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On Writing Mathematical Papers in English

The article is intended to provide practical help for authors of mathematical papers. It is written mainly for non-English speaking writers but should prove useful even to native speakers of English who are beginning their mathematical writing and may not yet have developed a template of the structure of mathematical discourse. The article is oriented mainly to research mathematics but applies to almost all mathematics writing, except more elementary texts. There is no intention whatsoever to impose any uniformity of mathematical style. Quite the contrary, the aim is to encourage prospective authors to write structurally correct manuscripts as expressively and flexibly as possible, but without compromising certain basic and universal rules.

Keywords: language of mathematics, advice and rules on writing mathematical texts, research mathematics.

Mathematics writing is different from ordinary writing and harder - in addition to all the requirement of ordinary good writing, there are additional constraints and conventions in mathematics. An additional constraint is that mathematics follows much more demanding rules of logic than ordinary discourse, and your writing must follow and display this logic. However, this does not mean that there is only one right way to present a mathematical argument. Some of the additional conventions are those for defining new concepts and those for organizing the material through theorems and examples.

Communicating your ideas and knowledge through writing and other media is a very important skill to mathematicians. Mathematics often includes concepts and ideas that you cannot easily express using equations and formulae. Mathematicians must write their ideas down to add to the body of mathematical knowledge. They must communicate their thought processes to non-mathematicians, often their employers. The ability to write clearly is as important a skill to mathematicians as solving equations.

Writing mathematics is not the same as showing your work. You do not write papers to demonstrate that you have done your work, but rather to demonstrate how well you understand the ideas and concepts. A list of calculations without any context or explanation demonstrates that you have spent time doing computations, but it omits ideas. It contains no mathematics. Writing good, clear mathematical explanations will also help you improve your knowledge and understanding of the mathematical ideas and concepts you encounter. The act of writing the explanation will force you to think more carefully about what you are doing. This means clear, carefully-written mathematics will more likely be correct, and the process will help you learn and retain the concepts. This paper discusses some of the basic ideas involved in writing a mathematical paper. For more information, you can consult reference books such as Krantz [1997], Higham [1998], and Knuth [1989]. You might also consult Turner [1998] for a brief introduction to proofs [1].

A good mathematical paper has a fairly standard format. Most short papers are divided up into about a half dozen sections, which are numbered and titled. Most papers have an abstract, an introduction, a number of sections of discussion, and a list of references, but no formal table of contents or index. On occasion, papers have appendices, which give special detailed information or provide necessary general background to secondary audiences. In some fields, papers routinely have a conclusion. This section is not present simply to balance the introduction and to close the paper. Rather, the conclusion discusses the results from an overall perspective, brings together the loose ends, and makes recommendations for further research. In mathematics, these issues are almost always treated in the introduction, where they reach more readers; so a conclusion is rare [2].

A well-organized paper is easier to read than a disorganized one. Fortunately, there are some standard ways to order a mathematics essay. A math paper should be punctuated, spelled, and organized clearly, just as in papers for disciplines besides math.

The title of your paper should be informative. It should be informative without being too long. Choosing a good title in a mathematics paper is not so easy. Often a paper hinges on a concept that is defined only within the paper itself, so using the name of that concept in the title will convey no meaning at all. Generally, titles should have no more than ten words, although, admittedly, I have not followed this advice on several occasions.

The abstract is the most important section. First it identifies the subject; it repeats words and phrases from the title to corroborate a reader's first impression, and it gives details that didn't fit into the title. Then it lays out the central issues, and summarizes the discussion to come. The abstract includes no general background material. It is essentially a table of contents in a paragraph of prose. The abstract allows readers to decide quickly about reading on. While many will decide to stop there, the potentially interested will continue. The goal is not to entice all, but to inform the interested efficiently. Remember, readers are busy. They have to decide quickly whether your paper is worth their time. They have to decide whether the subject matter is of interest to them, and whether the presentation will bog them down. A well-written abstract will increase the readership.

Some writers list *key words* supplied by the author, usually after abstract. The number of key words is usually ten or less. Since the key words may be used in computer searches, you should try to anticipate words for which reader might search and make them specific enough to give a good indication of paper's content.

The Introduction is the most important part of your paper. Although some mathematicians advise that the Introduction be written last, I advocate that the Introduction be written first. I find that writing the Introduction first helps me to organize my thoughts. However, I return to the Introduction many times while writing the paper, and after I finish the paper, I will read and revise the Introduction several times.

Get to the purpose of your paper as soon as possible. Don't begin with a pile of notation. Even at the risk of being less technical, inform readers of the purpose of your paper as soon as you can. Readers want to know as soon as possible if they are interested in reading your paper or not. If you don't immediately bring readers to the objective of your paper, you will lose readers who might be interested in your work but, being pressed for time, will move on to other papers or matters because they do not want to read further in your paper.

To state your main results precisely, considerable notation and terminology may need to be introduced. At this point, you do not want the reader to be bogged down with technical definitions and notation, and so it is therefore preferable to informally describe your results in such instances. Try to be as informative and precise as possible without drifting off into too much technical jargon.

Why are you writing this paper? The logic in climbing a mountain, «because it is there», does not apply to writing and publishing a paper. Just because you can prove a theorem does not mean that you should publish it and its proof. For example, the theorem may be of interest to no one else, the proof may involve no new ideas, or, despite a proof not being in the literature, the theorem can be easily proved by many, in particular, students.

Put your paper in an historical context. Indicate what you have done in relation to what others have done. Briefly survey the pertinent results of others to your work. On the other hand, as you place your results in an historical perspective, do not name drop. Ramanujan, J.P. Serre, and P. Deligne are common names that writers like to drop to enhance their own particular work. Readers will recognize that you are referring to these famous mathematicians in an attempt to bring attention to your work, which likely may not receive any notice otherwise [3].

Write a good introduction. Most people who read a mathematics paper will only read the introduction and skim the theorems.

The paper body discusses the various aspects of the subject individually. First, present the material in small digestible portions. Second, beware of jumping haphazardly from one detail to another, and of illogically making some details specific and others generic. Third, if possible, follow a sequential path through the subject. If such a path simply doesn't exist, then break the subject down into logical units, and present them in the order most conducive to understanding. If the units are independent, then order them according to their importance to the primary audience.

After stating what the problem is, we usually then state the answer, even before we show how we got it. Sometimes we even state the answer right along with the problem. It's uncommon, although not so uncommon as to be exceptional, to read a math paper in which the answer is left for the very end. Explaining the solution and then the answer is usually reserved for cases where the solution technique is even more interesting than the answer, or when the writers want to leave the readers in suspense. But if the solution is messy or boring, then it's typically best to hook the readers with the answer before they get bogged down in details.

Math is difficult enough that the writing around it should be simple. «Beautiful» math papers are the ones that are the easiest to read: clear explanations, uncluttered expositions on the page, well-organized presentation. For that reason, mathematical writing is not a creative endeavor the same way that, say, poetry is:

you shouldn't be spending a lot of time looking for the perfect word, but rather should be developing the clearest exposition. Unlike humanities students, mathematicians don't have to worry about over-using 'trite' phrases in mathematics. In fact, at the end of this article is a list of trite but useful advice that you may want to use in your papers, either in this class or in the future.

This guide, together with the advice, should serve as a reference while you write and will also be referred to when I comment on the drafts of the problems for your writing paper. If you can master these basic areas, your writing may not be spectacular, but it should be clear and easy to read - which is the goal of mathematical writing, after all.

You should use this article as a guide while you write because you will be graded directly on the criteria outlined on the checklist. What follows here is a more detailed explanation of the criteria that will be used for grading your papers.

1. Clearly restate the problem to be solved. Do not assume that the reader knows what you're talking about. The person you're writing to might be out on vacation, for example, or have a weak memory. You don't have to restate every detail, but you should explain enough so that someone who's never seen the assignment can read your paper and understand what's going on, without any further explanation from you.

2. State the answer in a complete sentence which stands on its own. If you can avoid variables in your answer, do so; otherwise, remind the reader what they stand for. If your answer is at the end of the paper and you've made any significant assumptions, restate them, too. Do not assume that the reader has actually read every word and remembers it all (do you?).

3. The burden of communication lies on you, not on your reader. It is your job to explain your thoughts; it is not your reader's job to guess them from a few hints. You are trying to convince a skeptical reader who doesn't believe you, so you need to argue with airtight logic in crystal clear language; otherwise he/she will continue to doubt. If you didn't write something on the paper, then (a) you didn't communicate it, (b) the reader didn't learn it, and (c) the grader has to assume you didn't know it in the first place.

4. Clearly state the assumptions which underlie the formulas. For example, what mathematical assumptions do you have to make? No friction, no air resistance? That something is lying on its side, or far away from everything else? Sometimes things are so straightforward that there are no assumptions, but not often.

5. Use diagrams, tables, or graphs, to help explain the math and clearly label them (if these are used). In math, even more than in literature, a picture is worth a thousand words, especially if it's well labeled. Label all axes, with words, if you use a graph. Give diagrams a title describing what they represent. It should be clear from the picture what any variables in the diagram should represent. The whole idea is to make everything as clear and self-explanatory as possible.

6. Define all variables used. (a) Even if you label your diagram (and you should), you should still explain in words what your variables are. (b) If there's a quantity you use only a few times, see if you can get away with not assigning it a variable.

7. Explain how each formula is derived, or where it can be found. Don't pull formulas out of a hat, and don't use variables which you don't define. Either derive the formula yourself in the paper, or explain exactly where you found it, so other people can find it, too. Put important or long formulas on a line of their own, and then center them; it makes them much easier to read.

8. In this paper, are the spelling, grammar, and punctuation correct? It may surprise you that it is on spelling and grammar that people tend to lose most of their points on their mathematics papers. Please spell-check and proofread your work for grammar mistakes.

9. Use English words. Although there will usually be equations or mathematical statements in your proofs, use English sentences to connect them and display their logical relationships. If you look in your textbook, you'll see that each proof consists mostly of English words. In general, you should try to minimize your use of symbols, such as $\Rightarrow$, $\forall$, $\exists$, in order to make your writing easier for the reader to parse. These symbols are fine in homework or in scratch work, but should only be used if necessary in your portfolio problems.

10. Use complete sentences. If you wrote a history essay in sentence fragments, the reader would not understand what you meant; likewise in mathematics you must use complete sentences, with verbs to convey your logical train of thought. Some complete sentences can be written purely in mathematical symbols, such as equations (like $a^3 = b^{-1}$), inequalities (like $o(a) < 5$), and other relations (like $5,7$ or $10 \in Z$). These statements usually express a relationship between two mathematical objects, like numbers (e.g., 10), matrices, or sets (e.g., R).

11. Show the logical connections among your sentences. Use phrases like «therefore» or «because» or «if . . . then . . .» or «if and only if» to connect your sentences.

12. Use whitespace. Don't cram your answer into a few lines of the paper, filled from left margin to right margin. Let your writing breathe! When you start a new thought, start a new line. Use indentation to organize your sentences. This helps the reader understand your thought much better, and it also encourages you to be clearer.

13. Use scratch paper. Finding your proof will be a long, messy process, full of false starts and dead ends. Does all that on scratch paper until you find a real proof, and Rules and Tips for Writing Mathematics Discrete Math Page 4 of 5 only then break out your clean paper to write your final proof carefully. Do not hand in your scratch work! Only sentences that actually contribute to your proof should be part of the proof. Do not just perform a brain dump," throwing everything you know onto the paper before showing the logical steps that prove the conclusion. That is what scratch paper is for.

14. «=» means equals. Don't write $A = B$ unless you mean that A actually equals B. This rule seems obvious, but there is a great temptation to be sloppy. For example in Calculus a student might write something like this: $f'(x) = x^2 = 2x$ when they really mean that the derivative of the function is $2x$.

15. Avoid weasel words. There are some notorious phrases that advertise that you don't really understand the logic you need. Be wary of phrases like, «clearly», «obviously», and, «the only way this can happen is». There is a time and a place for all of these phrases, but most likely that time and place will not be in your proofs in this class.

Types of Mathematical Results. In mathematics, results are labelled as a theorem, a lemma, or a corollary. What's the difference?

– A theorem is a main result.

– A lemma is a result whose primary purpose is to be used in the proof of a theorem but which, on its own, is not considered significant or as interesting.

– A corollary is a result that follows from a theorem. It could be a special case of the theorem or a particularly important consequence of it.

So theorems stand on their own, a lemma always comes before a theorem, and corollaries always come after a theorem. The order, in which these appear, then, is always Lemma, Theorem and Corollary.

There is no reason a theorem must have a lemma before it or a corollary after it. But if you have a string of lemmas which don't lead to a theorem, for instance, then it will look strange to anyone experienced with mathematical writing.

Here are two examples. First we give a lemma and a theorem whose proof depends on the lemma.

Lemma 1. In the integers, if d is a factor of a and b then d is a factor of $ax + by$ for any integers x and y.

P*roof.* Since d is a factor of both a and b, we can write $a = dm$ and $b = dn$ for some integers m and n. Then for any x and y we have

$$ax + by = dmx + dny = d(mx + ny),$$

which shows d is a factor of $ax + by$.

Theorem 2. If a and b are integers and $ax + by = 1$ for some integers x and y, then a and b have no common factor greater than 1.

Proof. This will be a proof by contradiction. Suppose there is a common factor $d > 1$ of a and b. Applying Lemma 1 to the particular combination $ax + by$, d is a factor of $ax + by$, so d is a factor of 1. But there are no factors of 1 which are greater than 1, so we have a contradiction. Therefore a and b have no common factor greater than 1.

Theorem 2 uses Lemma 1, but the statement of Lemma 1 was deemed (by the author writing it) to be worth isolating on its own. So it becomes a lemma rather than appear completely inside the proof of Theorem 2. Perhaps the author anticipates other uses of Lemma 1, and so wants to state it separately [4].

The following advice is aimed at students starting to write mathematical reports or theses, for undergraduate projects, MSc dissertations, PhD theses, or papers for publication. Most of these notes are informal suggestions, aimed at helping students develop a good written style.

Advice on Writing Mathematics

Tone and Style:

1. In most technical writing, the word I should be avoided unless the author's persona is important. Instead, use the word we when writing mathematics. We here mean you and me together. Often the word we can be omitted by rewriting the sentence.

Bad: I proved that S is closed and bounded, and thus compact.

Good: We proved that the set S is closed and bounded, so S is compact.

Good: The set S is compact since S is closed and bounded.

2. Try to write in the active voice and present tense.

Bad: S has the compactness property since S was closed and bounded.

Good: The set S is compact since S is closed and bounded.

3. Don't use the style of homework papers, in which a sequence of formulas is merely listed. Tie the concepts together with a running commentary. The commentary should scan smoothly if all but the simplest mathematical objects or expressions are replaced with noun. However, relations (like $=$ or $\Leftrightarrow$) should be read using their English equivalents. For instance, since $x = 2\theta$, the double angle formula shows that $\sin x = 2 \sin \theta \cos \theta$ scans fine as since noun equals noun, the noun shows that noun equals noun.

4. Less is more so eliminate unnecessary words. Pascal wrote, «I have made this letter longer than usual because I lack the time to make it shorter».

5. In general it is acceptable to omit simple steps in your arguments when your readers can fill in the details; however, it is important to explain your method. For example, rather than giving detailed calculations, you might just say using the chain rule, we find ... Of course, what is appropriate for one group of readers is not appropriate for another group, so keep your audience in mind. Your readers should be able to check your argument and replicate your analysis using the work that you present.

Organization:

1. A reader understands best when a paper is well organized. Your argument should be well organized and you should communicate this organization to the reader. Good organization can help you decide whether some interesting fact is actually extraneous and should be omitted.

Composition = Organization + Simplification

2. An important aspect of good organization is to clearly state your definitions and assumptions.

3. Figures and tables should be labeled. They should be referred to in the text before the figure appears. LaTeX makes it easy to manage references to figures and tables.

4. Sources must always be cited. One good way to do this is to use the author's last name, followed by a reference label.

Notation:

1. Symbols in different formulas should be separated by words and no sentence should start with a symbol. Avoid the use of the symbols $\Rightarrow$, $\forall$, $\exists$, $\ni$; replace them with the corresponding words.

2. Some mathematical expressions should be written in-line, such as $x = 2$, but others are important enough to be displayed, such as

$$D_1 = \int_0^t \frac{1}{(t-\tau)^{\beta}} \cdot \exp\left(-\frac{{x_0}^2}{4(t-\tau)}\right) d\tau = \int_{\frac{1}{t}}^{\infty} \xi^{\beta-2} \cdot \exp\left(-\frac{x_0^2}{4}\xi\right) d\xi \qquad (1)$$

or even displayed with a reference tag (1).

(Note the punctuation above.) In particular, long computations are usually displayed. Moreover, you should label equations that will be referred to later. When referring to previous equations, use the format Equation (1) implies ...

3. It is good form to indicate the type of a symbol when using it in a sentence.

Bad: For each D_1 and $D_2 \in E$, $D_1 + D_2 \in E$.

Good: If D_1 and D_2 are any two points on the elliptic curve E, then their sum $D_1 + D_2$ is also a point on E.

4. Avoid subscripts, particularly when indexing the elements of a set. Indeed, don't name the elements of a set X unless necessary. Then you can refer to elements x and y of X or a subset S of X in your subsequent discussion.

5. Don't use the same notation for two different things. Conversely, use consistent notation for the same thing, when it appears in several places.

Bad: … using A_j for $1 \le j \le n$ in one place and A_k for $1 \le k \le n$ in another place.

Bad: ... for $a,b \in R^n$ and $c,\delta \in S$.

Good: ... for $a,b \in R^n$ and $\gamma,\delta \in S$.

6. Choose names for indices and ranges consistent with the usual ordering of the alphabet.

Bad: The index i ranges from 1 to n and j ranges from 1 to m.

Good: The index i ranges from 1 to m and j ranges from 1 to n.

Warning: This convention is commonly ignored when dealing with 4-dimensional space, where most authors call the components x, y, z and t.

Tricks and Pitfalls:

1. Many people use the two words which and that interchangeably, but this is not correct. The word which is used to introduce an idea or qualifier that isn't necessary to understand the sentence. Since we like to use as few words as possible in technical writing, such phrases shouldn't appear in your writing very often. So a safe rule is to use that unless it is obviously inappropriate.

Bad: Groups which have prime order are cyclic.

Good: Groups that have prime order are cyclic.

Good: Navy beat Army 18-14, which is good news.

2. One of the meanings of the word respectively is in the order given. This is a very useful word that allows you to define several things at once.

Good: Let O_1 and O_2 be the centers of circles γ_1 and γ_2, respectively.

3. Capitalize when referring to a particular theorem, e.g., this proves Theorem 6.

4. Use a spellchecker.

Language:

Papers are not written in Mathematese, they are written in English. Remember: language is the main tool to convey information. Endless bright ideas have been rendered obscure and impenetrable by poor language.

1. First and foremost, even if your native language isn't English, avoid poor or careless linguistic presentation. Be sensitive to the language, its idiom and cadences.

2. Presentation shouldn't be overly flowery or informal: this is not a paper in literary criticism and you are judged on your ideas and their clear presentation, not on linguistic virtuosity. The language should be clear, unambiguous and informative.

3. Avoid the sort of lifeless formalism and dry linguistic economy.

4. Occasional flash of lighthearted humor or informal lingo is fine. Mathematical style is not such.

5. Be verbose enough to be clear – yet concise enough to privilege your core mathematical argument over its presentation.

6. Not using a spell-checker is major folly. Relying totally on a spell-checker is carelessness: no spell-checker will distinguish between «some» and «same» [5].

The book of Knuth [1] contains a lot of good advice from Don Knuth's technical writing class at Stanford University. Trzeciak [3] is also a good reference text. I've collected some of their best advice above, though I added a few thoughts of my own too.

As you write a mathematics paper remember that, unlike you, the reader has not been thinking intensely about the material for an extended period of time. Therefore, provide the reader with references, include useful comments, and give additional explication so that someone unfamiliar with the work can follow it.

Write a paper that you yourself would want to read. Make it accessible. Bear in mind that the referee for your paper will be a busy person who has no patience for a tract that he cannot fathom. Lay out the material so that it is rapidly apparent what your main result is, what the background for that result is, and how you are going to go about proving it. If the proof is long and complicated, then break it up into digestible pieces. Tell the reader what is going to happen before it happens. Tell the reader what has just happened before you go on to the next step. At the end of a long argument, summarize it.

Writing mathematics is not the easiest thing to do. Writing mathematics is a skill which takes practice and experience to learn. There are many resources here at Purdue Calumet which is available to you to help you with your mathematical writing. Among these are the Math Lab and the Writing Lab.

If you have not written mathematics much before, it may feel frustrating at first. But learning to write mathematics can only be done by actually doing it. It may be hard at first, but it will get easier with time and you will get better at it. Do not get discouraged! Being able to write mathematics well is a good skill to learn, and one which you will keep for a lifetime.

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Ағылшын тілінде математикалық мәтіндердің жазылуы туралы

Мақала математикалық жұмыстар авторларына практикалық көмек ретінде арналған. Негізінде, ағылшын тілін еркін меңгермеген авторларға жазылған, бірақ ағылшын тілінде еркін сөйлейтіндерге пайдасы көп болуы мүмкін, яғни өзінің математикалық бағытын бастаған және математикалық мәлімдеме құрылымын әзірлемеген. Сонымен қатар бұл мақала ғылыми математикаға бағытталған, алайда оны жеңіл әрі оңай мәтіндерден басқа, барлық математикалық жұмыстардың мәлімдемелерінде қолдануға болады. Авторлар математикалық стильдің бірізділігін қолдануға қатаң тәртіп қоймайды. Керісінше, бұл мақаланың мақсаты құрылымы дұрыс жазбаларды жазу үшін әдебиеттерді және икемді, бірақ негізгі, әмбебап ережелерді бұзбай сақтау үшін арналуы мүмкін.

Кілт сөздер: математика тілі, математикалық мәтіндерді жазу бойынша кеңестер мен ережелер, ағылшын тілі, дұрыс жазбалар, зерттеу математикасы.

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О написании математических текстов на английском языке

Статья предназначена для того, чтобы оказать практическую помощь авторам (в основном не англоязычным) математических работ. Отмечено, что она может оказаться полезной даже для носителей английского языка, которые начинают свое математическое письмо и, возможно, еще не разработали для себя шаблон структуры математического изложения. Статья ориентирована главным образом на исследовательскую математику, но ее также можно использовать почти в любом изложении математических работ, за исключением слишком элементарных текстов. Цель статьи состоит не в навязывании какого-либо единообразия математического стиля, а в поощрении потенциальных авторов писать структурно правильные рукописи настолько литературно и гибко, насколько это возможно, но без нарушения некоторых основных и универсальных правил.

Ключевые слова: язык математики, советы и правила написания математических текстов, исследовательская математика.

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