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On the calculation of rectangular plates by the variation method

The article presents the calculation of rectangular plates by a variational method. For considered rectangular plate the research was conducted by the equilibrium conditions of the elementary strip isolated from the plate, by the method that was used in the works of V.Z.Vlasov. To illustrate the above variational method were given specific examples of the calculation of a square plate, hinged along the contour and loaded equally by distributed loading of given intensity, as well as a square plate, rigidly clamped along the contour. A comparative analysis of the results was carried out.

Key words: crectangular plate, the deflection function, effort, bending moments, torque, shear forces, cylindrical rigidity of the plate, hinge support, equilly distributed loading.

A variational method by Vlasov-Kantorovich

Consider a rectangular plate. In contrast to considered classical variational methods in which the required function of the plate deflection $W(x,y)$ was set up in advance accurate to the constants a expansion

$$W_n(x,y) = a_1\varphi_1(x,y) + a_2\varphi_2(x,y) + ... + a_n\varphi_n(x,y),$$

in the Vlasov-Kantorovich method, this function is sought in the form

$$W(x,y) = \sum_{i=1}^{n} W_i(y)\chi_i(x), \qquad (1)$$

where $W_i(y)$ are functions to be determined, and $\chi_i(x)$ is the function selected in advance in accordance with the boundary conditions given at the longitudinal edges of the plate $x = 0$ and $x = a$.

The required functions $W_i(y)$ with the dimension of the deflection can be called the generalized plate deflections, and the dimensionless $\chi_i()x$ functions are coordinate functions or functions of the transverse distribution of deflections.

Comparing the expansion (1) with the formula of method of single trigonometric series, $W(x,y)$, where the required function of plate deflection can be presented here as single series

$$W(x,y) = \sum_{n=1}^{\infty} Y_n \sin\frac{n}{a}, \qquad (2)$$

where $Y_n = Y_n(y)$ is a unknown function of one variable, which is selected so that the expression (2) satisfied a resulting equation and conditions of fixing on the edges $y = 0$ and $y = b$, it is easy to notice that a variational method of Vlasov-Kantorovich can be considered as generalization of M.Levy method. Indeed, in the method of M. Levy as functions of the coordinates are chosen trigonometric expressions, which correspond only the case of hinged support the longitudinal edges of the plate [1].

Three different approaches can be used to define functions $W_i(y)$. These functions can be found by considering, for example, the conditions of equilibrium of elementary strip isolated from the plate sections parallel to the axis O_x. For this purpose, we can also use an expression of potential energy

$$E = \frac{D}{2}\int\int\{(\frac{\partial^2 W}{\partial x^2})^2 + (\frac{\partial^2 W}{\partial y^2})^2 + 2\nu(\frac{\partial^2 W}{\partial x^2}\cdot\frac{\partial^2 W}{\partial y^2}) + 2(1-\nu)(\frac{\partial^2 W}{\partial x \partial y})^2\}dxdy - \int\int q(x,y)W(x,y)dxdy,$$

if to add the expansion to it (1) and to equate to zero, the first variation of the energy caused by the variation δW_i of unknown function. We can finally find the functions $W_i(y)$, having used the Bubnov-Galerkin method for solving the basic equations of the problem.

$$D\nabla^2\nabla^2 W = q(x_1, x_2), \tag{3}$$

where q — the intensity of the external distributed load; $\nabla^2\nabla^2 W$ — the biharmonic operator. For this we need add the expression (1) in the differential equation (3) and require the operator to all orthogonal functions $\chi_i(x)$ obtained. Naturally, in all these cases, we get the same result, namely, a system of ordinary differential equations about the functions $W_i(y)$ [2].

Consider the first of these approaches, which was used in the works of V.Z.Vlasov and his disciples. Meanwhile the conditions of elementary strips equilibrium will be understood in the sense of the principle of virtual displacements, equating to zero the total operation of all internal and external forces on the strip possible for its movements. Let us take as a possible displacement displacements described by functions $\chi_i(x)$ at $W_j = 1$. Because at our disposal there is linearly independent functions $\chi_i(x)$ we can make the equilibrium equations, from which identify all n the required functions $W_i(y)$.

External forces to the selected strip are stated loading $q(x, y)$,bending moments M_y and listed shear forces, Q_y^* and their increments, and centered angular forces equal to twice the value of the torque in the same corner points. Possible work of these forces on the displacement $\chi_i(x)$,divided into dy, is

$$\int_0^a \frac{\partial Q_y^*}{\partial y}\chi_j dx - 2\left(\frac{\partial H}{\partial y}\chi_j\right)_0^a + \int_0^a q(x, y)\chi_j dx. \tag{4}$$

Note that the work of bending moments M_y is equal to zero due to the fact that the strip dy bent on a cylindrical surface. The expression in square brackets, part (4), should be understood as the difference between the values of the quantity in pa-rentheses to the extreme points of the strip $x = a$, $x = 0$.

Bending moments M_x and given shear forces Q_x^* are internal forces of the elementary strip. Work of shear forces will be zero due to the previously accepted hypothesis about the absence of progress in the vertical plane. Work of bending moments is

$$-\int_0^a M_x(-\chi_j'')dx = \int_0^a M_x\chi_j'' dx. \tag{5}$$

Here, the minus sign before the integral is common in the determination of the internal forces; the minus sign before χ_j'' the minus sign depends on different direction of curvature at positive χ_j'' and positive M_x .

In view of (4) and (5) the condition of the elementary strip balance corresponding to any n possible displacements χ_j, takes the form

$$\int_0^a \frac{\partial Q_y^*}{\partial y}\chi_j dx - 2\left(\frac{\partial H}{\partial y}\chi_j\right)_0^a + \int_0^a M_x\chi_j'' dx + \int_0^a q(x, y)\chi_j dx = 0. \tag{6}$$

Included in the equation (6) given shear forces Q_y^*, torque H and bending moments M on the basis of the expansion (1) can be written in the form

$$\begin{cases} Q_y^* = -D\sum_{i=1}^{n}(W_i''\chi_i + (2-\nu)W_i'\chi_i''); \\ H = -D\sum_{i=1}^{n}(1-\nu)W_i'\chi_i'; \\ M_x = -D\sum_{i=1}^{n}(\nu W_i''\chi_i + W_i\chi_i''). \end{cases} \quad (7)$$

Having substituted (7) into the equation (6), having made the necessary differentiation and simple transformations, we finally obtain a system n of ordinary differential equations of the following form

$$\sum_{i=1}^{n}[a_{ji}W_i^{IV} - 2b_{ji}W_i'' + c_{ji}W_i] = \frac{G_j}{D}; \quad (j = 1, 2, ...n), \quad (8)$$

where

$$a_{ji} = \int_0^a \chi_i\chi_j dx; \quad b_{ji} = \int_0^a \chi_i'\chi_j' dx - \frac{\nu}{2}\left(\chi_i\chi_j' + \chi_i'\chi_j\right)\Big|_0^a; c_{ji} = \int_0^a \chi_i''\chi_j'' dx; G_j = \int_0^a q(x,y)\chi_j dx. \quad (9)$$

The coefficients (x) (9) are determined only by the selected system of functions of the transverse distribution of the deflection χ_i. Due to Betty theorem of reciprocity work they have reciprocity property: $a_{ij} = a_{ji}; b_{ij} = b_{ji}; c_{ij} = c_{ji}$, that gives the symmetry of the system of equations (8).

Free terms (9) of the equations (8) depend on the given loading and selected functions $\chi_i(x)$. If the composition of given loading includes and point forces P_k, applied to m the plate sections $x = k$, we should add their work on displacements $\chi_j(k)$. As a result

$$G_j = \int_0^a q(x,y)\chi_j dx + \sum_{k=1}^{n} P_k\chi_j(k).$$

Integrating the system of equations (8), we can find n functions $W_i(y)$ accurate to $4n$ arbitrary constants of integration. To determine them on the cross edges of the plate $y = 0$ and $y = b$ we should put $4n$ of boundary conditions defined through generalized displacements, or through generalized efforts.

Generalized deflections $W_i(y)$ and rotations $\varphi_i(y)$ are generalized plate movements (10).

Under the generalized moment M_{yj} we understand the work of bending moments of M_y sections $y = const$ on corresponding displacements: $\dfrac{\partial W}{\partial y} = \varphi x_j$ in the generalized rotation angle $\varphi = I$, i.e.

$$\begin{cases} W(x,y) = \sum_{i=1}^{n} W_i(y)\chi_i(x); \\ \dfrac{\partial W(x,y)}{\partial y} = \sum_{i=1}^{n} \varphi_i(y)\chi_i(x); \end{cases} \quad (10)$$

$$M_{ij} = \int_0^a M_y x_j dx. \quad (11)$$

Under generalized shearing force Q_{yj} is meant work of shear forces Q_y and torques H of section $y = const$ on a possible displacement x_j or work of given shear forces Q_y and concentrated angular forces are numerically equal H

$$Q_{yj} = \int_0^a Q_y^* x_j dx - 2[H\chi]_0^a. \tag{12}$$

Including into the formula (11) and (12) values of moments and shear forces, expressed in terms of (1) (see, for ex., (7)), for generalized efforts of plate section $y = const$, we get the following expressions:

$$M_{yj} = -D \sum_{i=1}^{n} \left[W_i \int_0^a \chi_i \chi_j dx - \nu \left(\int_0^a \chi_i' \chi_j' dx - [\chi_i' \chi_j]_0^a \right) W_i \right]; \tag{13}$$

$$Q_{yj} = -D \sum_{i=1}^{n} \left[W_i \int_0^a \chi_i \chi_j dx - \left((2-\nu) \int_0^a \chi_i' \chi_j' dx - [\chi_i' \chi_j]_0^a \right) W_i \right]. \tag{14}$$

Formulas (10) and (13) make it possible to formulate on each of the cross edges of the plate $2n$ of the boundary condition, set by generalized displacement (built-in edge), generalized efforts (edge free from built-in) or in mixed form (hinged edge).

Because in the considered variational method a required function of deflections $W(x, y)$ is given a priori only in one direction and in the other is sought precisely because of solving differential equations (8), this method is, in principle, more accurate than the classical variational methods described previously. The accuracy of the solution here will depend, naturally, on the number of terms withheld in the series (1), as well as on how well are chosen the approximating functions $\chi_i(x)$.

When selecting a system of functions $\chi_i(x)$ we should bear in mind that they must be linearly independent and have the necessary fullness. In addition, they should satisfy geometrical conditions given on the longitudinal edges of the plate. Satisfying of static boundary conditions as in the Ritz method is in principle not required. But if functions $\chi_i(x)$ satisfy both geometric and static conditions on the edges $x = 0$ and $x = a$, the accuracy of the solution can be significantly increased. In case if given loading $q(x, y)$ is quite smoothly distributed over most part of the plate surface, in the expansion of (1) with sufficient accuracy for practical purposes we can restrict only the first one or, as a last resort, the first few terms. Moreover, for selection of functions $\chi_i(x)$ can be recommended so-called static method, the essence of which is that the function $\chi(x)$ is selected as the line of the beam deflection of length a , having on the ends the same fastenings, as the plate on the lines $x = 0$ and $x = a$. In the capacity of loading causing beam deflection may be taken loading, similar to that applied to the considered plate.

To illustrate the above variational method, consider a few specific examples of calculation. As the first example, take a square plate of size $a \times a$, hinged all over the contour and loaded uniformly distributed load of intensity q_0. Take also $\nu = 0.3$.

Solving the problem in the first approximation, i.e., limiting in the expansion (1) of one term, choose a function $\chi(x)$ as a line of beam deflection loaded uniform loading. In dimensionless form, this function will be written in the following way:

$$\chi(x) = \frac{x}{a} \left(1 - 2\frac{x^2}{a^2} + \frac{x^3}{a^3} \right), \tag{15}$$

and will, of course, satisfy the boundary conditions specified on the edges of the plate $x = 0$ and $x = a$. At one term of the expansion (1) a system of differential equations (8) contains only one equation

$$a_{11}W^{IV} - 2b_{11}W'' + c_{11}W = \frac{G_1}{D}, \tag{16}$$

the coefficients of which and a free term G with regard to (15) have the following meanings

$$a_{11} = \int_0^a \chi^2 dx = 0.04921a; \quad b_{11} = \int_0^a (\chi')^2 dx - \nu[\chi\chi']_0^a;$$

$$c_{11} = \int_0^a (\chi'')^2 dx = \frac{4.8}{a^3}; G_1 = q_0 \int_0^a \chi dx = 0.2q_0 a. \tag{17}$$

Rewrite equitation (16) in a more convenient form for integration

$$W^{IV} - 2r^2 W'' + s^4 W = P, \tag{18}$$

where $r^2 = \dfrac{b_{11}}{a_{11}} = \dfrac{9.87088}{a^2}$; $s^4 = \dfrac{c_{11}}{a_{11}} = \dfrac{97.54839}{a^4}$; $P = \dfrac{G_1}{a_{11}D} = 4.06454\dfrac{q_0}{D}$.

The roots of the characteristic equation corresponding to equation (18) will be equal

$$k = \pm\alpha \pm \beta i, \quad \alpha = \sqrt{(s^2 + r^2)/2} = 3.14226/a, \beta = \sqrt{(s^2 - r^2/2)} = 0.05375/a. \tag{19}$$

According to types of the roots (19) the solution of the differential equation (18) will be written as

$$W = C_1\Phi_1 + C_2\Phi_2 + C_3\Phi_3 + C_4\Phi_4 + W_0, \tag{20}$$

where

$$\begin{cases} \Phi_1 = chay \cdot sin\beta y; & \Phi_2 = chay \cdot cos\beta y; \\ \Phi_3 = shay \cdot cos\beta y; & \Phi_1 = shay \cdot sin\beta y \end{cases}$$

and particular solution $W_0 = \dfrac{P}{s^4} = 0.0416\dfrac{qa^4}{D}$.

We will take the form of integration $C_1, ...C_4$ occurring in (20), are determined from the boundary conditions defined on the transverse edges of the plate: at $y = 0$ and $y = a$ $W = W'' = 0$.

After determination of integration constants deflection of the plate can be found by the formula

$$W(x, y) = W(y)\chi(x) \tag{21}$$

and internal forces — according to the formulas (13)–(14), in which under the function $\chi(x)$ should be understood the expression (15), and under the function $W(x)$ — the expression (20).

The calculations show that in the center of the plate (at $x = a/2, y = a/2$) deflection and bending moments will take the following values:

$$W = 0.00406\frac{q_0 a^4}{D}; \quad M_x = 0.0498q_0 a^2; \quad M_y = 0.0480q_0 a^2.$$

If in the above method as a function χ has been accepted a function $sin\dfrac{\pi x}{a}$, the resulting decision would be exactly coincided with the decision of the method M.Levi [3].

As the second example, consider the same uniformly loaded square a plate with a side but rigidly clamped all over the contour.

Again solving a problem only in the first approximation we represent required deflection of the plate in the form (21), and the function of the transverse distribution of the deflection $\chi(x)$ will choose as a function of beam deflection, rigidly clamped at both ends.

At the same time, compared with the previous case, here we will change only the values of the coefficients (17), included in the basic differential equation of the problem (18), the solution of which again will be written in the form (20). Determining the arbitrary constants of integration $C_1, ..., C_4$, we can write the following boundary conditions at:

$$y = a \quad and \quad y = aW = W' = 0,$$

which will allow to find the final expression for the function $W(y)$.

Omitting the intermediate calculations, we give the values of the deflection and bending moments in the center of the plate, at $x = \frac{a}{2}$ and $y = \frac{a}{2}$.

$$\begin{cases} W = 0.00130\dfrac{qa^4}{D}, \quad \left(0.00126\dfrac{qa^4}{D}\right); \\ M_x = 0.0247qa^2, \quad \left(0.0247qa^2\right); \\ M_y = 0.0239qa^2, \quad \left(0.0231qa^2\right), \end{cases}$$

as well as the values of the reduced shear forces for the middle of clamped edges of the plate.

$$X = 0 \quad \text{and} \quad y = \frac{a}{2}; \quad Q_x^* = 0.498qa \quad (0.452qa);$$

$$x = \frac{a}{2} \quad \text{and} \quad y = 0; \quad Q_y^* = 0.482qa \quad (0.452qa).$$

The numbers in parentheses are shown the exact (table) values of the corresponding quantities.

We can see that and in this case, the first approximation of considered method gives good results: an error in the value of the bending moments does not exceed 6.9 % in the value of shear forces — 10.2 %. At the same time the results are more exact than those which are determined by means of the Bubnov-Galerkin method [3].

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Тіктөртбұрышты пластиналарды вариациялық әдіспен есептеу туралы

Мақалада Власов-Канторовичтің вариациялық әдісімен пластинаның ізделінетін, иілу функциясы тұрақтысына дейінгі дәлдікпен белгіленген тіктөртбұрышты пластина қарастырылған. Вариациялық әдісте нақты тіктөртбұрышты пластинаның мысалын зерттеп, оған бірнеше нақты контур бойымен топсалы тіркелген және қарқындылық жүктемесі біркелкі үлестірілген шаршы пластина алынған. Анықталған нәтижелерге салыстырмалы түрде қорытынды жасалған.

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О расчете прямоугольных пластин вариационным методом

В статье для рассматриваемой прямоугольной пластины проведено исследование через условия равновесия элементарной полоски, выделенной из пластины, методом, который применялся в работах В.З.Власова. Для иллюстрации изложенного вариационного метода приведены конкретные примеры расчета квадратной пластины, шарнирно опертой по всему контуру и загруженной равномерно распределенной нагрузкой заданной интенсивности, а также квадратной пластины, жестко защемленной по всему контуру. Проведен сравнительный анализ полученных результатов.

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