lambda_k}. \tag{25}$$

If the number t_0 does not satisfy condition (24), then there exists a number k_l, $l \in \mathbb{N}$, such that the estimate (25) holds for all $k > k_l$.

Proof. We begin by estimating $\Delta_k(t_0)$ from below. From its definition, it consists of a sum of two integrals. For the first and second integrals, using Lemma 4 and Lemma 5, we get:

$$\int_{-\alpha}^0 g(s) e^{\lambda_k(-\alpha-s)}\, ds \geq m\frac{1 - e^{-\lambda_k \alpha}}{\lambda_k}, \quad \int_0^{t_0} s^{\rho-1} E_{\rho,\rho}(-\lambda_k s^\rho) g(t_0 - s)\, ds \leq M\frac{1 - E_{\rho,1}(-\lambda_k t_0^\rho)}{\lambda_k}.$$

Hence,

$$\Delta_k(t_0) \geq \frac{E_{\rho,1}(-\lambda_k t_0^\rho)}{\lambda_k}\left[m(1 - e^{-\lambda_k \alpha}) + (\lambda - e^{-\lambda_k \alpha})M\right] - \frac{M}{\lambda_k},$$

which implies

$$\Delta_k(t_0) \geq -\frac{M}{\lambda_k},$$

where $C_1 = M$.

Next, to estimate $\Delta_k(t_0)$ from above using Lemma 2, we obtain:

$$\Delta_k(t_0) \leq \left(\frac{1-e^{-\lambda_k \alpha}}{\lambda_k}\right)\left(\frac{C_0(M+m)}{\lambda_k t_0^\rho} - m\right). \tag{26}$$

Note that the expression in parentheses becomes negative under the assumption:

$$t_0^\rho > \frac{C_0}{\lambda_1}\left(1+\frac{M}{m}\right).$$

Thus, for all $k \in \mathbb{N}$, we have:

$$\Delta_k(t_0) \leq -\frac{C_2}{\lambda_k},$$

where $C_2 = \left(\frac{M+m}{\lambda_1 t_0^\rho} - m\right) > 0$.

Hence, there exists a constant $C = \min\{C_1, C_2\}$ such that the required lower bound holds.

Now let $\lambda \geq 1$ and assume that, condition (24) not be satisfied for the given values of the parameter. However, there exists an index k_l, such that for all $k > k_l$ the condition $t_0^\rho > \frac{C_0}{\lambda_k}\left(1+\frac{M}{m}\right)$ is satisfied, since $\frac{C_0}{\lambda_k}\left(\frac{M+m}{m}\right) \to 0$ as $k \to \infty$, (see (26)). Therefore, for all $k > k_l$ the estimate (25) holds. Lemma 10 is proved. □

Lemma 11. Let $0 < \lambda < 1$, $g(t) \in C[-\alpha, \beta]$ and $g(t) \neq 0$, $t \in [-\alpha, \beta]$. Then for all $k > k_r$, $r \in \mathbb{N}$ the following estimate

$$|\Delta_{k,\rho}(t_0)| \geq \frac{C}{\lambda_k} \tag{27}$$

is valid, where a constant $C > 0$ depends on ρ, t_0 and α.

Proof. Since $\delta_k \neq 0$, it follows that $\lambda_k \neq \lambda_0$ for all k. Therefore, we consider only the following two cases.

Case 1. Let $\lambda_k < \lambda_0$. In this case, based on the proof of Lemma 9, it is not difficult to see that for all $k < k_0$, the following estimate holds:

$$\Delta_{k,\rho}(t_0) > c_0,$$

where $c_0 > 0$ is a constant depending on α, t_0, and ρ.

Case 2. Let $\lambda_k > \lambda_0$. We prove this case of the lemma similarly to the proofs of the previous lemmas. The lower bound of $\Delta_k(t_0)$ has the form (see Lemma 10)

$$\Delta_k(t_0) \geq -\frac{C_1}{\lambda_k}.$$

Now, we establish an upper bound for $\Delta_k(t_0)$. To this end, using Lemma 4, Lemma 5, and Lemma 2, we obtain:

$$\Delta_k(t_0) \leq \frac{C_0}{\lambda_k^2 t_0^\rho}\left(M(1-e^{-\lambda_k\alpha}) + (\lambda - e^{-\lambda_k\alpha})m\right) - \frac{(\lambda - e^{-\lambda_k\alpha})m}{\lambda_k}.$$

Thus, for all $k > k_r$, we have

$$\Delta_k(t_0) \leq -\frac{C_3}{\lambda_k},$$

where $C_3 = \left(\lambda - e^{-\lambda_k\alpha}\right)m > 0$.

Therefore, there exists a constant $C = \min\{c_0, C_1, C_3\}$ such that for all $k > k_r$ the required lower bound holds. Lemma 11 is proved. □

The above estimates (25) and (27) allows to determine explicitly the index from which they hold. For example, according to the proof of the second condition of Lemma 10, the index k_l is given by

$$k_l = \min\left\{k : t_0^\rho > \frac{1}{\lambda_k}\left(1 + \frac{M}{m}\right)\right\}.$$

Similarly, for estimate (27), the index k_r can be determined in the same way.

Hence, we introduce the set:

$$\mathbb{K}_0 = \{k \in \mathbb{N} : \Delta_k(t_0) = 0\}.$$

Remark 1. Note that if $k \in \mathbb{K}_0$, then obviously $\delta_k \neq 0$.

Lemma 12. The set $\mathbb{K}_0$ is either empty or contains only finitely many elements.

Proof. From the proof of Lemma 10, it follows that if there exists an index $k \in \mathbb{K}_0$, then necessarily $k \leq k_l$. Therefore, $\mathbb{K}_0$ is a finite set. Moreover, as mentioned in Section 1, the sequence $\{\lambda_k\}$ consists of discrete values. Hence, $\Delta_k(t_0)$ can vanish only at isolated indices, and it is possible that no such index exists. In this case, the set $\mathbb{K}_0$ is empty. A similar argument is valid for the elements of the set $\mathbb{K}_0$ when $k \leq k_r$. This completes the proof of Lemma 12. □

Theorem 5. Let $g(t) \in C[-\alpha, \beta]$ and $g(t) \neq 0$, $t \in [-\alpha, \beta]$. Let $\lambda < 0$ and the function $\varphi_0(x)$ satisfies the conditions (9). Then there exists a unique solution of the inverse problem (1)–(5) and it can be represented as:

$$\begin{aligned} u(x,t) = \textstyle\sum_{k=1}^{\infty} & \left(\frac{\varphi_{0k}}{\Delta_k(t_0)} E_{\rho,1}(\lambda_k t^\rho) \int\limits_{-\alpha}^{0} g(s) e^{\lambda_k(-\alpha-s)} ds\right) v_k(x) \\ & + \textstyle\sum_{k=1}^{\infty} \left(\frac{\varphi_{0k}(e^{-\lambda_k \alpha} - \lambda)}{\Delta_k(t_0)} \int\limits_{0}^{t} s^{\rho-1} E_{\rho,\rho}\left(-\lambda_k s^\rho\right) g(t-s)\, ds\right) v_k(x), \quad t > 0, \end{aligned} \tag{28}$$

$$u(x,t) = \sum_{k=1}^{\infty} \left(\frac{\varphi_{0k}}{\Delta_k(t_0)} e^{\lambda_k t} \int\limits_{-\alpha}^{0} g(s) e^{\lambda_k(-\alpha-s)} ds - \frac{\varphi_{0k}(e^{-\lambda_k \alpha} - \lambda)}{\Delta_k(t_0)} \int\limits_{t}^{0} g(s) e^{\lambda_k(t-s)} ds\right) v_k(x), \quad t < 0.$$

$$f(x) = \sum_{k=1}^{\infty} \frac{\varphi_{0k}(e^{-\lambda_k \alpha} - \lambda)}{\Delta_k(t_0)} v_k(x). \tag{29}$$

Proof. We write the series (28) as sums of two series: $I_1(x,t)$ and $I_2(x,t)$. If $I_1^j(x,t)$ and $I_2^j(x,t)$ are the corresponding partial sums, then we have:

$$-\Delta I_1^j(x,t) = \sum_{k=1}^{j} \left(\frac{\lambda_k \varphi_{0k}}{\Delta_k(t_0)} E_{\rho,1}(\lambda_k t^\rho) \int\limits_{-\alpha}^{0} g(s) e^{\lambda_k(-\alpha-s)} ds\right) v_k(x),$$

$$-\Delta I_2^j(x,t) = \sum_{k=1}^{j} \left(\frac{\lambda_k \varphi_{0k}(e^{-\lambda_k \alpha} - \lambda)}{\Delta_k(t_0)} \int\limits_{0}^{t} s^{\rho-1} E_{\rho,\rho}\left(-\lambda_k s^\rho\right) g(t-s)\, ds\right) v_k(x).$$

Next, applying the identity $\hat{A}^{-\sigma} v_k(x) = \lambda_k^{-\sigma} v_k(x)$ and using Lemma 1, Lemma 2, and by applying Parseval's equality, we obtain:

$$\left\|-\Delta I_1^j(x,t)\right\|_{C(\Omega)}^2 \leq \frac{MCt^{-2\rho}}{\lambda_1} \sum_{k=1}^{j} |\varphi_{0k}|^2 \lambda_k^{2\sigma}, \quad \tau = 2\sigma > \frac{N}{2}.$$

By the Lemma 4 and Lemma 2 we have

$$\left\|-\Delta I_2^j(x,t)\right\|^2_{C(\Omega)} \le \frac{MC}{\lambda_1}\left(\sum_{k=1}^{j}\lambda_k^{2\sigma}\left|\varphi_{0k}\right|^2\right), \quad \tau = 2\sigma > \frac{N}{2}.$$

It is easy to see that

$$\|f(x)\|^2_{C(\Omega)} \le C\sum_{k=1}^{j}\lambda^{2\sigma}\left|\varphi_{0k}\right|^2, \quad \tau = 2\sigma > \frac{N}{2}.$$

Therefore, if the function $\varphi_0(x)$ satisfies the conditions (10), then the following estimates hold:

$$\left\|-\Delta I_1^j(x,t)\right\|^2_{C(\Omega)} \le C, \quad \left\|-\Delta I_2^j(x,t)\right\|^2_{C(\Omega)} \le C, \quad \|f(x)\|^2_{C(\Omega)} \le C, \quad t>0.$$

Thus, we conclude that $\Delta u(x,t) \in C(\overline{\Omega}\times(0,\beta])$. In particular, $u(x,t) \in C(\overline{\Omega}\times[0,\beta])$, and $f(x)\in C(\overline{\Omega})$. Theorem 5 is proved. □

Theorem 6. Let $\varphi_0(x)$ satisfy the conditions (9) and $g(t)\in C[-\alpha,\beta]$, $g(t)\neq 0$, $t\in[-\alpha,\beta]$ and let $\delta_k \neq 0$ for all k. Moreover, let the assumptions of Lemma 10 or Lemma 11 hold.

1) If the set $\mathbb{K}_0$ is empty, then there exists a unique solution of the inverse problem (1)–(5) and it can be represented as the series in Theorem 5.

2) If the set $\mathbb{K}_0$ is not empty, then for the existence of a solution to the inverse problem (1)–(5), it is necessary and sufficient that the following conditions

$$\varphi_{0k} = (\varphi_0, v_k) = 0, \ k\in\mathbb{K}_0$$

be satisfied. In this case, the solution to inverse problem (1)–(5) exists, but is not unique:

$$f(x) = \sum_{k\notin\mathbb{K}_0}\frac{\delta_k\varphi_{0k}}{\Delta_k(t_0)}v_k(x) + \sum_{k\in\mathbb{K}_0} f_k v_k(x), \tag{30}$$

$$u(x,t) = \sum_{k=1}^{\infty} f_k\left(\frac{\int\limits_{-\alpha}^{0} g(s)e^{\lambda_k(-\alpha-s)}ds}{\delta_k}E_{\rho,1}(-\lambda_k t^\rho) + \int\limits_0^t s^{\rho-1}E_{\rho,\rho}\left(-\lambda_k s^\rho\right)g(t-s)\,ds\right)v_k(x), \quad t>0, \tag{31}$$

$$u(x,t) = \sum_{k=1}^{\infty} f_k\left(\frac{\int\limits_{-\alpha}^{0} g(s)e^{\lambda_k(-\alpha-s)}ds}{\delta_k}e^{\lambda_k t} - \int\limits_t^0 g(s)e^{\lambda_k(t-s)}ds\right)v_k(x), \quad t<0, \tag{32}$$

where if $k\notin\mathbb{K}_0$ then f_k has the form (23) and if $k\in\mathbb{K}_0$, then f_k are arbitrary real numbers.

Proof. To prove the theorem we need to show that the series (30), (31) and (32) satisfy all the conditions of Definition 1. This follows directly from the proof of Theorem 5, and the proof is almost the same when any of the conditions of Lemma 10 or Lemma 11 hold. For clarity, let us suppose that the assumptions of Lemma 10 are satisfied. Series (31) and (32) are divided into two parts, following the structure given in (30). The second part of both these series, as stated in Lemma 12, is a finite sum of smooth functions. In the first part, the satisfaction of the series of the conditions of Definition 1 can be proved in the same way as for the series (28). Here we use the lower bound (25) for $\Delta_k(t_0)$. The convergence of the first part of (30) is shown similarly to that of the series (29), while the second part is a sum of finitely many smooth functions. □

Conclusion

In this work, a subdifusion equation with the Caputo fractional derivative of order $\rho \in (0,1)$ is studied for $t > 0$, while a classical parabolic equation is considered for $t < 0$. Following the work [3], forward and inverse problems ($f(x)$ is unknown) are considered with a non-local Dezin type condition. The solutions are constructed using the classical Fourier method. The main contribution of the authors is that such non-local direct and inverse problems for mixed-type equations with a fractional order have not been previously studied. In the process of studying these problems, we investigate the effect of the parameter λ in Dezin's condition, on the existence and uniqueness of the solution. As proved, it is shown that for certain values of λ, the uniqueness of the solution may fail, and in order to recover the solution, orthogonality conditions on the given functions $\varphi_0(x)$ and $F(x,t)$ are required.

In the future, it would be of interest to consider other types of fractional derivatives instead of the Caputo derivative, in order to investigate whether similar effects occur. Another promising direction is the study of inverse problems aimed at determining fractional orders in mixed-type equations for such nonlocal problems.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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