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Some non-standard quasivarieties of lattices

The questions of the standardness of quasivarieties have been investigated by many authors. The problem "Which finite lattices generate a standard topological prevariety?" was suggested by D.M. Clark, B.A. Davey, M.G. Jackson and J.G. Pitkethly in 2008. We continue to study the standardness problem for one specific finite modular lattice which does not satisfy all Tumanov's conditions. We investigate the topological quasivariety generated by this lattice and we prove that the researched quasivariety is not standard, as well as is not finitely axiomatizable. We also show that there is an infinite number of lattices similar to the lattice mentioned above.

Keywords: lattice, quasivariety, basis of quasi-identities, profinite algebra, topological quasivariety, profinite quasivariety.

Introduction

The problems concerning finite axiomatizability and standardness of (quasi)varieties of algebras are among the most researched and relevant topics in universal algebra.

According to R. McKenzie [1], each finite lattice has a finite identity basis. The analogous statement for quasi-identities is incorrect. V.P. Belkin in [2] proved that there is a finite lattice which has no finite quasi-identity basis. In this regard, the problem "Which finite lattices have finite quasi-identity bases" was proposed by V.A. Gorbunov and D.M. Smirnov [3]. A sufficient two-part condition under which a locally finite quasivariety of lattices does not have a finite (independent) quasi-identity basis was found by V.I. Tumanov [4].

In [5] the concept of a standard (topological) quasivariety was introduced, and the basic properties were investigated and many examples of standard and non-standard quasivarieties were provided. The standardness of algebras was further studied by D.M. Clark, B.A. Davey, R.S. Freese and M.G. Jackson in [6], who established a general condition guaranteeing the standardness of a set of finite algebras. In [7] sufficient conditions were found under which a quasivariety contains a continuum of non-standard subquasivarieties. In [6] it was proved that any finite lattice generates a standard variety. However, in [8] it was established that Belkin's lattice generates non-standard quasivariety. These naturally arose the problem "Which finite lattices generate standard topological quasivarieties?" that was suggested by D.M. Clark, B.A. Davey, M.G. Jackson and J.G. Pitkethly in [8].

In [9, 10] one specific lattice was studied and it was proved that this lattice has no finite basis of quasi-identities [9] and generates non-standard quasivariety [10], respectively. The special feature of this lattice is that it does not satisfy one of the two-part Tumanov's condition (see Theorem 2).

In this paper we continue to study the standardness problem for one specific finite modular lattice. This lattice does not satisfy all Tumanov's conditions [4] and the quasivariety generated by this lattice is not standard, as well as is not finitely based (Theorem 3). At the end we show that there is an infinite number of lattices similar to this lattice (Theorem 4).

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1 Basic concepts and preliminaries

We recall some basic definitions and results for quasivarieties that we will refer to. For more information on the basic notions of general algebra and topology introduced below and used throughout this paper, we refer to [11–13].

We assume that all classes of algebras the same fixed finite signature σ and abstract, unless we specify otherwise. Also an algebra $\langle A;\sigma\rangle$ and its carrier (its basic set) A will be identified and denoted by the same way, namely A.

A class of algebras which is closed with respect to subalgebras, direct products (including the direct product of an empty family), and ultraproducts is a *quasivariety*. In other words, a class of algebras axiomatized by a set of quasi-identities is a quasivariety. A *quasi-identity* is a universal Horn sentence with the non-empty positive part

$$(\forall\bar{x})[p_1(\bar{x})\approx q_1(\bar{x})\wedge\cdots\wedge p_n(\bar{x})\approx q_n(\bar{x})\rightarrow p(\bar{x})\approx q(\bar{x})],$$

where $p,q,\ p_1,q_1,\ldots,p_n,q_n$ are terms. A quasivariety closed with respect to homomorphisms is a *variety*. In other words, a variety is a class of similar algebras axiomatized by a set of identities, according to Birkhoff theorem [14]. An *identity* is a sentence of the form $(\forall\bar{x})[s(\bar{x})\approx t(\bar{x})]$ for some terms $s(\bar{x})$ and $t(\bar{x})$. A quasivariety $\mathbf{K}$ has a finite basis of quasi-identities (finitely axiomatizable) if there is a finite set Σ of quasi-identities such as $\mathbf{K}=\mathrm{Mod}(\Sigma)$. Otherwise $\mathbf{K}$ has no finite basis of quasi-identities.

By $\mathbf{Q}(\mathbf{K})$ ($\mathbf{V}(\mathbf{K})$) we denote the smallest quasivariety (variety) containing a class $\mathbf{K}$. $\mathbf{Q}(\mathbf{K})$ is called finitely generated if $\mathbf{K}$ is a finite family of finite algebras. In case when $\mathbf{K}=\{A\}$ we write $\mathbf{Q}(A)$ instead of $\mathbf{Q}(\{A\})$. By Maltsev-Vaught theorem [15], $\mathbf{Q}(\mathbf{K})=\mathbf{SPP}_u(\mathbf{K})$, where $\mathbf{S}$, $\mathbf{P}$ and $\mathbf{P}_u$ are operators of taking subalgebras, direct products and ultraproducts, respectively.

Let $\mathbf{K}$ be a quasivariety. A congruence α on algebra A is called a $\mathbf{K}$-*congruence* provided $A/\alpha\in\mathbf{K}$. The set $\mathrm{Con}_{\mathbf{K}}A$ of all $\mathbf{K}$-congruences of A forms an algebraic lattice with respect to inclusion $\subseteq$. An algebra $A\in\mathbf{K}$ is *subdirectly* $\mathbf{K}$-*irreducible* if an intersection of any number of nontrivial $\mathbf{K}$-congruences is nontrivial. Since for any class $\mathbf{R}$ we have $\mathbf{Q}(\mathbf{R})=\mathbf{SPP}_u(\mathbf{R})=\mathbf{P}_s\mathbf{SP}_u(\mathbf{R})$, where $\mathbf{P}_s$ is operator of taking subdirect products, then for finitely generated quasivariety $\mathbf{Q}(A)$, every subdirectly $\mathbf{Q}(A)$-irreducible algebra is isomorphic to some subalgebra of A.

A finite algebra A with discrete topology generates a topological quasivariety consisting of all topologically closed subalgebras of non-zero direct powers of A endowed with the product topology. An algebra A is *profinite* with respect to quasivariety $\mathbf{R}$ if A is an inverse limit of finite algebras from $\mathbf{R}$. A topological quasivariety $\mathbf{Q}_\tau(A)$ is *standard* if every Boolean topological algebra (compact, Hausdorf and totally disconnected) with the algebraic reduct in $\mathbf{Q}(A)$ is profinite with respect to $\mathbf{Q}(A)$. In this case, we say that algebra A generates a *standard topological quasivariety*. For more information on the topological quasivarieties we refer to [6] and [8].

We say that X is *pointwise non-separable* with respect to quasivariety $\mathbf{R}$ if the following condition holds: There exist $a,b\in X$, $a\neq b$, such that, for each $n\in N$, each finite structure $M\in\mathbf{R}$ and each homomorphism $\varphi:X_n\to M$, we have $\varphi(a)=\varphi(b)$.

The following theorem provides non-standardness of quasivariety.

Theorem 1.(Second inverse limit technique [8])

Let $X=\varprojlim\{X_n\mid n\in N\}$ be a surjective inverse limit of finite structures, and let $\mathbf{K}$ be a quasivariety. Assume that $X\in\mathbf{K}$ is pointwise non-separable with respect to $\mathbf{K}$ and each substructure of X_n that is generated by at most n elements belongs to $\mathbf{K}$ for all $n\in N$. Then $\mathbf{K}$ is non-standard, as well as is not finitely axiomatizable.

To formulate our main result (Theorem 3) we need some special preliminaries.

Let $(a] = \{x \in L \mid x \leq a\}$ ($[a) = \{x \in L \mid x \geq a\}$) be a principal ideal (coideal) of a lattice L. A pair $(a, b) \in L \times L$ is called *splitting* (*semi-splitting*) if $L = (a] \cup [b)$ and $(a] \cap [b) = \varnothing$ ($L = (a] \cup [b)$ and $(a] \cap [b) \neq \varnothing$).

For any semi-splitting pair (a, b) of a lattice M we define a lattice

$$M_{a-b} = \langle \{(x, 0), (y, 1) \in M \times 2 \mid x \in (a], y \in [b)\}; \vee, \wedge \rangle \leq_s M \times \mathbf{2},$$

where $\mathbf{2} = \langle \{0, 1\}; \vee, \wedge \rangle$ is a two element lattice.

Theorem 2. (Tumanov's theorem [4])

Let a locally finite quasivarieties of lattices $\mathbf{M}$ and $\mathbf{N} \subset \mathbf{M}$ satisfy the following two conditions:

a) in any finitely subdirectly $\mathbf{M}$-irreducible lattice $M \in \mathbf{M} \backslash \mathbf{N}$ there is a semi-splitting pair (a, b) such that $M_{a-b} \in \mathbf{N}$;

b) there is a finite simple lattice $\mathcal{P} \in \mathbf{N}$ that is not a proper homomorphic image of any subdirectly $\mathbf{N}$-irreducible lattice.

Then the quasivariety $\mathbf{N}$ has no coverings in the lattice of subquasivarieties of $\mathbf{M}$. In particular, $\mathbf{N}$ has no finite (independent) basis of quasi-identities provided $\mathbf{M}$ is finitely axiomatizable.

A quasivariety is called *proper* if it is not variety. A subalgebra B of an algebra A is called *proper* if B is not one-element (trivial) and $B \ncong A$. For an algebra A and elements $a, b \in A$, by $\theta(a, b)$ we denote the least congruence on A containing pair (a, b).

2 Main result

Let A' and A are the modular lattices displayed in Figure 1. And let $\mathbf{Q}(A)$ and $\mathbf{V}(A)$ are quasivariety and variety generated by A, respectively. Since every subdirectly $\mathbf{Q}(A)$-irreducible lattice is a sublattice of A, and A' is simple and a homomorphic image of A, and A' is not a sublattice, then $A' \in \mathbf{V}(A) \backslash \mathbf{Q}(A)$, that is $\mathbf{Q}(A)$ is a proper quasivariety. One can check that A' has no semi-splitting pair. Thus, the condition *a*) of Tumanov's theorem does not hold on the quasivariety $\mathbf{Q}(A)$. It is easy to see that M_3 is unique non-distributive simple lattice in $\mathbf{Q}(A)_{SI}$ and it is a homomorphic image of A. Hence, the condition *b*) of Tumanov's theorem is not hold on quasivarieties $\mathbf{Q}(A)$ and $\mathbf{V}(A)$.

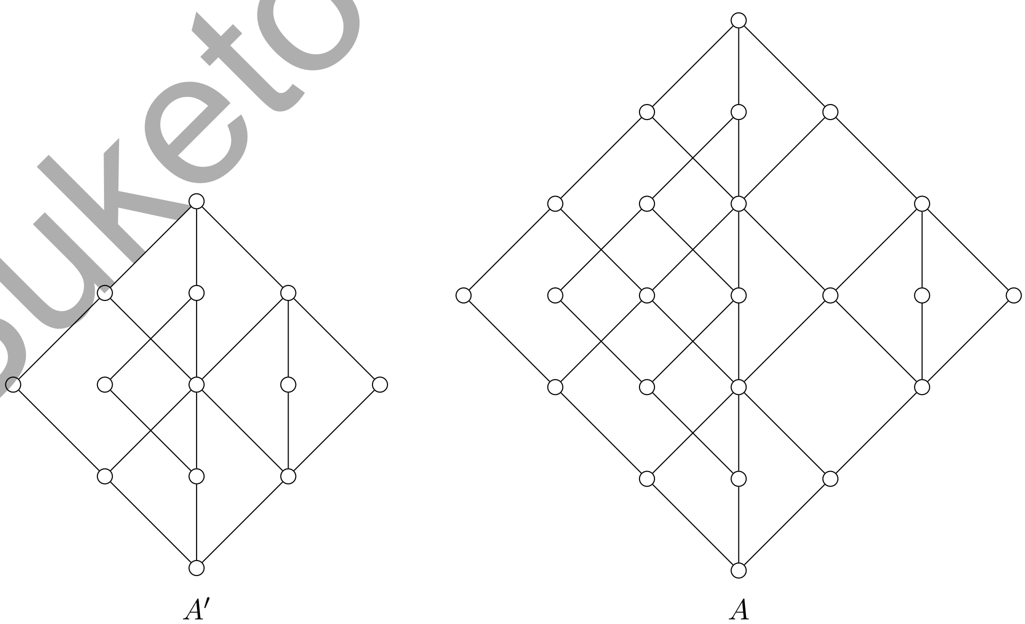

Figure 1: Lattices A' and A

The main result of the paper is

Theorem 3. The topological quasivariety generated by the lattice A is not standard, as well as is not finitely axiomatizable.

Proof of Theorem 3.

To prove the theorem we use Theorem 1. According to this theorem we will construct $L = \varprojlim\{L_n \mid n \in N\}$ a surjective inverse limit of the finite lattices such that every n-generated sublattice of L_n belongs to $\mathbf{Q}(A)$ and L is pointwise non-separable with respect to $\mathbf{Q}(A)$.

Let S be a non-empty subset of a lattice L. Denote by $\langle S\rangle$ the sublattice of L generated by S.

We define a modular lattice L_n by induction:

$n = 0$. $L_0 \cong M_{3-3}$ and $L_0 = \langle\{a_0, b_0, c_0, a^0, b^0, c^0\}\rangle$ (Fig. 2).

$n = 1$. L_1 is a modular lattice generated by $L_0 \cup \{a_1, b_1, c_1, a^1, b^1, c^1\}$ such that $\langle\{a_1, b_1, c_1, a^1, b^1, c^1\}\rangle \cong M_{3-3}$, and $c_0 = a^1$, $a^0 \wedge b^0 = c_0 \vee b^1 = c_0 \vee c_1$ (Fig. 3).

$n > 1$. L_n is a modular lattice generated by the set $L_{n-1} \cup \{a_n, b_n, c_n, a^n, b^n, c^n\}$ such that $\langle\{a_n, b_n, c_n, a^n, b^n, c^n\}\rangle \cong M_{3-3}$, and $c_{n-1} = a^n$, $a^0 \wedge b^0 = c_0 \vee b^n = c_0 \vee c_n$ (Fig. 4).

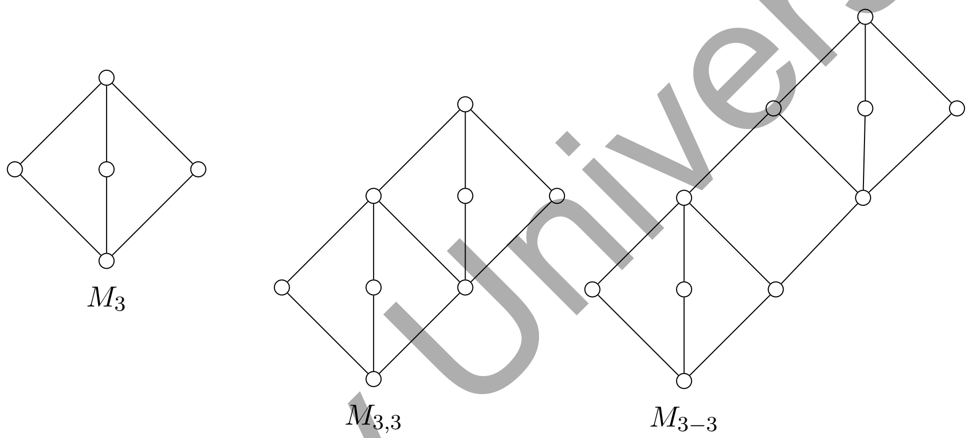

Figure 2: Lattices M_3, $M_{3,3}$ and M_{3-3}

Let L_n^- be a sublattice of L_n generated by the set $\{a_i, b_i, c_i, a^i, b^i, c^i \mid 0 < i \leq n\}$. One can see that $L_n^- \cong L_n/\theta(a_0, b_0)$ and $L_n^- \leq_s M_{3-3}^n$. Hence, $L_n^- \in \mathbf{Q}(A)$.

Claim 1. Every proper sublattice of L_n belongs to $\mathbf{Q}(A)$.

Proof of Claim 1.

It is enough to prove the claim for arbitrary maximal proper sublattices of L_n. Since L_n is generated by the set of double irreducible elements $S = \{a_0, b_0, b^0, c^0, c_n\} \cup \{b_i, b^i \mid 0 < i \leq n\}$ then every maximal proper sublattices L of L_n generated by $S - \{x\}$ for some $x \in S$, that is $L = \langle S - \{x\}\rangle$.

Suppose that $x \in \{a_0, b_0, b^0, c^0\}$. Then the lattice $\langle\{a_0, b_0, b^0, c^0\} \setminus \{x\}\rangle/\theta(c_0, a^0 \wedge b^0)$ be a homomorphic image of L with the kernel $\alpha = \theta(a_1, c_n)$ and belongs to $\mathbf{Q}(A)$.

One can see that for $\beta = \theta(a_0, b_0)$ if $x \in \{b^0, c^0\}$ and $\beta = \theta(b^0, c^0)$ if $x \in \{a_0, b_0\}$, L/β is isomorphic to a sublattice of $L_n^- \times \mathbf{2}$ and belongs to $\mathbf{Q}(A)$. Thus, α and β are $\mathbf{Q}(A)$-congruences. One can check that $\alpha \cap \beta = 0$. Hence $L \leq_s L/\alpha \times L/\beta$. Therefore, $L \in \mathbf{Q}(A)$.

Suppose that $x \in \{b_i, b^i \mid 0 < i \leq n\} \cup \{c_n\}$. Without loss of generality, assume that $x = b_n$. Let $\alpha = \theta(c_0, c_{n-1})$. Then L/α is isomorphic to the sublattice S of L_1 generated by the set $\{a_0, b_0, b^0, c^0, a_1, b_1, b^1\}$. Since the lattice $P = \langle\{a_0, b_0, b^0, c^0, b^1, c^1\}\rangle$ is a sublattice of A and $S \leq_s P \times \mathbf{2}^2$ we get $S \in \mathbf{Q}(A)$. On the other hand, $L/\theta(a_0, b_0)$ is a sublattice of L_n^-. Since $L_n^- \in \mathbf{Q}(A)$ then $L/\theta(a_0, b_0) \in \mathbf{Q}(A)$. One can see that $\alpha \cap \theta(a_0, b_0) = 0$. Hence, L is a subdirect product of two lattices from $\mathbf{Q}(A)$. Therefore, $L \in \mathbf{Q}(A)$.

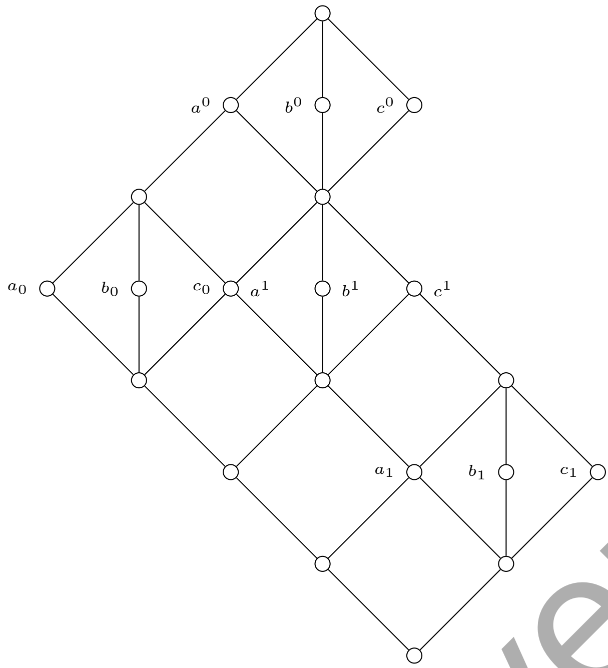

Figure 3: Lattice L_1

Let $\varphi_{n,n-1}$ be a homomorphism from L_n to L_{n-1} such that $\ker \varphi_{n,n-1} = \theta(a^n, b_n)$, and $\varphi_{n,n}$ an identity map for all $n > 1$ and $m < n$. And let $\varphi_{n,m} = \varphi_{m+1,m} \circ \cdots \circ \varphi_{n,n-1}$. It can be seen that $\{L_n; \varphi_{n,m}, N\}$ forms inverse family, where N is the linear ordered set of positive integers.

We denote $L = \varprojlim\{L_n \mid n \in N\}$ and show that $L \in \mathbf{Q}(A)$.

Claim 2. The lattice L belongs to $\mathbf{Q}(A)$.

Proof of Claim 2.

Let α be a quasi-identity of the following form

$$\&_{i \le r} p_i(x_0, \dots, x_{n-1}) \approx q_i(x_0, \dots, x_{n-1}) \to p(x_0, \dots, x_{n-1}) \approx q(x_0, \dots, x_{n-1}).$$

Assume that α is valid on $\mathbf{Q}(A)$ and

$$L \models p_i(a_0, \dots, a_{n-1}) = q_i(a_0, \dots, a_{n-1}) \quad \text{for all} \quad i < r,$$

for some $a_0, \dots, a_{n-1} \in L$. From the definition of inverse limit we have that $L \le_s \prod_{i \in I} L_i$. Therefore

$$L_s \models p_i(a_0(s), \dots, a_{n-1}(s)) = q_i(a_0(s), \dots, a_{n-1}(s)) \quad \text{for all} \quad i < r.$$

Each at most n generated subalgebra of L_s belongs to $\mathbf{Q}(A)$ for all $s > n$, by Claim 1. Hence α is true in L_s for all $s > n$. And this in turn entails

$$L_s \models p(a_0(s), \dots, a_{n-1}(s)) = q(a_0(s), \dots, a_{n-1}(s)).$$

Since $a_i(m) = \varphi_{s,m}(a_i(s))$ for all $0 \le i < n$ and $m < s$, we get

$$L_m \models p(a_0(m), \dots, a_{n-1}(m)) = q(a_0(m), \dots, a_{n-1}(m)) \quad \text{for all} \quad m < s.$$

So

$$L \models p(a_0, \dots, a_{n-1}) = q(a_0, \dots, a_{n-1}).$$

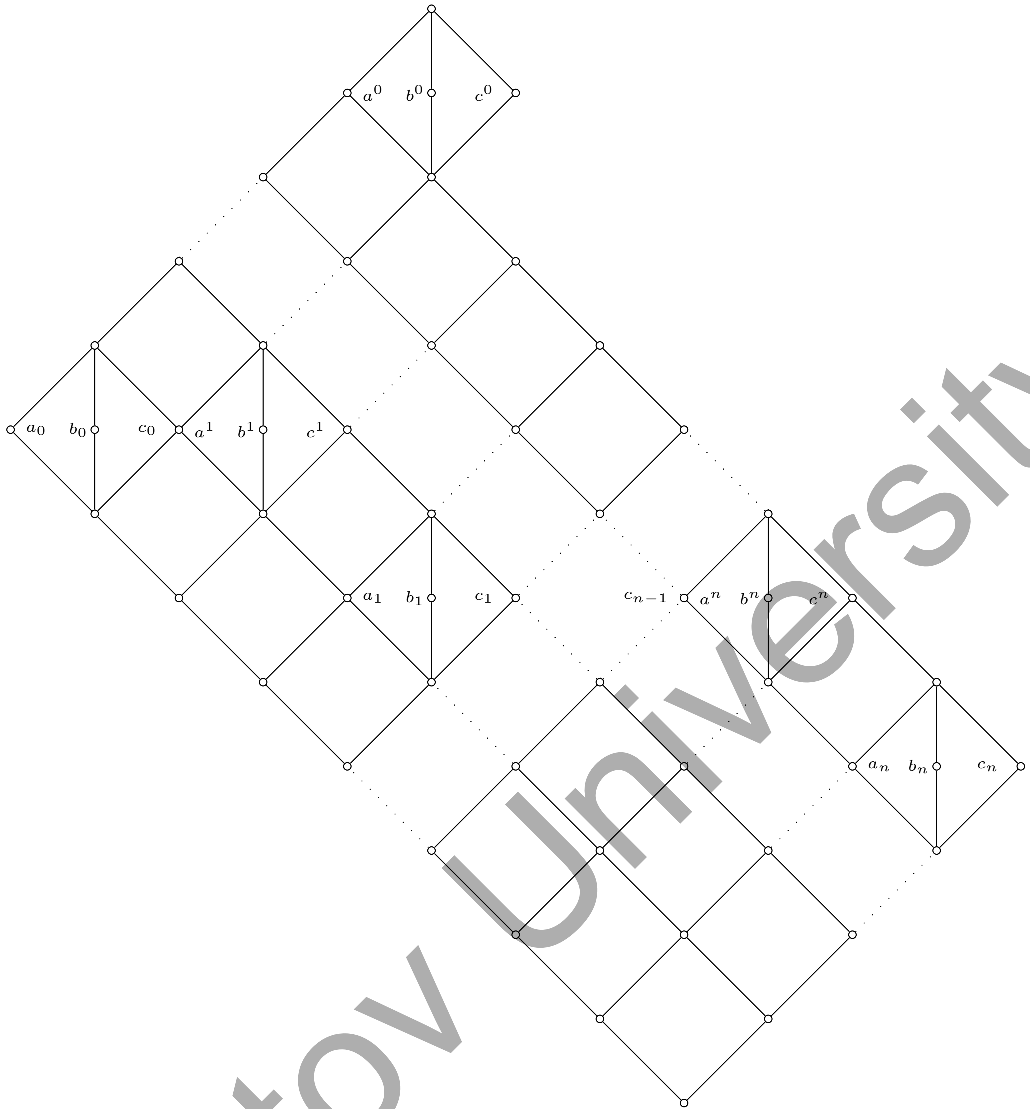

Figure 4: Lattice L_n, $n \geq 2$

Hence $L \models \alpha$, for every α that is valid on $\mathbf{Q}(A)$. This proves that $L \in \mathbf{Q}(A)$.

Claim 3. The lattice L is point-wise separable with respect to $\mathbf{Q}(A)$.

Proof of Claim 3.

We obtain $\varphi_{n,m}(a_0) = a_0$ and $\varphi_{n,m}(b_0) = b_0$, by definition of $\varphi_{n,n-1}$. And $a = (a_0, \ldots, a_0, \ldots)$, $b = (b_0, \ldots, b_0, \ldots) \in L$, by definition of inverse limit. Let $\alpha : L \to M$ be a homomorphism, $M \in \mathbf{Q}(A)$ and M finite. There is $n > 2$ and homomorphism $\psi_M : L_n \to M$ such that $\alpha = \varphi_n \circ \psi_M$ for some surjective homomorphism $\varphi_n : L \to L_n$ (by universal property of inverse limit). It is not difficult to see that any non-trivial homomorphic image of L_n is isomorphic to L_m, $m < n$, or contains $M_{3,3}$ as a sublattice. Since $L_m, M_{3,3} \notin \mathbf{Q}(A)$ and $\psi_M(L_n) \leq M \in \mathbf{Q}(A)$, then we obtain that $\psi_M(L_n)$ is trivial. That is $\psi_M(x) = const$ for all $x \in L_n$. So we get $\alpha(a) = \alpha(b)$.

Thus, the Claims 1–3 state that the conditions of Theorem 1 holds on $\mathbf{Q}(A)$. Therefore, the quasivariety $\mathbf{Q}(A)$ generated by A is not standard, as well as not finitely axiomatizable.

Remark. In the paper [16] it has been proved that the quasivariety generated by the lattice A is not finitely based. We would like to point out that we presented the proof of the Claim 1 for the sake of completeness of the proof of the main result. We also note that Claims 2 and 3 were proved by arguments of [17].

We note that there is an infinite number of lattices similar to the lattice A. This is the context of the following.

Theorem 4. Let L be a finite lattice such that $M_{3,3} \not\leq L$, $A \leq L$ and $L_n \not\leq L$ for all $n > 1$. Then the topological quasivariety generated by the lattice L is not standard, as well as is not finitely axiomatizable.

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Кейбір стандартты емес торлардың квазикөпбейнелері

Квазикөпбейнелердің стандарттылық мәселелерін көптеген авторлар зерттеді. Д.М. Кларк, Б.А. Дэйви, М.Г. Джексон және Дж.Г. Питкетли «Қандай соңғы торлар стандартты топологиялық предкөпбейнені тудырады?» деген мәселені 2008 жылы ұсынды. Тумановтың барлық жағдайларын қанағаттандырмайтын бір нақты модульдік тордың стандарттылық мәселесін зерттеу жалғастырылған. Осы тордан пайда болған топологиялық квазикөпбейне зерттелген және зерттелетін квазикөпбейне стандартты емес, сонымен қатар әрине аксиоматизацияланбайтыны дәлелденген. Сондай-ақ жоғарыда аталған торға ұқсас торлардың шексіз саны бар екені көрсетілген.

Кілт сөздер: тор, квазикөпбейне, квазисәйкестіктердің базисі, профиниттік алгебра, топологиялық квазикөпбейне, профиниттік квазикөпбейне.

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Некоторые нестандартные квазимногообразия решеток

Вопросы стандартности квазимногообразий исследовались многими авторами. Проблема «Какие конечные решетки порождают стандартное топологическое предмногообразие?» была предложена Д.М. Кларком, Б.А. Дэйви, М.Г. Джексоном и Дж.Г. Питкетли в 2008 году. Мы продолжаем изучать проблему стандартности для одной конкретной конечной модулярной решетки, которая не удовлетворяет всем условиям Туманова. Исследуем топологическое квазимногообразие, порожденное этой решеткой, и доказываем, что исследуемое квазимногообразие не является стандартным и конечно аксиоматизируемым. Кроме того, показываем, что существует бесконечное число решеток, подобных упомянутой выше.

Ключевые слова: решетка, квазимногообразие, базис квазитождеств, профинитная алгебра, топологическое квазимногообразие, профинитное квазимногообразие.

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