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EXPERIMENTAL STUDY OF AN OSCILLATOR ENSEMBLE WITH GLOBAL AND NONLINEAR COUPLING

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We analyze collective dynamics of a population of electronic limit-cycle oscillators with a linear or nonlinear phase shifting unit in the global feedback loop. We conduct both physical experiment and mathematical modeling. Besides the standard Kuramoto synchronization transition in case of linear coupling we demonstrate a transition to a self-organized quasiperiodic state, predicted in [M. Rosenblum and A. Pikovsky, Phys. Rev. Lett., 98, 064101 (2007)]. In this state frequency of oscillators differs from the frequency of the mean field and dependence of the order parameter on the coupling strength is not monotonic. We demonstrate a good correspondence between the experiment and previously developed theory.

Keywords: oscillator ensemble, self-organized quasiperiodic state, *frequency, synchronization transition,* globally coupled system.

Introduction

Mean field approximation is widely used in description of oscillator networks with high degree of connectivity. Models of oscillator ensembles with mean field coupling, also known as global or all-to-all coupling, provide characterization of collective dynamics of oscillating objects of various nature, including fireflyes, spontaneously beating atrial cells, pedestrians on the footbridges, handclapping individuals in a large audience, Josephson junctions, lasers, electrochemical oscillators, spiking or bursting neurons, to name just a few. Analysis of collective behavior of such systems poses a number of problems which are highly nontrivial from the standpoint of nonlinear dynamics. Due to these reasons, this topic remains in the focus of interest within last three decades. Basic theory and further references can be found in the following books, book chapters, and review articles [1]. The main effect of global coupling is emergence of collective synchrony, reflected in the increase of the mean field amplitude with the interaction strength. This effect is well-understood within the framework of the analytically solvable model of sine coupled phase oscillators [2] and is often referred to as the Kuramoto transition. Depending on the distribution of the natural frequencies within the population, this transition can be either smooth [2, 3] or abrupt [4]. Further well-known effects are clustering [5] and chaotization of the mean field [6, 7]. A subject of recent interest is partial synchrony in networks of pulse coupled integrate-and-fire units [7, 8] and in ensembles of phase oscillators with global nonlinear coupling [9]. The latter systems exhibit an interesting state of self-organized quasiperiodicity (SOQ), when the frequency of the mean field differs from the frequency of oscillators; this effects will be discussed in details below. Experimental demonstration of the SOQ states in a globally coupled population is the primary goal of this paper. We briefly review the experimental studies of globally coupled systems. First of all, there is a number of observations of synchronous collective dynamics in systems, where the coupling is assumed to be of all-to-all type, although it is most likely not homogeneous. These observations include synchronous emission of optical or acoustical pulses by groups of insects [10], rhythmical hand clapping in opera houses [11], glycolytic oscillation in populations of yeast cells [12, 13], etc. A well-known example is pedestrian synchrony on the London Millennium Bridge; the experiments with the pedestrian groups of deferent size demonstrated that collective synchrony is a threshold phenomenon [14], in correspondence with the theoretical results [2]. Next, we mention a brilliant demonstration of collective synchrony in a very simple experiments with metronomes, performed by B. Daniels within a framework of student research . Well-controlled experiment on globally coupled oscillators have been performed by J. Hudson, I. Kiss, and collaborators. Using an array of 64 electrochemical oscillators they have confirmed practically all theoretical predictions. In particular, they have demonstrated Kuramoto transition in ensembles of periodic and chaotic oscillators. Other laboratory experiments have been conducted with Josephson junctions, photochemical oscillators, and vibrating motors on a common support . In this paper we present the results of an experiment with electronic oscillators, globally coupled via a common feedback loop which contains a phase shifting unit. The coupling is nonlinear in the sense that phase shift depends on the amplitude of the collective oscillation. We demonstrate that in this setup we can observe, with increase of the strength of the global coupling, a transition from asynchronous state to collective synchrony and then to SOQ state. The paper is organized as follows. Section I presents the experimental setup. In Section II we derive equations of the mathematical model. In Section III we demonstrate the latter state experimentally and in Section IV we conclude our results.

Experimental setup

We perform experiments with 20 electronic generators, coupled via a global feedback loop, see Fig. 1. Coupling is organized via a common resistor R_c; a fraction of the voltage across this resistor is fed to the input of the phase-shifting unit. The output of this unit is fed back to all oscillators via resistors R_1. Scheme of an individual generator is given in Fig. 2; it represents a Wien bridge oscillator with a saturation of the amplitude.

The saturation is ensured by the non-linear circuit in the negative feedback loop of the operational amplifier; this circuit is built by diodes $D_{1,2}$ and resistor R_7. With the help of the trimmer resistor R_6 amplitudes of all uncoupled oscillators were tuned to approximately output voltage ≈ 1V. The experimentally obtained input-output characteristics of the operational amplifier along with its analytical approximation is shown in Fig. 3.

The scheme of the phase-shifting unit is depicted in Fig. 4. It consists of two identical linear subunits and one nonlinear. The linear subunit is a standard RC-circuit. The nonlinear part represents a high-pass first order filter, where nonlinear properties of diodes provide a dependence of the phase shift between input and output on the amplitude of the input, see Fig. 5.

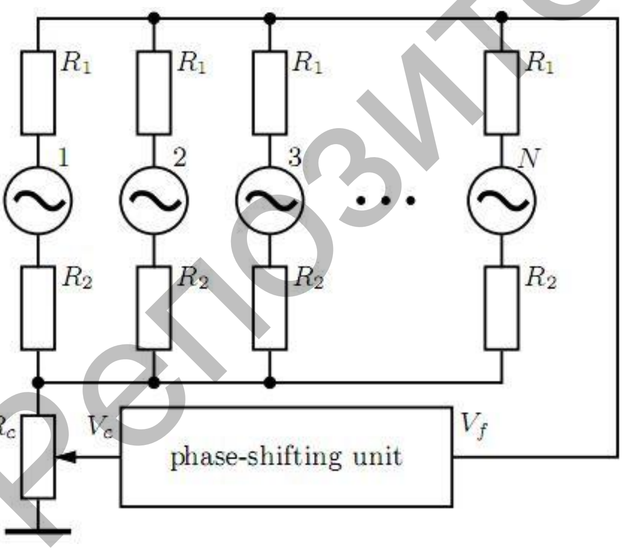

Fig. 1. Scheme of the globally coupled system. Individual generators are shown here by one symbol, there detailed scheme is given in Fig. 2, whereas the scheme of the phase- shifting unit is given in Fig. 4.

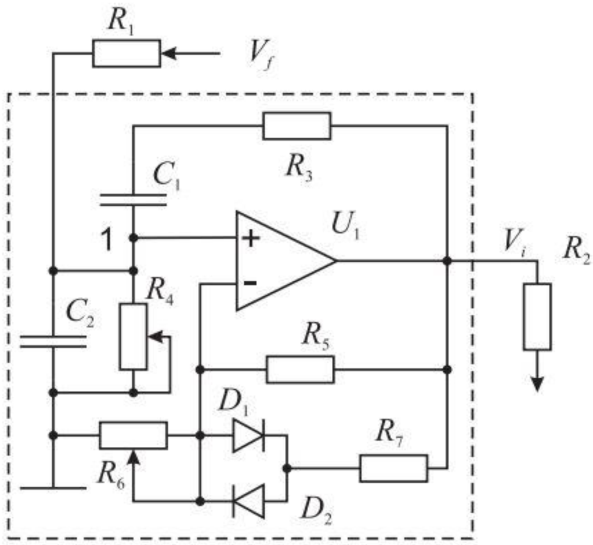

Fig. 2. Wien bridge oscillator. Here V_i is the output voltage of the i-th oscillator, V_f is the output voltage of the global feedback loop. We used following components (see [12]).

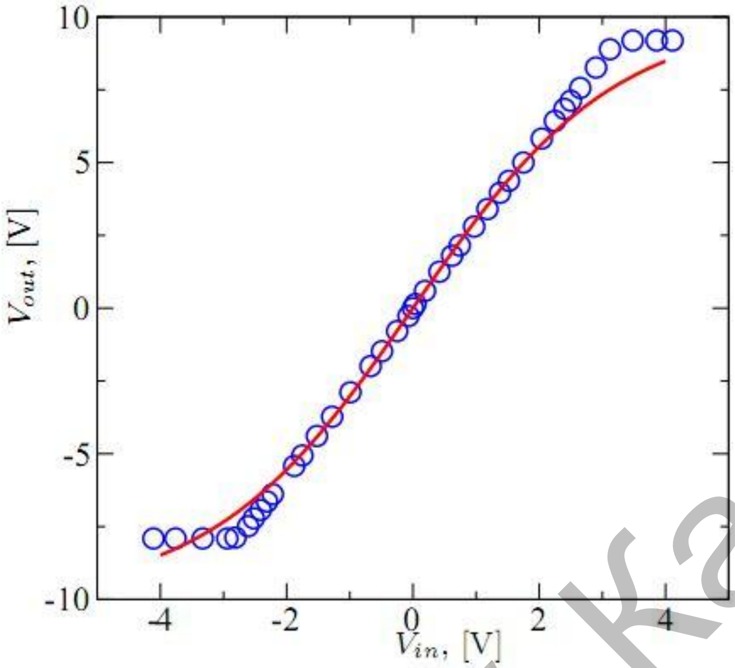

Fig. 3. Experimentally obtained input-output characteristics of the operational amplifier (symbols) and its approximation by V_{out} = *10 tanh(V_{in}/3.2)* (solid line).

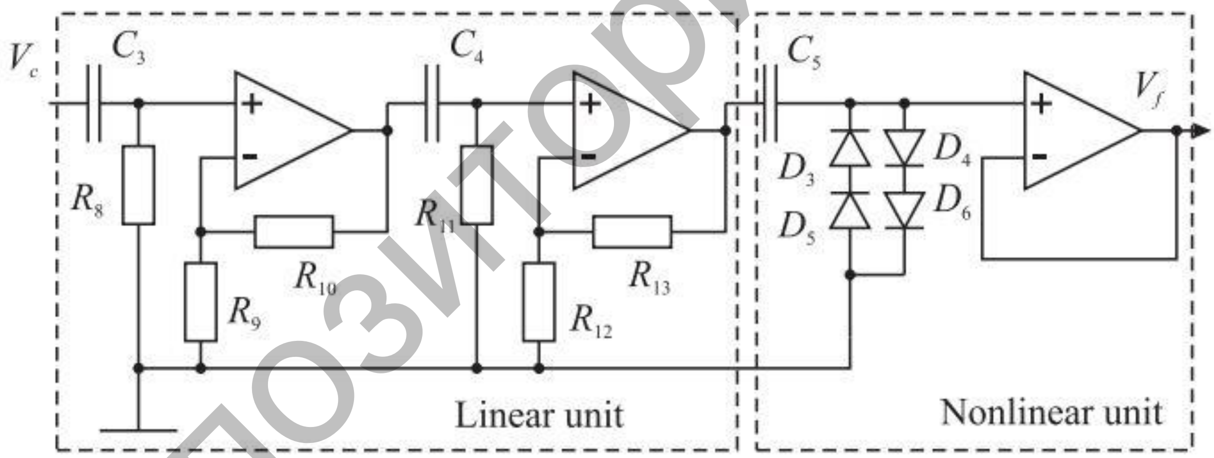

Fig. 4. Phase-shifting unit consists of two linear and one nonlinear shifting subunits.

Mathematical model

We first write the equations for one oscillator, driven by the voltage V_f. Denoting the input and output voltages of the operational amplifier as u and V, respectively, and the input-output characteristics of the amplifier as $V = f(u)$,we write the Kirchhoff laws for balance of

currents and voltages at point 1in Fig.(2):

$$I = C_2\dot{u} + \frac{u}{R_4} + \frac{u - V_f}{R_1},$$

$$f(u) = u + IR_3 + \frac{1}{C_1}\int I dt,$$

where I is the current through the R_3C_1 grid. Excluding I and differentiating with respect to time, we obtain:

$$\ddot{u} + \left[3\omega + \nu - \omega f'(u)\right]\dot{u} + \Omega^2 u = \omega\nu V_f + \nu\dot{V}_f, \quad (1)$$

Where we use that $R_3 = R_4$, $C_1 = C_2$, and denote $\omega = 1/R_3C_1$, $\nu = 1/R_1C_1$, and $\Omega^2 = \omega(\omega + \nu)$. The function $f(u)$ can be well approximated as:

$$f(u) = ku_s \tanh(u/u_s), \quad (2)$$

see Fig.3. Here u_s determines the range of the input voltages where the amplifier works without saturation and k is the slope of the characteristics in the linear regime, $k = f'(0)$.

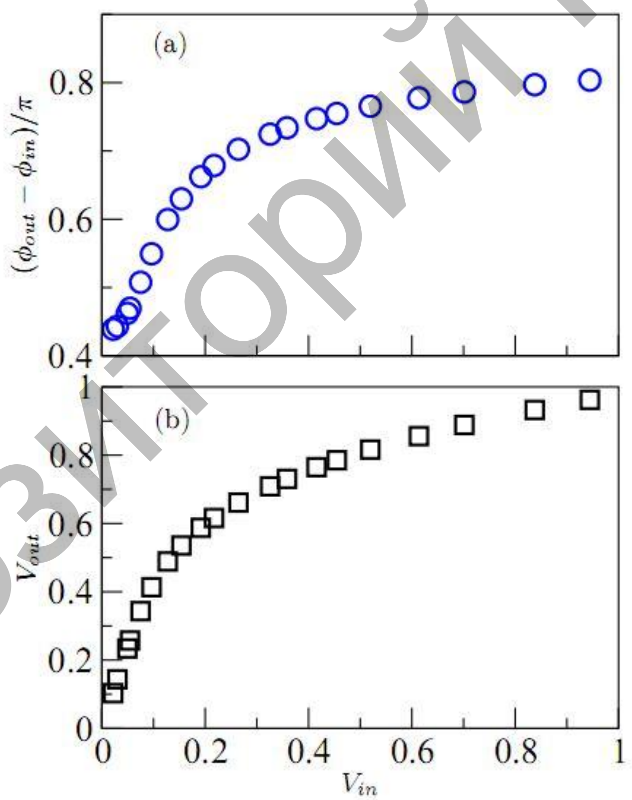

Fig. 5. Characteristics of the phase-shifting unit: (a) Phase shift and (b) output voltage vs the input voltage.

Using Eq.(8)and introducing dimensionless variable $x = u/u_s$, we obtain an equation of the van der Pol type:

$$\ddot{x} + \mu\left(1 - \frac{\gamma}{\cosh^2 x}\right)\dot{x} + \Omega^2 x = \frac{\nu}{u_s}\left(\omega V_f + \dot{V}_f\right). \quad (3)$$

where $\mu = 3\omega + \nu$ and $\gamma = \omega k/\mu$. For $V_f = 0$, the system exhibits a limit cycle solution if $\gamma > 1$, i.e. if $k > 3 + \nu/\omega$. Note that for given parameters $\nu << \omega$. Now we consider N systems and label the corresponding variables by the index $i = 1,...,N$, e.g., the output voltage of the i-th oscillator is V_i. Systems are coupled via the common load R_c, as shown in Fig.1. The voltage across the common load can be easily found. A fraction of this voltage is applied to the input of the feedback loop:

$$V_c = \varepsilon \frac{\sum_{i=1}^{N} V_i}{N + R_2 / R_c}, \tag{4}$$

Where $0 << \varepsilon << 1$ has the meaning of the coupling strength. Since $R_2 << NR_c$, we obtain pure mean field coupling:

$$V_c = \varepsilon k u_s \langle \tanh x \rangle, \tag{5}$$

where $\langle \cdot \rangle$ denotes averaging. Finally, we write the equations of the coupled system:

$$\ddot{x}_i + \mu \left(1 - \frac{\gamma}{\cosh^2 x} \right) \dot{x}_i + \Omega^2 x_i = \frac{\nu}{u_s} \left(\omega V_f + \dot{V}_f \right) = \Re \tag{6}$$

And derive the feedback term $\Re$ for three following cases.
No phase shifting unit. Here we simply have $V_f = V_c$ and

$$\Re = \varepsilon k \nu \left(\omega \langle \tanh x \rangle + \left\langle \frac{\sinh x}{1 + \cosh^2 x} \right\rangle \right). \tag{7}$$

Linear phase shifting unit. We have two *RC*-circuits with amplifier, amplification k_1.Then the equations are

$$\dot{V}_1 = k_1 \dot{V}_c - \alpha_1 V_1, \tag{8}$$

$$\dot{V}_f = k_1 \dot{V}_1 - \alpha_2 V_f, \tag{9}$$

where $\alpha_1 = 1/R_8 C_3$, $\alpha_2 = 1/R_{11} C_4$, and V_1 is the output of the first phase-shifting unit. Then

$$\Re = \frac{\nu}{u_s} \left(\omega V_f + k_1 \dot{V}_1 - \alpha_2 V_f \right) \tag{10}$$

$$\Re = \frac{\nu}{u_s} \left[\left(\omega - \alpha_2 \right) V_f + k_1^2 \dot{V}_c - k_1 \alpha_1 V_1 \right] \tag{11}$$

Equations (7,8,9,10,11) represent a close system which can be simulated numerically.

Nonlinear phase shifting unit. Now we have two linear units and one nonlinear. For the linear units we have

$$\dot{V}_1 = k_1 \dot{V}_c - \alpha_1 V_1, \tag{12}$$

$$\dot{V}_2 = k_1 \dot{V}_1 - \alpha_2 V_2, \tag{13}$$

For the nonlinear unit we have:

$$V_2 - V_3 = \frac{1}{C_5} \int I_D dt,$$

where V_3 is the voltage at the input of the amplifier, I_D is the current through the diods. Generally, the current through a diod can be written as

$$I_D = I_S \left(e^{V/V_T} - 1 \right),$$

where V is the applied voltage, and I_S and V_T are theoretical reverse current and the thermal voltage, respectively. For our setup the current is a sum of the direct and reverse currents, and applied voltage is $\pm V_3/2$. Hence, we have

$$I_D = I_S \left(e^{\frac{V_3}{2V_T}} - 1 + e^{-\frac{V_3}{2V_T}} - 1 \right) = 2 I_S \left[\cosh\left(\frac{V_3}{2V_T} \right) - 1 \right] \tag{14}$$

Then

$$\dot{V}_2 - \dot{V}_3 = \frac{2 I_S}{C_5} \left[\cosh\left(\frac{V_3}{2V_T} \right) - 1 \right]. \tag{15}$$

Suppose the amplifier here is characterized by k_2 .Then $V_f = k_2 V_3$, so that

$$\dot{V}_f = k_2 \dot{V}_2 - \frac{2 k_2 I_S}{C_5} \left[\cosh\left(\frac{V_f}{2 k_2 V_T} \right) - 1 \right]. \tag{16}$$

Equations (12,13,14,15,16) represent a close system which can be simulated numerically.

Experimental results

We performed 3 experiments. In the first one the phase shifting unit was excluded so that the signal from the common load was directly applied to the inputs of oscillators, i.e. $V_f = V_c$. In the second experiment only the linear phase shifting unit was included (see Fig4., first part of the phase-shifting unit), and in the third run we had both linear and nonlinear units, as shown in Fig. 4. In each experiment we gradually changed the input to the feedback loop V_c from zero to its maximal value V_L and recorded the outputs of all oscillators, V_i, $i = 1, \ldots, N$, and the mean field voltage V_L [13]. In each recording we obtained 10^5 points per channel, with the sampling rate *65 kHz*. For each value of the coupling strength $\varepsilon = V_c/V_L$ we performed 10 recordings.

For the presentation of results we have computed, for each ε, the following quantities: (i) instantaneous phases ϕ_i of all oscillators and instantaneous phase and amplitude A_{mf} of the mean field V_L were obtained with the help of the Hilbert transform; (ii) frequencies f_i of all oscillators and frequency f_{mf} of the mean field were computed from the unwrapped phases for each recording and then averaged over 10 recordings; (iii) the order parameter R was obtained by averaging the quantity $N^{-1}\sum_{j=1}^{N} e^{i\phi_j}$ over time and over 10 measurements; (iv) the minimal (over all 10 measurements) value A_{min} of the instantaneous mean field amplitude A_{mf} and (v) the fraction η of the data points where the instantaneous frequency of the mean field is negative. Typically, synchronization transition in a globally coupled system is traced by plotting R vs. ε. In the limit $N \to \infty$, $R = 0$ in the incoherent state. However, since in our case $N = 20$, the finite-size fluctuations of the mean field in this state are quite large (they are known to scale as $1/\sqrt{N}$ and therefore R is not small either. We find that the distinction between incoherent (fluctuating mean field) and coherent (oscillatory mean field) states can be better revealed by A_{min} and η. In the first and second experiments (no phase shifting unit and linear unit, respectively), we observed standard Kuramoto transitions to collective synchrony. These transitions occurred at $\varepsilon \approx 0.85$ and $\varepsilon \approx 0.17$, respectively (see Fig.6, where the second experiment is illustrated), and were characterized by a monotonic dependence of R and A_{min} on ε. In the third, main, experiment, we observed an on-monotonic dependence of R and A_{min} on ε (Fig.7).We have found, that with increase of ε, 10 oscillators formed a cluster at $\varepsilon \approx 0.12$, while other 10 remained asynchronous. Next, the frequency locked oscillators leaved the cluster one by one. Finally, SOQ state appeared at $\varepsilon \approx 0.72$.

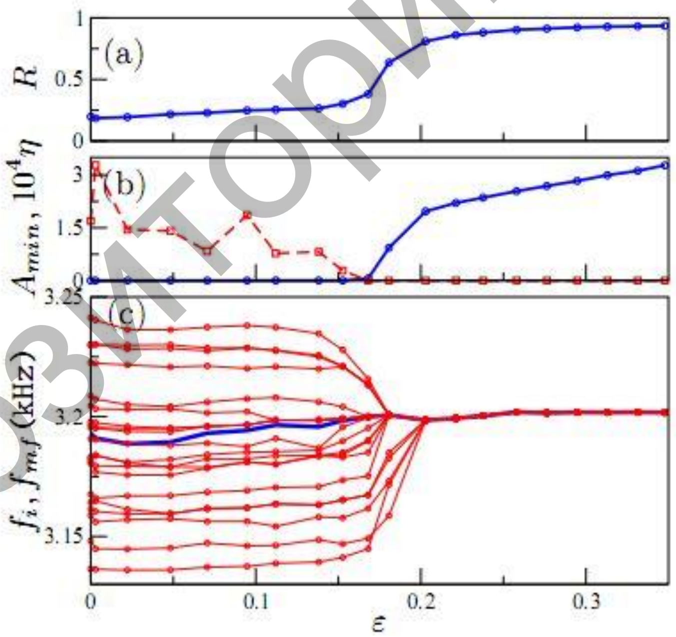

Fig.6. Results of the experiment with the linear phase-shifting unit. Order parameter R (a) and minimal mean field amplitude A_{min} (b, blue circles) reveal synchronization transition at the coupling strength $\varepsilon \approx 0.17$. This is also confirmed by the plot of η (b, red squares): this quantity shows that for $\varepsilon \gtrsim 0.17$, the instantaneous frequency of the mean field is always positive, as expected for a coherent, oscillatory mean field. The transition can be also very good seen from the frequency plot in (c): at $\varepsilon \approx 0.17$ several oscillators form a synchronous cluster and for $\varepsilon \gtrsim 0.2$ full frequency locking is observed, with R close to 1. Here circles show frequencies of oscillators f_i and bold blue line shows the mean field frequency f_{mf}. Notice that for sub-threshold coupling R is not small due to finite size effect; A_{mf} is here more efficient for determination of the threshold.

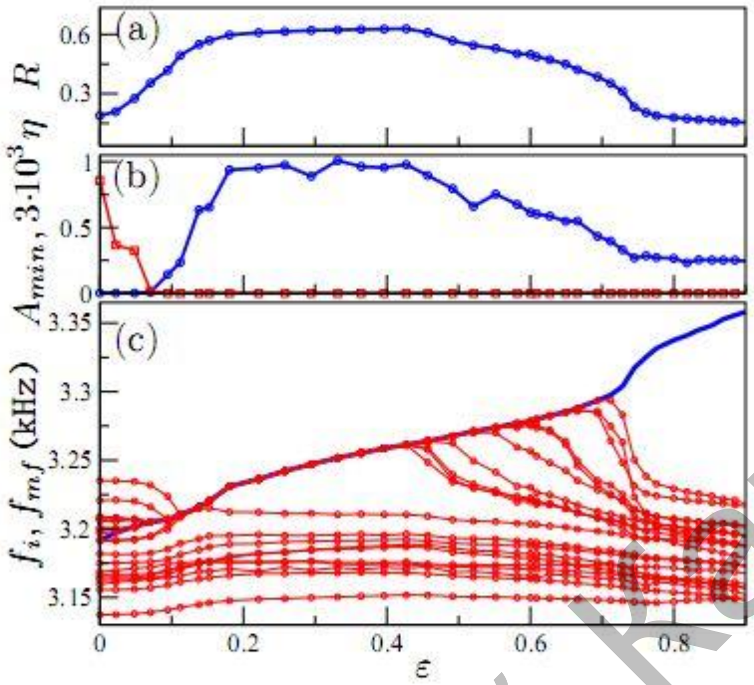

Fig. 7. The same as in Fig. 6 but for the experiment with the linear and nonlinear phase-shifting units. At $\varepsilon \approx 0.12$ we observe the transition to a partially synchronous state, where the fastest oscillators lock to each other and to the mean field. Between $\varepsilon \approx 0.43$ and the $\varepsilon \approx 0.72$ oscillators leave the cluster and for $\varepsilon \gtrsim 0.72$ the SOQ state is observed: mean field is faster than all oscillators. Although the values of the order parameter in the asynchronous ($\varepsilon \lesssim 0.12$) and SOQ states are almost the same, these states are qualitatively different, it can be easily distinguished by the quantities A_{min} and η (see text).

Thus, we have experimentally demonstrated a state where oscillators are synchronized neither with each other nor with the mean field, but the amplitude of the latter is, nevertheless, non-zero. This peculiar coherent state is possible because phases of oscillators, though not locked, are coordinated in a way that their distribution is non-uniform (Fig.8).

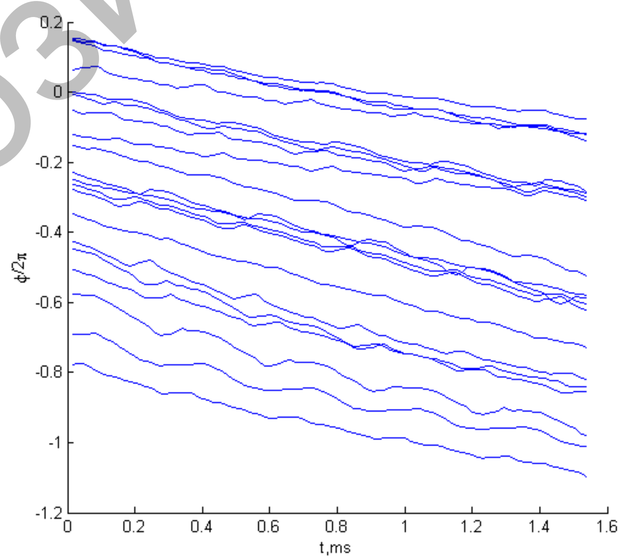

Fig. 8. After the transition to SOQ, the phase difference φ rotates non-uniformly, epochs of nearly constant φ are intermingled with some slips.

Conclusions

Our results well correspond to analytical results for phase oscillators [9,14]. The SOQ regime we observe emerges when the system is brought, due to the phase shift, close to the point where attractive interaction becomes repulsive. Thus, we expect SOQ to be observed in other physical systems where the global coupling is characterized by an amplitude-dependent phase shift or time delay.

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