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# Generalized Hankel shifts and exact Jackson–Stechkin inequalities in $L_2$

In this paper, we have solved several extremal problems of the best mean-square approximation of functions f on the semiaxis with a power-law weight. In the Hilbert space $L^2$ with a power-law weight $t^{2\alpha+1}$ we obtain Jackson–Stechkin type inequalities between the value of the $E_\sigma(f)$-best approximation of a function $f(t)$ by partial Hankel integrals of an order not higher than $\sigma$ over the Bessel functions of the first kind and the $k$-th order generalized modulus of smoothnes $\omega_k(B^r f, t)$, where $B$ is a second–order differential operator.

*Keywords:* best approximation, generalized modulus of smoothness of $m$-th order, Hilbert space.

## *Introduction*

At present, there is a number of meaningful papers [1–3] devoted to the theory of approximation of a function from $L_2[0, 2\pi]$. Let $\alpha > -\frac{1}{2}$. For $p = 2$ by $L_{2,\mu_\alpha}$ we denote the space consisting of measurable functions $f$ on $[0, \infty)$, for which the norm is finite

$$\|f\|_{2,\mu_\alpha} = \left(\int_0^\infty |f(x)|^2 d\mu_\alpha(x)\right)^{\frac{1}{p}},$$

where

$$d\mu_\alpha(x) = \frac{x^{2\alpha+1}}{2^\alpha \Gamma(\alpha+1)} dx.$$

Consider the Hankel transform defined for the function $f$:

$$h_\alpha(f)(\lambda) = \int_0^\infty x^{2\alpha+1}(x\lambda)^{-\alpha} J_\alpha(x\lambda) f(x) dx, \quad \lambda \in (0, \infty),$$

where $J_\alpha(z)$ is the Bessel function of the first kind of an order $\alpha \geq -\frac{1}{2}$, $\Gamma(x)$ is the gamma-function.

In particular, for $\alpha = \frac{1}{2}$ and $\alpha = -\frac{1}{2}$ the Hankel transforms turn into the sine transform and the cosine Fourier transform, respectively:

$$F_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin(\lambda x) dx,$$

$$F_c(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos(\lambda x) dx,$$

since the formulas $J_{\frac{1}{2}} = \sqrt{\frac{2}{\pi}} \sin x$ and $J_{-\frac{1}{2}} = \sqrt{\frac{2}{\pi}} \cos x$ hold.

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For a function $f \in L_{2,\mu_\alpha}$ the expansion into the Hankel integral [4], is valid:

$$\hat{H}_\alpha(f)(\lambda) = \int_0^\infty f(x) j_\alpha(\lambda x) d\mu_\alpha(x),$$

and

$$f(x) = \int_0^\infty \hat{H}_\alpha(f)(\lambda) j_\alpha(\lambda x) d\mu_\alpha(\lambda).$$

Let $T > 0$ and we denote by $S_T(f, x)$ the partial Hankel integral of a function $f \in L_{2,\mu_\alpha}$ i.e.

$$S_T(f, x) = \int_0^T \hat{H}_\alpha(f)(\lambda) j_\alpha(\lambda x) d\mu_\alpha(\lambda), \quad x \in (0, \infty).$$

For functions $f, g \in L_{2,\mu_\alpha}$, the generalized Plancherel's theorem holds [5]

$$(f, g) = (\hat{f}, \hat{g}),$$

where $(f, g) = \int_0^\infty f(x)\overline{g(x)} d\mu_\alpha$ is the scalar product of $f$ and $g$.

In the space $L_{p,\alpha}$ consider the generalized shift operator of functions $f(x)$ [6]

$$(T^h f)(x) = \frac{\Gamma(\alpha + 1)}{\sqrt{\pi}\Gamma(\alpha + 1/2)} \int_0^\pi f(\sqrt{x^2 + h^2 - 2xh\cos\varphi})(\sin\varphi)^{2\alpha} d\varphi.$$

For a function $f \in L_{2,\mu_\alpha}$, $\Delta_h^k f(x)$ finite differences of the $k$-th order with a step $h > 0$ are defined as follows (see [7]):

$$\Delta_h^1 f(x) = (I - T^h)(x), \Delta_h^k f(x) = (I - T^h)^k f(x), k > 1.$$

The value

$$\omega_k(f, \delta)_{2,\mu_\alpha} = \sup_{0 \le h \le \delta} \|\Delta_h^k f(x)\|_{2,\mu_\alpha} = \sup_{0 \le h \le \delta} \left\{ \int_0^\infty (1 - j_\alpha(\lambda h))^{2k} |\hat{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda) \right\}^{\frac{1}{2}} \tag{01}$$

will be called the generalized modulus of smoothness of the $k$-th order of a function$f \in L_{2,\mu_\alpha}$. We denote by $M(\nu, 2, \alpha), \nu > 0$ the set of all functions $Q_\nu(x)$ satisfying the following conditions (see [7]):

1. $Q_\nu(x)$ is an even entire function of exponential type $\nu$;
2. $Q_\nu(x)$ belongs to the class $L_{2,\mu_\alpha}$.

The best approximation of a function $f \in L_{2,\mu_\alpha}$ from the class $M(\sigma, 2, \alpha), \nu > 0$ is defined as follows:

$$E_\sigma(f)_{2,\mu_\alpha} = \inf\{\|f - Q_\sigma\|_{2,\mu_\alpha} : Q_\sigma \in M(\sigma, 2, \alpha)\} = \left\{ \int_\sigma^\infty |\hat{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda) \right\}^{\frac{1}{2}}. \tag{02}$$

Let

$$B = B_t = \frac{d^2}{dt^2} + \frac{2\alpha + 1}{t}\frac{d}{dt}$$

be a differential Bessel operator. We denote by $j_\alpha(\lambda t)$ the normalized Bessel function

$$j_\alpha(\lambda t) = \frac{2^\alpha \Gamma(\alpha + 1) J_\alpha(\lambda t)}{(\lambda t)^\alpha}.$$

The function $j_\alpha(\sqrt{\lambda} t)$ is a solution to the problem

$$\frac{d^2 y}{dt^2} + \frac{2\alpha + 1}{t}\frac{dy}{dt} + \lambda y = 0,$$

$$y(0) = 1, y'(0) = 0.$$

In [8], when solving problems of the theory approximations in the space $L_{2,\mu_\alpha}$ associated with finding the exact constants in the Jackson–Stechkin inequality

$$E_\sigma(f) \le \omega_r(f, \frac{\tau}{\sigma})$$

it is considered the following extreme characteristic:

$$K_{\sigma,r,m,\tau} = sup\left\{\frac{E_\sigma(f)}{\omega_r(f, \frac{\tau}{\sigma})} : f \in L_2(R^m)\right\}.$$

In this article, we want to get the exact constant in Jackson's inequality

$$E_\sigma(f) \le K\sigma^{-2r}\omega_r(B^r f, \frac{\tau}{\sigma})$$

for the functions $f \in W^r_{2,\mu_\alpha}(B)$. For the goal, we introduce an extremal approximate characteristic of the following form

$$\Xi_{\sigma,r,m,p,s}(\varphi, h) = \sup_{f\in W^r_{2,\mu_\alpha}(B)} \frac{E_\sigma(f)}{\left(\int_0^h \omega_m^p(B^r f, t)\varphi(t)dt\right)^s}, \tag{03}$$

where $r, m \in \mathbb{N},\ 0 < p < 2,\ h > 0,\ \sigma > 0,\ \varphi(t) \ge 0$ is an arbitrary integrable, not equivalent to zero on the segment $[0, h]$, weight function and $W^r_{2,\mu_\alpha}(B), r = 1, 2, ...$ is a Sobolev space, constructed by the differential operator B, i.e.

$$W^r_{2,\mu_\alpha}(B) = \left\{f \in L_{2,\mu_\alpha} : B^j f \in L_{2,\mu_\alpha}, j = 1, 2, ..., r\right\}.$$

Note that values $\Xi_{\sigma,r,m,p,s}(\varphi, h)$ for different values of the parameters therein and specific weight functions were examined by Chernykh, Taykov, Yudin, Esmaganbetov, Ivanov, Babenko, Shalaev, Vakarchuk, Shabozov, Tukhliev and many others (see., e.g., [6-11] and the literature cited therein).

In the case of approximation of $2\pi$-periodic function from $L_2$ by the subspace of trigonometric polynomials of an order $(n-1)$ in the metric $L_2$, similar problems were solved in [9] by Taikov, in [10] by M. Esmaganbetov, and in [11] by Sh.Shabozov and K. Tukhliev.

The extension of this question to the case of the best mean-square approximation by entire functions of exponential $\sigma > 0$ type in space $L_2$ with a power-law weight was carried out in [8] by A.G. Babenko and in [12] by D. V. Gorbachev, in [5] by V.I. Ivanov.

## *1 Auxiliary results*

*Lemma 1.* Let $q_{\alpha+1,1}$ be the smallest positive zero of the function $j_{\alpha+1}(t)$. Let $\sigma > 0$ and $t \in (0, \frac{q_{\alpha+1,1}}{\sigma}]$, $\alpha \ge -\frac{1}{2}$. Then

$$\sup_{0<h\le t}(1 - j_\alpha(\sigma h)) = 1 - j_\alpha(\sigma t).$$

*Proof of Lemma 1.* Since

$$j'_\alpha(t) = -\frac{t}{2(\alpha+1)}j_{\alpha+1}(t), 0 \le t \le \infty$$

(see [5]), then from $j_{\alpha+1}(0) > 0$ and $j_{\alpha+1}(q_{\alpha+1,1}) = 0$ we obtain for all $t \in [0, q_{\alpha+1,1}]$ values $(1 - j_\alpha(t))' = \frac{t}{2(\alpha+1)}j_{\alpha+1}(t) > 0$. It follows that the function $1 - j_\alpha(t)$ increases on $[0, q_{\alpha+1,1}]$. Hence, for all $t \in (0, q_{\alpha+1,1}]$ we have

$$\sup_{0<h\le t}(1 - j_\alpha(h)) = 1 - j_\alpha(t).$$

Therefore, for all $t \in (0, \frac{q_{\alpha+1,1}}{\sigma}]$ we get

$$\sup_{0<h\leq t}(1-j_\alpha(\sigma h)) = 1-j_\alpha(\sigma t). \tag{1}$$

Lemma 1 is proved.

*Lemma 2.* Let $q_{\alpha+1,1}$ be the first positive zero of the function $j_{\alpha+1}(t),\ h \in (0, \frac{q_{\alpha+1,1}}{\sigma}],\ \alpha \geq -\frac{1}{2}$ and $\sigma > 0$. Let

$$\Psi(y) = y^{4r}\int_0^h (1-j_\alpha(yt))^{2k}dt, \quad y\in G, \quad \text{where} \quad G=\{y:\sigma\leq y<\infty\}.$$

Then

$$\min\{\Psi(y): y\in G\} = \sigma^{4r}\int_0^h (1-j_\alpha(\sigma t))^{2k}dt.$$

*Proof of Lemma 2.* Since $j'_\alpha(t) = -\frac{t}{2(\alpha+1)}j_{\alpha+1}(t),\ 0\leq x\leq\infty$, then for $y\in G$ we have

$$\Psi'(y) = 4ry^{4r-1}\int_0^h (1-j_\alpha(yt))^{2k}dt + y^{4r}\int_0^h \frac{\partial}{\partial y}\left((1-j_\alpha(yt))^{2k}\right)dt. \tag{2}$$

Since it is not difficult to verify by direct verification that the equality is true

$$\frac{1}{y}\frac{\partial}{\partial t}\left((1-j_\alpha(yt))^{2k}\right) = \frac{1}{t}\frac{\partial}{\partial y}\left((1-j_\alpha(yt))^{2k}\right), \tag{3}$$

where $t, y$ are non-zero, then from (2) by virtue of equality (3) we have

$$\Psi'(y) = y^{4r-1}\left[4r\int_0^h (1-j_\alpha(yt))^{2k}dt + \int_0^h t\frac{\partial}{\partial t}\left((1-j_\alpha(yt))^{2k}\right)dt\right]. \tag{4}$$

Applying the method of integration by parts to calculate the second integral in the right-hand side of (4), we conclude

$$\Psi'(y) = y^{4r-1}\left[(4r-1)\int_0^h (1-j_\alpha(yt))^{2k}dt + h(1-j_\alpha(yh))^{2k}\right]. \tag{5}$$

Since $|j_\alpha(u)|\leq 1, \forall u\geq 0$ (see [8], formula (21)) and (1), then by virtue of (5), we have $\Psi'(y)>0$ for all $y\geq\sigma$. Lemma is proved.

### *2 Main results*

The main results of this work are the following theorems.

*Theorem 1.* For any function $f\in W^r_{2,\mu_\alpha}(B)$ for any $h>0$, the following estimate holds:

$$E_\sigma(f)_{2,\mu_\alpha} \leq \frac{\left(\int_0^h \omega_k^2(B^r f,t)_{2,\mu_\alpha}dt\right)^{\frac{1}{2}}}{\sigma^{2r}\left(\int_0^h (1-j_\alpha(\sigma t))^{2k}dt\right)^{\frac{1}{2}}}.$$

*Proof of Theorem 1.* Let $f\in W^r_{2,\mu_\alpha}(B)$. Then from Parseval's equality, we have

$$\omega_k^2(B^r f,t)_{2,\mu_\alpha} \geq \int_\sigma^\infty (1-j_\alpha(\lambda t))^{2k}\lambda^{4r}|\hat{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda).$$

Integrating both sides of this inequality variable $t$ over the range $t = 0$ and $t = h$, we obtain

$$\int_0^h \omega_k^2(B^r f,t)_{2,\mu_\alpha}dt \geq \int_0^h \left(\int_\sigma^\infty (1-j_\alpha(\lambda t))^{2k}\lambda^{4r}|\hat{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda)\right)dt =$$

$$= \int_\sigma^\infty |\hat{H}_\alpha(f)(\lambda)|^2 \left(\int_0^h \lambda^{4r}(1-j_\alpha(\lambda t))^{2k}\right) dt d\mu_\alpha(\lambda). \tag{6}$$

From (6) by virtue of lemma 2, we have

$$\int_0^h \omega_k^2(B^r f,t)_{2,\mu_\alpha}dt \geq \sigma^{4r}\int_0^h (1-j_\alpha(\sigma t))^{2k}dt \int_\sigma^\infty |\hat{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda).$$

It follows that

$$\int_\sigma^\infty |\hat{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda) \leq \frac{\int_0^h \omega_k^2(B^r f,t)_{2,\mu_\alpha}dt}{\sigma^{4r}\int_0^h (1-j_\alpha(\sigma t))^{2k}dt}. \tag{7}$$

Further, given the following equality

$$\|f - S_\sigma(f,x)\|_{2,\mu_\alpha} = E_\sigma(f)_{2,\mu_\alpha}$$

in view of the inequality (7) we get

$$E_\sigma^2(f)_{2,\mu_\alpha} \leq \frac{\int_0^h \omega_k^2(B^r f,t)_{2,\mu_\alpha}dt}{\sigma^{4r}\int_0^h (1-j_\alpha(\sigma t))^{2k}dt}.$$

Theorem 1 is proved.

*Theorem 2.* For any function $f \in W^r_{2,\mu_\alpha}(B)$ for any $h > 0$, the following estimate holds:

$$\sup_{f\in W^r_{2,\mu_\alpha}(B)} \frac{\sigma^{2r}E_\sigma(f)_{2,\mu_\alpha}}{\left(\int_0^h \omega_k^2(B^r f,t)_{2,\mu_\alpha}dt\right)^{\frac{1}{2}}} = \frac{1}{\left(\int_0^h (1-j_\alpha(\sigma t))^{2k}dt\right)^{\frac{1}{2}}}. \tag{8}$$

*Proof of Theorem 2.* Let $f \in W^r_{2,\mu_\alpha}(B)$. Arguing in the same way as in Theorem 1, for $f \in W^r_{2,\mu_\alpha}(B)$ we have

$$\frac{\sigma^{2r}E_\sigma(f)_{2,\mu_\alpha}}{\left(\int_0^h \omega_k^2(B^r f,t)_{2,\mu_\alpha}dt\right)^{\frac{1}{2}}} \leq \frac{1}{\left(\int_0^h (1-j_\alpha(\sigma t))^{2k}dt\right)^{\frac{1}{2}}}.$$

Hence we get

$$\sup_{f\in W^r_{2,\mu_\alpha}(B)} \frac{\sigma^{2r}E_\sigma(f)_{2,\mu_\alpha}}{\left(\int_0^h \omega_k^2(B^r f,t)_{2,\mu_\alpha}dt\right)^{\frac{1}{2}}} \leq \frac{1}{\left(\int_0^h (1-j_\alpha(\sigma t))^{2k}dt\right)^{\frac{1}{2}}}. \tag{9}$$

To obtain a lower estimate, we construct the function $f_\epsilon \in W^r_{2,\mu_\alpha}(B)$ so that it satisfies the inequality:

$$\sup_{f\in W^r_{2,\mu_\alpha}(B)} \frac{\sigma^{2r}E_\sigma(f)_{2,\mu_\alpha}}{\left(\int_0^h \omega_k^2(B^r f,t)_{2,\mu_\alpha}dt\right)^{\frac{1}{2}}} \geq \frac{\sigma^{2r}E_\sigma(f_\epsilon)_{2,\mu_\alpha}}{\left(\int_0^h \omega_k^2(B^r f_\epsilon,t)_{2,\mu_\alpha}dt\right)^{\frac{1}{2}}}.$$

To do this, we use function $f_\epsilon \in W^r_{2,\mu_\alpha}(B)$ constructed by Babenko in [9] and such that

$$\hat{H}_\alpha(f_\epsilon)(\lambda) = \begin{cases} |\lambda|^{-\alpha-\frac{1}{2}} & \text{if} \quad \sigma < |\lambda| < \sigma+\epsilon, \\ 0, & \text{otherwise.} \end{cases}$$

Relations (2) and using the properties of the Hankel transform (see [7]) imply the equality

$$E_\sigma^2(f_\epsilon)_{2,\mu_\alpha} = \int_\sigma^\infty |\hat{H}_\alpha(f_\epsilon)(\lambda)|^2 d\mu_\alpha(\lambda) = \int_\sigma^{\sigma+\epsilon} |\hat{H}_\alpha(f_\epsilon)(\lambda)|^2 d\mu_\alpha(\lambda) = \frac{\epsilon}{2^\alpha\Gamma(\alpha+1)}.$$

Therefore

$$E_\sigma(f_\epsilon)_{2,\mu_\alpha} = \sqrt{\frac{\epsilon}{2^\alpha\Gamma(\alpha+1)}}. \tag{10}$$

In virtue of the equality (01) and using the properties of the Hankel transform (see [7], [4])

$$\hat{H}_\alpha(B^r f_\epsilon)(\lambda) = \lambda^{2r}\hat{H}_\alpha(f_\epsilon)(\lambda)$$

we write:

$$\omega_k^2(B^r f_\epsilon, t)_{2,\mu_\alpha} = \int_\sigma^{\sigma+\epsilon} \lambda^{4r}|\hat{H}_\alpha(f_\epsilon)(\lambda)|^2(1-j_\alpha(\lambda t))^{2k} d\mu_\alpha(\lambda) \le$$

$$\le (\sigma+\epsilon)^{4r}(1-j_\alpha((\sigma+\epsilon)t))^{2k}\frac{\epsilon}{2^\alpha\Gamma(\alpha+1)}. \tag{11}$$

Integrating both parts of the inequality (11), we have

$$\left\{\int_0^h \omega_k^2(B^r f_\epsilon, t)_{2,\mu_\alpha} dt\right\}^{\frac{1}{2}} \le (\sigma+\epsilon)^{2r}\sqrt{\frac{\epsilon}{2^\alpha\Gamma(\alpha+1)}}\left\{\int_0^h (1-j_\alpha((\sigma+\epsilon)t)^{2k} dt\right\}^{\frac{1}{2}}. \tag{12}$$

Using (10), (12) we write

$$\frac{\sigma^{2r}E_\sigma(f_\epsilon)_{2,\mu_\alpha}}{\left(\int_0^h \omega_k^2(B^r f_\epsilon, t)_{2,\mu_\alpha} dt\right)^{\frac{1}{p}}} \ge \frac{\sigma^{2r}}{(\sigma+\epsilon)^{2r}\{\int_0^h (1-j_\alpha((\sigma+\epsilon)t))^{2k} dt\}^{\frac{1}{p}}}. \tag{13}$$

Since $f_\epsilon \in W_{2,\mu_\alpha}^r(B)$, then from (13) and from left side of equality (8) we obtain

$$\sup_{f\in W_{2,\mu_\alpha}^r(B)} \frac{\sigma^{2r}E_\sigma(f)_{2,\mu_\alpha}}{\left(\int_0^h \omega_k^2(B^r f, t)_{2,\mu_\alpha} dt\right)^{\frac{1}{2}}} \ge \frac{\sigma^{2r}}{(\sigma+\epsilon)^{2r}\{\int_0^h (1-j_\alpha((\sigma+\epsilon)t))^{2k} dt\}^{\frac{1}{2}}}. \tag{14}$$

Obviously, the left side of inequality (14) does not depend on $\epsilon$, and the expression located on its right side is the function of $\epsilon$ (with fixed values of other parameters). Since the left side of inequality (14) does not depend on $\epsilon$, then calculating the supremum with respect to $\epsilon$ from its right side, we write

$$\sup_{f\in W_{2,\mu_\alpha}^r(B)} \frac{\sigma^{2r}E_\sigma(f)_{2,\mu_\alpha}}{\left(\int_0^h \omega_k^2(B^r f, t)_{2,\mu_\alpha} dt\right)^{\frac{1}{2}}} \ge \frac{\sigma^{2r}E_\sigma(f_\epsilon)_{2,\mu_\alpha}}{\left(\int_0^h \omega_k^2(B^r f_\epsilon, t)_{2,\mu_\alpha} dt\right)^{\frac{1}{2}}} = \frac{1}{\left(\int_0^h (1-j_\alpha(\sigma t))^{2k} dt\right)^{\frac{1}{2}}}. \tag{15}$$

Comparing the upper estimate (9) and the lower estimate (15), we obtain the required equality. Theorem 2 is proved.

*Theorem 3.* Let $m, n \in \mathbb{N},\ r \in \mathbb{Z}_+,\ 0 < p \le 2,\ h > 0,\ \alpha \ge -\frac{1}{2}$. Then the following estimate is valid

$$\sup_{f\in W_{2,\mu_\alpha}^r(B)} \frac{\sigma^{2r}E_\sigma(f)_{2,\mu_\alpha}}{\left(\int_0^h \omega_k^p(B^r f, t)_{2,\mu_\alpha} dt\right)^{\frac{1}{p}}} = \frac{1}{\{\int_0^h (1-j_\alpha(\sigma t))^{kp} dt\}^{\frac{1}{p}}}. \tag{16}$$

*Proof of Theorem 3.* Let $0 < p \le 2$, then, arguing as in the previous theorem, we have

$$\omega_k^2(B^r f, t)_{2,\mu_\alpha} \ge \int_\sigma^\infty \lambda^{4r}(1 - j_\alpha(\lambda t))^{2k} |\hat{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda).$$

Raising both sides of this inequality by the power $p/2$, integrating the variable $t$ over the range $t = 0$ and $t = h$ we obtain

$$\left(\int_0^h \omega_k^p(B^r f, t)_{2,\mu_\alpha} dt\right)^{\frac{1}{p}} \ge \left\{\int_0^h \left(\int_\sigma^\infty \lambda^{4r}(1 - j_\alpha(\lambda t))^{2k} |\hat{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda)\right)^{\frac{p}{2}} dt\right\}^{\frac{1}{p}} = I.$$

Applying the inverse Minkowski inequality for $\frac{p}{2} \le 1$, we have

$$I \ge \left\{\int_\sigma^\infty |\hat{H}_\alpha(f)(\lambda)|^2 \left(\int_0^h \lambda^{2rp}(1 - j_\alpha(\lambda t))^{kp} dt\right)^{\frac{2}{p}} d\mu_\alpha(\lambda)\right\}^{\frac{1}{2}}. \tag{17}$$

Then from inequality (17) and in view of Lemma 2, we obtain

$$I \ge \sigma^{2r} \left\{\int_\sigma^\infty |\hat{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda)\right\}^{\frac{1}{2}} \left(\int_0^h (1 - j_\alpha(\sigma t))^{kp} dt\right)^{\frac{1}{p}} =$$

$$= \sigma^{2r} E_\sigma(f)_{2,\mu_\alpha} \left(\int_0^h (1 - j_\alpha(\sigma t))^{kp} dt\right)^{\frac{1}{p}}. \tag{18}$$

So combining (17) and (18), we get

$$\left(\int_0^h \left(\omega_k^2(B^r f, t)_{2,\mu_\alpha}\right)^{\frac{p}{2}} dt\right)^{\frac{1}{p}} \ge \sigma^{2r} E_\sigma(f)_{2,\mu_\alpha} \left(\int_0^h (1 - j_\alpha(\sigma t))^{kp} dt\right)^{\frac{1}{p}}.$$

Hence it follows that for all $f \in W^r_{2,\mu_\alpha}(B)$ the inequality holds

$$\frac{\sigma^{2r} E_\sigma(f)_{2,\mu_\alpha}}{\left(\int_0^h \omega_k^p(B^r f, t)_{2,\mu_\alpha} dt\right)^{\frac{1}{p}}} \le \frac{1}{\left(\int_0^h (1 - j_\alpha(\sigma t))^{kp} dt\right)^{\frac{1}{p}}}.$$

For all $f \in W^r_{2,\mu_\alpha}(B)$, we have

$$\sup_{f \in W^r_{2,\mu_\alpha}(B)} \frac{\sigma^{2r} E_\sigma(f)_{2,\mu_\alpha}}{\left\{\int_0^h \omega_k^p(B^r f, t)_{2,\mu_\alpha} dt\right\}^{\frac{1}{p}}} \le \frac{1}{\left\{\int_0^h (1 - j_\alpha(\sigma t))^{kp} dt\right\}^{\frac{1}{p}}}. \tag{19}$$

Thus, the upper estimate is proved.

To obtain a lower estimate, we construct a function $f_\epsilon \in W^r_{2,\mu_\alpha}(B)$ so that the inequality is fulfilled:

$$\sup_{f \in W^r_{2,\mu_\alpha}(B)} \frac{\sigma^{2r} E_\sigma(f)_{2,\mu\alpha}}{\left\{\int_0^h \omega_k^p(B^r f, t)_{2,\mu_\alpha} dt\right\}^{\frac{1}{p}}} \ge \frac{\sigma^{2r} E_\sigma(f_\epsilon)_{2,\mu\alpha}}{\left\{\int_0^h \omega_k^p(B^r f_\epsilon, t)_{2,\mu_\alpha} dt\right\}^{\frac{1}{p}}}. \tag{20}$$

To do this, we use function $f_\epsilon \in W^r_{2,\mu_\alpha}(B)$ constructed by Babenko in [8] and such that

$$\hat{H}_\alpha(f_\epsilon)(\lambda) = \begin{cases} |\lambda|^{-\alpha-\frac{1}{2}} & \text{if} \quad \sigma < |\lambda| < \sigma + \epsilon, \\ 0, & \text{otherwise.} \end{cases}$$

Raising both sides of the inequality (11) by the power $\frac{p}{2}$ and integrating the variable $t$ over the range $t=0$ to $t=h$, we have

$$\left\{\int_0^h \omega_k^p(B^r f_\epsilon, t)_{2,\mu_\alpha} dt\right\}^{\frac{1}{p}} \leq (\sigma+\epsilon)^{2r}\sqrt{\frac{\epsilon}{2^\alpha\Gamma(\alpha+1)}}\left\{\int_0^h (1-j_\alpha((\sigma+\epsilon)t)^{kp}dt\right\}^{\frac{1}{p}}. \tag{21}$$

Using (21), (10) we write

$$\frac{\sigma^{2r}E_\sigma(f_\epsilon)_{2,\mu_\alpha}}{\left(\int_0^h \omega_k^p(B^r f_\epsilon, t)_{2,\mu_\alpha} dt\right)^{\frac{1}{p}}} \geq \frac{\sigma^{2r}}{(\sigma+\epsilon)^{2r}\{\int_0^h (1-j_\alpha((\sigma+\epsilon)t))^{kp}dt\}^{\frac{1}{p}}}. \tag{22}$$

In view of the fact that the function $f_\epsilon$ belongs to the class $W^r_{2,\mu_\alpha}(B)$ and from the right-hand side of equality (16) and by virtue of the inequality (22), (20) we obtain

$$\sup_{f\in W^r_{2,\mu_\alpha}(B)} \frac{\sigma^{2r}E_\sigma(f)_{2,\mu_\alpha}}{\left(\int_0^h \omega_k^p(B^r f, t)_{2,\mu_\alpha} dt\right)^{\frac{1}{p}}} \geq \frac{\sigma^{2r}}{(\sigma+\epsilon)^{2r}\{\int_0^h (1-j_\alpha((\sigma+\epsilon)t))^{kp}dt\}^{\frac{1}{p}}}. \tag{23}$$

Obviously, the left side of inequality (23) does not depend on $\epsilon$, and the expression located on its right side is the function of $\epsilon$ (with fixed values of other parameters). Since the left side of inequality (23) does not depend on $\epsilon$, then calculating the supremum with respect to $\epsilon$ from its right side, we write

$$\sup_{f\in W^r_{2,\mu_\alpha}(B)} \frac{\sigma^{2r}E_\sigma(f)_{2,\mu\alpha}}{\left(\int_0^h \omega_k^p(B^r f, t)_{2,\mu_\alpha} dt\right)^{\frac{1}{p}}} \geq \frac{1}{\{\int_0^h (1-j_\alpha(\sigma t))^{kp}dt\}^{\frac{1}{p}}}. \tag{24}$$

Comparing the upper estimate (19) and the lower estimate (24), we obtain the required equality. The theorem 3 is proved.

*Theorem 4.* Let $m,n\in\mathbb{N}$, $r\in\mathbb{Z}_+$, $0<p\leq 2$, $h>0$, $\alpha\geq -\frac{1}{2}$ and $\varphi(t)\geq 0$ be a measurable function on the interval (0, h). Then the inequality

$$\left\{\gamma_{\sigma,r,m,p,\frac{1}{p}}(\varphi,h)\right\}^{-1} \leq \Xi_{\sigma,r,m,p,\frac{1}{p}}(\varphi,h) \leq \left\{\inf_{\sigma\leq\lambda<\infty}\gamma_{\lambda,r,m,p,\frac{1}{p}}(\varphi,h)\right\}^{-1}$$

holds, where

$$\gamma_{\lambda,r,m,p,\frac{1}{p}}(\varphi,h) = \left(\lambda^{2rp}\int_0^h (1-j_\alpha(\sigma t))^{kp}\varphi(t)dt\right)^{\frac{1}{p}}, \quad \lambda\geq\sigma.$$

*Proof of Theorem 4.* Let $0<p\leq 2$ then, arguing as in the previous theorem, we have

$$\omega_k^2(B^r f, t)_{2,\mu_\alpha} \geq \int_\sigma^\infty \lambda^{4r}(1-j_\alpha(\lambda t))^{2k}|\hat{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda).$$

Raising both sides of this inequality by the power $p/2$ and multiplying them by a function $\varphi(t)$ and integrating the variable $t$ over the range $t=0$ to $t=h$ we get

$$\left(\int_0^h \omega_k^p(B^r, t)_{2,\mu_\alpha}\varphi(t)dt\right)^{\frac{1}{p}} \geq \left\{\int_0^h \left(\int_\sigma^\infty \lambda^{4r}(1-j_\alpha(\lambda t))^{2k}|\hat{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda)\right)^{\frac{p}{2}}\varphi(t)dt\right\}^{\frac{1}{p}} = I. \tag{25}$$

Applying the inverse Minkowski inequality for $\frac{p}{2} \leq 1$ and by virtue of Lemma 2 we obtain

$$I \geq \left\{ \int_{\sigma}^{\infty} |\hat{H}_{\alpha}(f)(\lambda)|^2 d\mu_{\alpha}(\lambda) \left( \int_0^h \lambda^{2rp}(1 - j_{\alpha}(\lambda t))^{kp} \varphi(t) dt \right)^{\frac{2}{p}} \right\}^{\frac{1}{2}} =$$

$$= \left\{ \int_{\sigma}^{\infty} |\hat{H}_{\alpha}(f)(\lambda)|^2 d\mu_{\alpha}(\lambda) \left\{ \gamma_{\lambda,r,k,p,\frac{1}{p}}(\varphi, h) \right\}^2 \right\}^{\frac{1}{2}} \geq$$

$$\geq E_{\sigma}(f)_{2,\mu_{\alpha}} \inf_{\sigma \leq \lambda < \infty} \gamma_{\lambda,r,k,p,\frac{1}{p}}(\varphi, h). \quad (26)$$

So combining (25) and (26) we get

$$\left( \int_0^h \omega_k^p(B^r, t)_{2,\mu_{\alpha}} \varphi(t) dt \right)^{\frac{1}{p}} \geq E_{\sigma}(f)_{2,\mu_{\alpha}} \inf_{\sigma \leq \lambda < \infty} \gamma_{\lambda,r,k,p,\frac{1}{p}}(\varphi, h).$$

Therefore, according the definition of quantity (03), by previous inequality we obtain an upper bound for the extremal characteristics $\Xi_{\sigma,r,k,p,\frac{1}{p}}(\varphi, h)$, namely

$$\Xi_{\sigma,r,k,p,\frac{1}{p}}(\varphi, h) = \sup_{f \in W_{2,\mu_{\alpha}}^r(B)} \frac{E_{\sigma}(f)_{2,\mu_{\alpha}}}{\left\{ \int_0^h \omega_k^p(B^r f, t)_{2,\mu_{\alpha}} \varphi(t) dt \right\}^{\frac{1}{p}}} \leq \frac{1}{\inf_{\sigma \leq \lambda < \infty} \gamma_{\lambda,r,k,p,\frac{1}{p}}(\varphi, h)}. \quad (27)$$

To obtain a lower estimate, we construct the function $f_{\epsilon} \in W_{2,\mu_{\alpha}}^r(B)$ so that the inequality would be fulfilled:

$$\Xi_{\sigma,r,k,p,\frac{1}{p}}(\varphi, h) = \sup_{f \in W_{2,\mu_{\alpha}}^r(B)} \frac{E_{\sigma}(f)_{2,\mu_{\alpha}}}{\left\{ \int_0^h \omega_k^p(B^r f, t)_{2,\mu_{\alpha}} \varphi(t) dt \right\}^{\frac{1}{p}}} \geq \frac{E_{\sigma}(f_{\epsilon})_{2,\mu_{\alpha}}}{\left\{ \int_0^h \omega_k^p(B^r f_{\epsilon}, t)_{2,\mu_{\alpha}} \varphi(t) dt \right\}^{\frac{1}{p}}}. \quad (28)$$

To do this, we use function $f_{\epsilon} \in W_{2,\mu_{\alpha}}^r(B)$ constructed by Babenko in [9] and such that

$$\hat{H}_{\alpha}(f_{\epsilon})(\lambda) = \begin{cases} |\lambda|^{-\alpha - \frac{1}{2}} & \text{if} \quad \sigma < |\lambda| < \sigma + \epsilon, \\ 0, & \text{otherwise.} \end{cases}$$

Raising both sides of the inequality (11) by the power $\frac{p}{2}$, multiplying them by the weight function $\varphi(t)$, and integrating the variable $t$ over the range $t = 0$ to $t = h$, we have

$$\left\{ \int_0^h \omega_k^p(B^r f_{\epsilon}, t)_{2,\mu_{\alpha}} \varphi(t) dt \right\}^{\frac{1}{p}} \leq (\sigma + \epsilon)^{2r} \sqrt{\frac{\epsilon}{2^{\alpha}\Gamma(\alpha + 1)}} \left\{ \int_0^h (1 - j_{\alpha}((\sigma + \epsilon)t)^{kp} \varphi(t) dt \right\}^{\frac{1}{p}}. \quad (29)$$

Using (29) and (10) we write

$$\frac{E_{\sigma}(f_{\epsilon})_{2,\mu_{\alpha}}}{\left( \int_0^h \omega_k^p(B^r f_{\epsilon}, t)_{2,\mu_{\alpha}} \varphi(t) dt \right)^{\frac{1}{p}}} \geq \frac{1}{(\sigma + \epsilon)^{2r} \{ \int_0^h (1 - j_{\alpha}((\sigma + \epsilon)t))^{kp} \varphi(t) dt \}^{\frac{1}{p}}}. \quad (30)$$

In view of the fact that the function $f_{\epsilon}$ belongs to the class $W_{2,\mu_{\alpha}}^r(B)$, by virtue of inequality (30) and relation (03), (28) we obtain

$$\sup_{f \in W_{2,\mu_{\alpha}}^r(B)} \frac{E_{\sigma}(f)_{2,\mu_{\alpha}}}{\left( \int_0^h \omega_k^p(B^r f, t)_{2,\mu_{\alpha}} \varphi(t) dt \right)^{\frac{1}{p}}} \geq \frac{1}{(\sigma + \epsilon)^{2r} \{ \int_0^h (1 - j_{\alpha}((\sigma + \epsilon)t))^{kp} \varphi(t) dt \}^{\frac{1}{p}}}. \quad (31)$$

Obviously, the left side of the inequality (31) does not depend on $\epsilon$, and the expression located on its right side is the function of $\epsilon$ (with fixed values of other parameters). Since the left side of inequality (31) does not depend on $\epsilon$, then calculating the supremum with respect to $\epsilon$ from its right side, we write

$$\sup_{f\in W^r_{2,\mu_\alpha}(B)} \frac{\sigma^{2r}E_\sigma(f)_{2,\mu\alpha}}{\left(\int_0^h \omega_k^p(B^rf,t)_{2,\mu_\alpha}\varphi(t)dt\right)^{\frac{1}{p}}} \geq \frac{1}{\{\int_0^h(1-j_\alpha(\sigma t))^{kp}\varphi(t)dt\}^{\frac{1}{p}}}. \tag{32}$$

Comparing the upper estimate (27) and the lower estimate (32), we obtain the required equality. Theorem 4 is proved.

Let us find: what differential properties the weight function $\varphi$ must possess in order that the following equality holds

$$\gamma_{\sigma,r,k,p,\frac{1}{p}}(\varphi,h) = \inf_{\sigma\leq\lambda<\infty}\gamma_{\lambda,r,k,p,\frac{1}{p}}(\varphi,h).$$

The following statement gives an answer to this question.

*Theorem 5.* Let $\varphi(t)$ be a non-negative continuously differentiable function on the interval $[0,h]$. If for some $p\in(0,2]$, $r\in\mathbb{N}$ any $t\in[0,h]$, $\alpha\geq-\frac{1}{2}$, $\varphi$ satisfies the differential inequality

$$(2rp-1)\varphi(t) - t\varphi'(t) \geq 0,$$

then for all $\sigma\in(0,\infty)$ and $0<h\leq\frac{q_{\alpha+1,1}}{\sigma}$ we have

$$\inf\left\{\gamma_{\lambda,k,r,p,\frac{1}{p}}(\varphi,h) : \sigma\leq\lambda<\infty\right\} = \gamma_{\sigma,k,r,p,\frac{1}{p}}(\varphi,h)$$

and there is a relation

$$\Xi_{\sigma,k,r,p,\frac{1}{p}}(\varphi,h) = \left(\gamma_{\sigma,k,r,p,\frac{1}{p}}(\varphi,h)\right)^{-1}.$$

*Proof of Theorem 5.* Since

$$\gamma_{\lambda,r,k,p,\frac{1}{p}}(\varphi,h) = \left\{\lambda^{2rp}\int_0^h(1-j_\alpha(\lambda t))^{kp}\varphi(t)dt\right\}^{\frac{1}{p}}$$

it is sufficient to prove that under the above assumptions on $\varphi(t)$ and the function

$$\eta(y) = y^{2rp}\int_0^h(1-j_\alpha(yt))^{kp}\varphi(t)dt$$

is strictly increasing on the interval $G=\{y : y\geq\sigma\}$. Since

$$\eta'(y) = 2rpy^{2rp-1}\int_0^h(1-j_\alpha(yt))^{kp}\varphi(t)dt + y^{2rp}\int_0^h\frac{d}{dy}(1-j_\alpha(yt))^{kp}\varphi(t)dt, \tag{33}$$

then, using the obvious identity

$$\frac{d}{dy}(1-j_\alpha(yt))^{kp} = \frac{t}{y}\frac{d}{dt}(1-j_\alpha(yt))^{kp} \tag{34}$$

from (33) and taking into account (34) we have

$$\eta'(y) = 2rpy^{2rp-1}\int_0^h(1-j_\alpha(yt))^{kp}\varphi(t)dt + y^{2rp-1}\int_0^h\frac{d}{dt}(1-j_\alpha(yt))^{kp}(t\varphi(t))dt.$$

Applying the method of integration by parts when calculating the second integral, we come to the conclusion

$$\eta'(y) = y^{2rp-1}\left((1-j_\alpha(yh))^{kp}h\varphi(h) + \int_0^h (1-j_\alpha(yt))^{kp}[(2rp-1)\varphi(t)+t\varphi'(t)]dt\right). \qquad (35)$$

Since $|j_\alpha(y)| \le 1$ for all $y \in [0,\infty)$, then by virtue of the

$$(2rp-1)\varphi(t) - t\varphi'(t) \ge 0,$$

taking into account the conditions $p \in (0,2], r \in \mathbb{N}$ from (35) we have $\eta'(y) \ge 0$, for $y \ge \sigma$. Whence follows $\inf\{\eta(y) : \sigma \le y < \infty\} = \eta(\sigma)$, which is equivalent to equality

$$\inf\left\{\gamma_{\lambda,k,r,p,\frac{1}{p}}(\varphi,h) : \sigma \le \lambda < \infty\right\} = \gamma_{\sigma,k,r,p,\frac{1}{p}}(\varphi,h).$$

Then by virtue of the double inequality from Theorem 4, we obtain the required equality. Theorem 5 is proved.

## *4 Approximation in* $L^2(\mathbb{R}^m)$

The exact inequality and its various generalizations have been the subject of study for many specialists in the last 50 years. Some historical information on the Jackson–Stechkin inequalities in $L^2(\mathbb{R}^m)$ can be found in [5, 8, 13–18].

Let $L^2 = L^2(R^m)$ be the Hilbert space of complex functions on $R^m$ with a scalar product and norm

$$(f,g) = \int_{R^m} f(x)g(x)dx, \quad \|f\| = \sqrt{(f,f)}.$$

The Fourier transform of the function $f \in L^2$ is defined by this formula

$$\hat{f}(y) = \frac{1}{(2\pi)^{m/2}}\int_{R^m} f(x)e^{-ix\cdot y}dx,$$

where $x \cdot y = \sum_{l=1}^m x_l \cdot y_l$ is the scalar product of vectors $x$ and $y$ of $R^m$.

The function $f$ can be decomposed through its Fourier transform $\hat{f}$ as:

$$f(x) = \frac{1}{(2\pi)^{m/2}}\int_{R^m} f(y)e^{ix\cdot y}dy. \qquad (36)$$

For the Fourier transform in $L^2$ space, the Plancherell formula applies

$$(f,g) = (\hat{f},\hat{g}), \quad f,g \in L^2.$$

Let us denote by $W_\sigma$ the class of exponential spherical integer functions $\sigma > 0$ belonging to the space. The class $W_\sigma$ of integer functions consists of integer functions $g \in L^2$ such that the support $supp\,\hat{g}$ of Fourier transform lies in a Euclidean ball $B_{\sigma^m} = \left\{x \in R^m : |x| = \sqrt{(x,x)} \le \sigma\right\}$ of a radius $\sigma > 0$ and with a center at the origin of the space $R^m$. The best approximation of the function $f$ of $L^2$ by the class $W_\sigma$ is

$$A_\sigma f = \inf\{\|f-g\| : g \in W_\sigma\}.$$

The spherical shift with a step $h$ is the operator $S_h$ acting according to the rule

$$S_h f(x) = \frac{1}{|\mathbb{S}^{m-1}|}\int_{\mathbb{S}^{m-1}} f(x+h\xi)d\xi,$$

where $\mathbb{S}^{m-1}$ is a unit Euclidean sphere in $R^m$, $|\mathbb{S}^{m-1}|$ is its surface area. Let $I$ be an identical operator, $k$ is a positive number. Following H.P. Rustamov's operator $(I - S_h f)^{\frac{k}{2}}$ (see [17]), will be called a difference operator of order $k$ with step $h$ and will be denoted by $\Delta_h^k$:

$$\Delta_h^k = \sum_{l=0}^{\infty} (-1)^l \binom{\frac{k}{2}}{l} S_h^l,$$

and the $k$-order continuity module of the function $f \in L^2(S^{m-1})$ will be the function of the variable $\tau > 0$:

$$\omega_k(f,\tau) = \sup\left\{\|\Delta_h^k f\| : 0 < h \le \tau\right\}.$$

Denote by $K_n(\tau,k,m), \tau > 0, k \ge 1, m = 2,3,...$ the exact constant $K$ the Jackson–Stechkin inequality in $L^2(S^{m-1})$

$$A_\sigma(f) \le K\omega_k(f,\tau), f \in L^2(S^{m-1}),$$

let's put

$$K_\sigma(\tau,k,m) = \sup\left\{\frac{A_\sigma(f)}{\omega_k(f,\tau)} : f \in L^2(S^{m-1})\right\}.$$

Using the Plancherell formula, it is easy to see that the value of the best approximation for the function $f \in L^2(S^{m-1})$ is expressed by

$$A_\sigma^2 f = \int_{|y|>\sigma} |\hat{f}(y)|^2 dy.$$

It is known ([19], [13; 176]) that the $S_h$ spherical shift operator with step $h > 0$ acts on the function $e_y(x) = e^{ix\cdot y}$ as follows:

$$S_h e_y(x) = \frac{1}{|\mathbb{S}^{m-1}|}\int_{\mathbb{S}^{m-1}} e^{i(x+h\xi)\cdot y} d\xi =$$

$$= \frac{e^{ix\cdot y}}{|S^{m-1}|}\int_{S^{m-1}} e^{ih\xi\cdot y} d\xi = j_{\frac{m-2}{2}}(h|y|) e_y(x). \quad (37)$$

Applying $k$ times to both parts of equality (36) the spherical shift operator and using relation (37) we have

$$S_h^k f(x) = \frac{1}{(2\pi)^{m/2}}\int_{\mathbb{S}^{m-1}} (j_{\frac{m-2}{2}}(h|y|))^k \hat{f}(y) e^{ix\cdot y} dy. \quad (38)$$

From the definition of the difference operator by virtue of (38) we obtain

$$\Delta_h^k f(x) = \int_{\mathbb{R}^m} (1 - j_{\frac{m-2}{2}}(h|y|))^{\frac{k}{2}} \hat{f}(y) e^{ix\cdot y} dy. \quad (39)$$

Hence, by virtue of the Plancherell formula from (39) we have

$$\|\Delta_h^k f\|^2 = \int_{\mathbb{R}^m} (1 - j_{\frac{m-2}{2}}(h|y|))^k |\hat{f}(y)|^2 dy.$$

## *5 The Jackson–Stechkin Theorem in $L^2(\mathbb{R}^m)$*

*Theorem 6.* Let $k \ge 1, \sigma > 0$. Then for any function $f \in L^2(\mathbb{S}^{m-1})$ it holds:

$$A_\sigma(f) \le \frac{\left(\int_0^{\frac{2q_{\alpha,1}}{\sigma}} \omega_k^{\frac{2}{k}}(B^r f,t) j_{\frac{m-2}{2}}(\sigma t) t^{m-1} dt\right)^{\frac{k}{2}}}{\sigma^{2r}\left(\int_0^{\frac{2q_{\alpha,1}}{\sigma}} j_{\frac{m-2}{2}}(\sigma t) t^{m-1} dt\right)^{\frac{k}{2}}},$$

where $q_{\frac{m-2}{2},1}$ is the first positive zero of the function $j_{\frac{m-2}{2}}(t)$.

*Proof of Theorem 6.* For any function $f \in L^2(\mathbb{R}^m)$ and by the equality

$$A_\sigma(f) = \left\{ \int_{|y|>\sigma} |\hat{f}(y)|^2 dy \right\}^{\frac{1}{2}}$$

and applying the Hölder inequality we have

$$A_\sigma^2(f) - \int_\sigma^\infty |\hat{f}(y)|^2 j_{\frac{m-2}{2}}(t|y|)dy = \int_\sigma^\infty |\hat{f}(y)|^2(1 - j_{\frac{m-2}{2}}(t|y|))dy =$$

$$= \int_\sigma^\infty |\hat{f}(y)|^{2-\frac{2}{k}} |\hat{f}(y)|^{\frac{2}{k}} (1 - j_{\frac{m-2}{2}}(t|y|))dy =$$

$$\leq A_\sigma^{2-\frac{2}{k}}(f) \left( \sigma^{-4r} \int_\sigma^\infty y^{4r} |\hat{f}(y)|^2 (1 - j_{\frac{m-2}{2}}(t|y|))^k dy \right)^{\frac{1}{k}}. \tag{40}$$

Since the equality holds

$$\omega_k^2(B^r f, t) = \int_0^\infty y^{4r} |\hat{f}(y)|^2 (1 - j_{\frac{m-2}{2}}(t|y|))^k dy$$

then from (40) we have

$$A_\sigma^2(f) - \int_\sigma^\infty |\hat{f}(y)|^2 j_{\frac{m-2}{2}}(t|y|)dy \leq A_\sigma^{2-\frac{2}{k}}(f) \sigma^{-\frac{4r}{k}} \omega_k^{\frac{2}{k}}(B^r f, t). \tag{41}$$

By multiplying both parts of the inequality (41) by the Babenko weight function (see [8]) $v(t) = t^{2\alpha+1} T_{\tau_{\alpha,1}} V(t),\ t \in R_+,\ \alpha > \frac{1}{2},\ \alpha = \frac{m-2}{2}$, where

$$V(t) = \begin{cases} j_{\frac{m-2}{2}}(\sigma t), & 0 < t < \frac{q_{\alpha,1}}{\sigma} \\ 0, & t > \frac{q_{\alpha,1}}{\sigma}, \end{cases}$$

$$T_h f(x) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)} \int_0^\pi f(\sqrt{x^2 + h^2 - 2xh\cos\varphi})(sin\varphi)^{2\alpha} d\varphi$$

and integrating them over $t$ to zero to $q_{\alpha,1} = q_{\frac{m-2}{2},1}$ we obtain

$$\int_0^{\frac{2q_{\alpha,1}}{\sigma}} A_\sigma^2(f) v(t)dt - \int_0^{\frac{2q_{\alpha,1}}{\sigma}} \int_\sigma^\infty |\hat{f}(y)|^2 j_{\frac{m-2}{2}}(t|y|)dy v(t)dt \leq$$

$$\leq \sigma^{-\frac{4r}{k}} \int_0^{\frac{2q_{\alpha,1}}{\sigma}} A_\sigma^{2-\frac{2}{k}}(f) \omega_k^{\frac{2}{k}}(B^r f, t) v(t)dt, \tag{42}$$

where $q_{\frac{m-2}{2},1}$ is the smallest root of the function $j_{\frac{m-2}{2}}(t)$. Since in [8] the inequality

$$\int_0^{\frac{2q_{\alpha,1}}{\sigma}} j_{\frac{m-2}{2}}(t|y|) v(t)dt < 0, \text{ for all } |y| > 1 \tag{43}$$

has been proved, so from (42) and (43), we obtain

$$A_\sigma^2(f) \int_0^{\frac{2q_{\alpha,1}}{\sigma}} v(t)dt \leq \sigma^{-\frac{4r}{k}} A_\sigma^{2-\frac{2}{k}}(f) \int_0^{\frac{2q_{\alpha,1}}{\sigma}} \omega_k^{\frac{2}{k}}(B^r f, t) v(t)dt.$$

Then, applying the properties of the generalized shift operator $T_h f$ (see [6–8]) we have

$$A_\sigma^{\frac{2}{k}}(f) \leq \frac{\int_0^{\frac{2q_{\alpha,1}}{\sigma}} \omega_k^{\frac{2}{k}}(B^r f,t) j_{\frac{m-2}{2}}(\sigma t) t^{m-1} dt}{\sigma^{\frac{4r}{k}} \int_0^{\frac{2q_{\alpha,1}}{\sigma}} j_{\frac{m-2}{2}}(\sigma t) t^{m-1} dt}.$$

It follows that

$$A_\sigma(f) \leq \frac{\left(\int_0^{\frac{2q_{\alpha,1}}{\sigma}} \omega_k^{\frac{2}{k}}(B^r f,t) j_{\frac{m-2}{2}}(\sigma t) t^{m-1} dt\right)^{\frac{k}{2}}}{\sigma^{2r}\left(\int_0^{\frac{2q_{\alpha,1}}{\sigma}} j_{\frac{m-2}{2}}(\sigma t) t^{m-1} dt\right)^{\frac{k}{2}}}.$$

Theorem 6 is proved.

*Corollary 1.* Let $k \in \mathbb{R}_+, k \geq 1,\ q_{\alpha,1} > 0, \sigma > 0, \alpha = \frac{m-2}{2}$. Then for any function $f \in L^2(\mathbb{R}^m)$ the inequality holds

$$A_\sigma(f) \leq \sigma^{-2r} \omega_k(B^r f, \frac{2q_{\alpha,1}}{\sigma}),$$

where $q_{\alpha,1}$ is the smallest root of the function $j_\alpha(t)$.

*Proof of Corollary 1.* Let's first show that the functionality of the

$$J_k(f, q_{\alpha,1}) = \frac{\left(\int_0^{\frac{2q_{\alpha,1}}{\sigma}} \omega_k^{\frac{2}{k}}(B^r f,t) j_{\frac{m-2}{2}}(\sigma t) t^{m-1} dt\right)^{\frac{k}{2}}}{\sigma^{2r}\left(\int_0^{\frac{2q_{\alpha,1}}{\sigma}} j_{\frac{m-2}{2}}(\sigma t) t^{m-1} dt\right)^{\frac{k}{2}}}$$

is smaller than $\omega_k(f, \frac{2q_{\alpha,1}}{\sigma})$. Indeed, it follows from the monotonicity of $\omega_k(f,t)$ that

$$J_k(f, \frac{2q_{\alpha,1}}{\sigma}) = \frac{\left(\int_0^{\frac{2q_{\alpha,1}}{\sigma}} \omega_k^{\frac{2}{k}}(B^r f,t) j_{\frac{m-2}{2}}(\sigma t) t^{m-1} dt\right)^{\frac{k}{2}}}{\sigma^{2r}\left(\int_0^{\frac{2q_{\alpha,1}}{\sigma}} j_{\frac{m-2}{2}}(\sigma t) t^{m-1} dt\right)^{\frac{k}{2}}} \leq \sigma^{-2r} \omega_k(B^r f, \frac{2q_{\alpha,1}}{\sigma}). \tag{44}$$

From Theorem 4 and by virtue of (44) we have

$$A_\sigma(f) \leq \frac{\left(\int_0^{\frac{2q_{\alpha,1}}{\sigma}} \omega_k^{\frac{2}{k}}(B^r f,t) j_{\frac{m-2}{2}}(\sigma t) t^{m-1} dt\right)^{\frac{k}{2}}}{\sigma^{2r}\left(\int_0^{\frac{2q_{\alpha,1}}{\sigma}} j_{\frac{m-2}{2}}(\sigma t) t^{m-1} dt\right)^{\frac{k}{2}}} = J_k(f, \frac{2q_{\alpha,1}}{\sigma}) \leq \sigma^{-2r} \omega_k(B^r f, \frac{2q_{\alpha,1}}{\sigma}).$$

*Remark.* Earlier in [5, 8, 12] similar results were obtained. The proof of Corollary 1 of Theorem 6 given here is new, i.e. it differs from the proofs of the theorems of A.G. Babenko [8], D.V. Gorbachev [12] and V.I. Ivanov [5]. The obtained result, which is a consequence of Theorem 6, coincides with the exact result of A.G. Babenko [8] at $k \geq 1$. In the works [20–22] , direct theorems of the theory of approximation were proved without refining the coefficients

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## $L_2$ метрикасындағы дәл Джексон–Стечкин теңсіздіктері және жалпыланған Ганкелдің ығыстыруы

Жұмыста $f$ функциясының ең жақсы орташа квадраттық жуықтауы бойынша, дәрежелі салмағы бар жарты осьте бірнеше экстремалды есептер шешілген. Гильберт кеңістігінде $L_2$ салмағы $t^{2\alpha+1}$ дәрежесі болатын, $f$ функциясының Бессельдің бірінші текті функциялары бойынша құрылған $\sigma$-ретті дербес Ганкел интегралдарымен ең жақсы жуықтауы $E_\sigma(f)$ және $k$-ретті үздіксіздіктің жалпыланған модулі $\omega_k f(B^r)f,t)$ арасындағы Джексон–Стечкин типті теңсіздіктер алынған, мұндағы $B$-екінші ретті дифференциалдық оператор.

*Кілт сөздер:* ең жақсы жуықтау, үзіліссіздік модулі, $m$-ретті жалпыланған, тегістік модулі, гильберт кеңістігі.

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## Обобщенные сдвиги Ганкеля и точные неравенства Джексона–Стечкина в $L_2$

В работе решено несколько экстремальных задач о наилучшем среднеквадратическом приближении функции $f$ на полуоси о степенным весом. В гильбертовом пространстве $L_2$ со степенным весом $t^{2\alpha+1}$ получены неравенства типа Джексона–Стечкина между величиной $E_\sigma(f)$ — наилучшего приближения функции $f$ частичными интегралами Ганкеля порядка не выше $\sigma$ по функциям Бесселя первого рода и обобщенным модулем непрерывности $k$-го порядка $\omega_k f(B^r f,t)$, где $B$ — дифференциальный оператор второго порядка.

*Ключевые слова:* наилучшее приближение, обобщенный модуль гладкости $m$-го порядка, гильбертово пространство.

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