

INVESTIGATION OF THE MODEL FOR THE ESSENTIALLY LOADED HEAT EQUATION

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The studied problem for the essentially loaded heat equation is connected with mathematical modeling of thermophysical processes in the electric arc of high-current disconnecting devices. Experimental studies of such phenomena are difficult due to their transience, and in some cases only a mathematical model is able to provide adequate information about their dynamics. The study of the mathematical model is carried out when the order of the derivative in the loaded summand is less than, equal to and greater than the order of the differential part of the heat equation, at a fixed point of the load and in the case when the load point moves at a variable speed. The article is focused mainly on scientific researchers engaged in practical applications of loaded differential equations.

Keywords: thermophysical processes, electric arc, loaded heat equation, boundary value problem, reduction to integral equation

Introduction

The theory of boundary value problems for loaded differential parabolic equations is very relevant for the modeling of physical, technical and applied processes, and also in experimental studies conducted in wide various fields of science. The loaded differential equations are used to model processes of different nature: physical, mechanical, chemical, biological, ecological, economic, etc. Such equations are also widely used in solving various engineering and technical problems.

This information capacity of loaded differential equations is due to the fact that they are based on the fundamental laws of nature, such as, for example, conservation laws. Due to this, processes, completely different in nature, can be described by the same form of equations.

Loaded differential equations arise naturally in the study of nonlinear equations, particles transport equations and optimal control problems, in the numerical solution of integral-differential equations, in the equivalent transformation of boundary value problems, etc. [1]

The modern trend in technology to use super-strong and super-weak currents in many electrical devices leads to the need to study phenomena outside the usual current range. When switching electrical devices or overvoltages an electric arc may appear in the circuit between the current-carrying parts (Fig.1).

Electrical safety has paramount importance for the maintenance of any effective and productive equipment, and one of the most serious threats to security is precisely the electric arc and arc flash. Situations where an electric arc is created in an uncontrolled environment, such as an arc flash, can cause injury, fire, and equipment damage.

In a number of devices the electric arc phenomenon is harmful and especially dangerous. These are, first of all, contact switching devices used in power supply and electric drive: high-voltage switches, circuit breakers, contactors, sectional insulators on the contact network of electrified railways and urban electric transport. When disconnecting the loads of the above mentioned devices, an arc arises between the opening contacts (Fig.2).

Electric arcs can have useful technological purposes when they are used correctly. For example, electric arcs are used in camera flashes, in spotlights for stage lighting, for fluorescent lighting, for arc welding, in arc furnaces (for the production of steel and substances such as calcium

carbide), in plasma cutters (in which compressed air is combined with a powerful arc and is converted into plasma, which has the ability to instantly cut steel), etc.

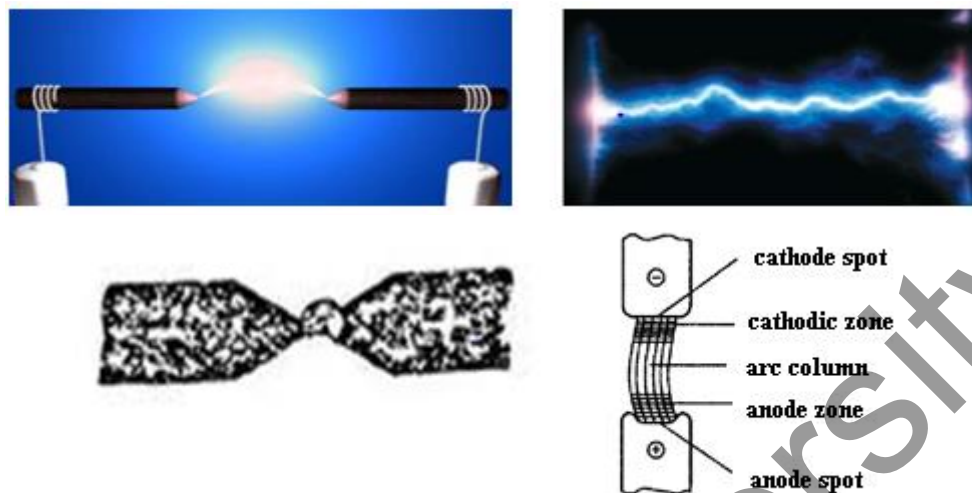


Fig.1. Electric arc



Fig.2. Emergence of electric arc

1. Formulation of the boundary value problem

In the domain $\Omega = \{(x, t) : x \in (0, \infty); t \in (0, \infty)\}$ the essentially loaded heat equation is investigated

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - \frac{1-2\beta}{x} \cdot \frac{\partial u}{\partial x} - \lambda \cdot \frac{\partial^k u}{\partial x^k} \Big|_{x=\bar{x}(t)} = f(x, t) \quad (1)$$

where $u = u(x, t)$ is an unknown function. β , λ и k are numerical parameters, and $0 < \beta < 1$,

$\lambda \in C$, $k = 0, 1, 2, \dots$ $\lambda \cdot \frac{\partial^k u}{\partial x^k} \Big|_{x=\bar{x}(t)}$ is a loaded summand, $f(x, t)$ is a known function defined in the

domain Ω . The load point moves according to a given law $x = \bar{x}(t)$ for $t \in (0, \infty)$, and the initial condition

$$u(x, 0) = g(x), \quad (2)$$

the boundary condition

$$u(0, t) = h(t) \quad (3)$$

are given. Functions $g(x)$ and $h(t)$ are given at $x \in (0, \infty)$ and $t \in (0, \infty)$ respectively.

As it was noted, when using the basic laws of conservation (energy, mass, etc.), mathematical modeling of various physical processes often leads to the same equations, in particular, to equations of parabolic type. Parabolic equations are most often encountered in the study of problems related to the propagation of heat, both in limited and in unlimited bodies, in the temperature problems of mechanics and physics, in problems of building heat engineering, in the task about the adiabatic filtration of gases and liquids in porous media, etc. Equations of parabolic type are also used in many technical issues, for example, in the study of the influence of the temperature field on the deformation of the railway rail [2]. At the same time, the heat equations are especially frequent.

Thermophysical processes in the electric arc of high-current disconnecting devices (Fig. 2) are described by the physical model, the mathematical interpretation of which is the studied problem (1) - (3). A tool for describing thermal processes in an arc is the heat equation (1).

As previously indicated, when an electrical circuit is disconnected, an electrical discharge occurs in the form of an electric arc. Experimental studies of such phenomena are complex and burdensome due to their temporal short duration, therefore, in many cases, only a mathematical model can give adequate information about their dynamics, so the problem under study is relevant in modern natural science.

2. The solution of the boundary value problem

The solution of the boundary value problem (1) - (3) provided $0 < \beta < 1$ has the form [3]

$$\begin{aligned} u(x, t) = & \frac{1}{2} \cdot \int_0^t \int_0^\infty \left[f(\xi, \tau) + \lambda \cdot \frac{\partial^k u}{\partial \xi^k} \Big|_{\xi=\bar{x}(\tau)} \right] \cdot \frac{x^\beta \cdot \xi^{1-\beta}}{t-\tau} \cdot \exp\left(-\frac{x^2 + \xi^2}{4(t-\tau)}\right) \cdot I_\beta\left(\frac{\xi \cdot x}{2(t-\tau)}\right) \cdot d\xi \cdot d\tau + \\ & + \frac{x^\beta}{2t} \cdot \int_0^\infty g(\xi) \cdot \xi^{1-\beta} \cdot \exp\left(-\frac{x^2 + \xi^2}{2t}\right) \cdot I_\beta\left(\frac{\xi \cdot x}{2t}\right) \cdot d\xi + \\ & + \frac{x^{2\beta}}{2^{2\beta+1} \cdot \Gamma(\beta+1)} \cdot \int_0^t h(\tau) \cdot \exp\left(-\frac{x^2}{4(t-\tau)}\right) \cdot \frac{d\tau}{(t-\tau)^{1+\beta}} \end{aligned} \quad (4)$$

or

$$u(x, t) = \lambda \cdot \int_0^t \int_0^\infty \frac{x^\beta \cdot \xi^{1-\beta}}{2(t-\tau)} \cdot \exp\left(-\frac{x^2 + \xi^2}{4(t-\tau)}\right) \cdot I_\beta\left(\frac{\xi \cdot x}{2(t-\tau)}\right) \cdot \frac{\partial^k u}{\partial \xi^k} \Big|_{\xi=\bar{x}(\tau)} \cdot d\xi \cdot d\tau + \bar{F}(x, t), \quad (5)$$

$$\bar{F}(x, t) = F_1(x, t) + F_2(x, t) + F_3(x, t),$$

$$F_1(x, t) = \frac{1}{2} \cdot \int_0^t \int_0^\infty f(\xi, \tau) \cdot \frac{x^\beta \cdot \xi^{1-\beta}}{t-\tau} \cdot \exp\left(-\frac{x^2 + \xi^2}{4(t-\tau)}\right) \cdot I_\beta\left(\frac{\xi \cdot x}{2(t-\tau)}\right) \cdot d\xi \cdot d\tau,$$

$$F_2(x, t) = \frac{x^\beta}{2t} \cdot \int_0^\infty g(\xi) \cdot \xi^{1-\beta} \cdot \exp\left(-\frac{x^2 + \xi^2}{2t}\right) \cdot I_\beta\left(\frac{\xi \cdot x}{2t}\right) \cdot d\xi,$$

$$F_3(x, t) = \frac{x^{2\beta}}{2^{2\beta+1} \cdot \Gamma(\beta+1)} \cdot \int_0^t h(\tau) \cdot \exp\left(-\frac{x^2}{4(t-\tau)}\right) \cdot \frac{d\tau}{(t-\tau)^{1+\beta}}.$$

3. Reduction of the boundary value problem to the integral equation

The equation (5) can be represented as an integral Volterra equation of the second kind. To do this, the equation (5) can be written as

$$u(x, t) = \lambda \cdot \int_0^t Q(x, t-\tau) \cdot \left. \frac{\partial^k u}{\partial \xi^k} \right|_{\xi=\bar{x}(\tau)} d\tau + \bar{F}(x, t). \quad (6)$$

$$Q(x, t-\tau) = \frac{x^\beta}{2(t-\tau)} \cdot \exp\left(-\frac{x^2}{4(t-\tau)}\right) \cdot P(x, t-\tau).$$

We differentiate (6) by variable x k times and substitute instead $x = \bar{x}(t)$. As a result, we get

$$\mu(t) - \lambda \cdot \int_0^t K(t, \tau) \cdot \mu(\tau) \cdot d\tau = F(t), \quad (7)$$

$$\mu(t) = \left. \frac{\partial^k u}{\partial x^k} \right|_{x=\bar{x}(t)}, \quad K(t, \tau) = \left. \frac{\partial^k Q(x, t-\tau)}{\partial x^k} \right|_{x=\bar{x}(t)}, \quad F(t) = \left. \frac{\partial^k \bar{F}(x, t)}{\partial x^k} \right|_{x=\bar{x}(t)}. \quad (8)$$

The relation (7) with (8) is a Volterra integral equation of the second kind. Thus, the solving the stated boundary value problem was reduced to the solving the integral equation (7).

4. Properties of the function $Q(x, t-\tau)$

The $Q(x, t-\tau)$ function defines the kernel of the integral equation (7). As it is known, the properties of the kernel play an important role in the question of solvability of the integral equation and dictate the methods of investigation of the integral equation, so a study of the properties of the function $Q(x, t-\tau)$ was carried out, the results of this study are presented in [4].

We list some properties of the function $Q(x, t-\tau)$, necessary for our research.

- 1) The function $Q(x, t-\tau)$, $0 < \tau < t < \infty$, is continuous.
- 2) The function $Q(x, t-\tau) \geq 0$, $0 < \tau < t < \infty$.
- 3) The function $Q(x, t-\tau)$ can be represented as

$$Q(x, t-\tau) = \frac{1}{\Gamma(\beta)} \cdot \gamma\left(\beta, \frac{x^2}{4(t-\tau)}\right), \quad (9)$$

where $\Gamma(\beta)$ is gamma function, $\gamma(v, x)$ is incomplete gamma function [5].

- 4) There is a relationship for the function $Q(x, t-\tau)$

$$\int_0^t Q(x, t-\tau) d\tau = \frac{1}{\Gamma(\beta)} \cdot \left[t \cdot \gamma\left(\beta, \frac{x^2}{4t}\right) + \frac{x^2}{4} \cdot \Gamma\left(\beta-1, \frac{x^2}{4t}\right) \right]. \quad (10)$$

4. Solvability of the integral equation

In the case when the load point is fixed: $\bar{x}(t) = x_0$, where $x_0 \in R_+$, we have the following theorem on the solvability of the integral equation (7).

Theorem 1. *If for each fixed value $k = 0, 1, 2, \dots$ with $\bar{x}(t) = x_0$, where $x_0 \in R_+$, and the function $F(t) \in C(0, \infty)$, then the integral equation (7) has a unique continuous solution peuehue [6].*

We investigate the solvability of the integral equation (7) in the case where the load point moves at a variable speed: $\bar{x}(t) = t^\omega$, $\omega \in R$.

Let $k = 0$, then the integral equation (7) takes the form

$$\begin{aligned} \tilde{\mu}_0(t) - \lambda \cdot \int_0^t \tilde{K}_0(t, \tau) \cdot \tilde{\mu}_0(\tau) \cdot d\tau &= \tilde{F}_0(t), \\ \tilde{\mu}_0(t) = u(x, t)|_{x=t^\omega}, \quad \tilde{K}_0(t, \tau) &= Q(x, t - \tau)|_{x=t^\omega}, \quad \tilde{F}_0(t) = \bar{F}(x, t)|_{x=t^\omega}. \end{aligned} \quad (11)$$

When using for the function $Q(x, t - \tau)$ of the representation (9), the kernel of the integral equation (11) is determined by the expression

$$\tilde{K}_0(t, \tau) = \frac{1}{\Gamma(\beta)} \cdot \gamma\left(\beta, \frac{t^{2\omega}}{4(t-\tau)}\right).$$

We calculate the integral

$$\int_0^t \tilde{K}_0(t, \tau) d\tau = \frac{1}{\Gamma(\beta)} \cdot \int_0^t \gamma\left(\beta, \frac{t^{2\omega}}{4(t-\tau)}\right) \cdot d\tau = \frac{1}{\Gamma(\beta)} \cdot \int_0^t d\tau \int_0^{\frac{t^{2\omega}}{4(t-\tau)}} e^{-\xi} \cdot \xi^{\beta-1} d\xi.$$

In the last a ratio producing the replacement of the integration order and the necessary calculations, we obtain the relation

$$\begin{aligned} \int_0^t \tilde{K}_0(t, \tau) d\tau &= \frac{1}{\Gamma(\beta)} \cdot \left[\int_0^{\frac{t^{2\omega-1}}{4}} e^{-\xi} \cdot \xi^{\beta-1} d\xi \int_0^t d\tau + \int_{\frac{t^{2\omega-1}}{4}}^\infty e^{-\xi} \cdot \xi^{\beta-1} d\xi \int_{\frac{t^{2\omega}}{4\xi}}^t d\tau \right] = \\ &= \frac{1}{\Gamma(\beta)} \cdot \left[t \cdot \gamma\left(\beta, \frac{t^{2\omega-1}}{4}\right) + \frac{t^{2\omega}}{4} \cdot \Gamma\left(\beta - 1, \frac{t^{2\omega-1}}{4}\right) \right]. \end{aligned} \quad (12)$$

In (12) the limiting transition as $t \rightarrow 0$ gives that

$$\lim_{t \rightarrow 0} \int_0^t \tilde{K}_0(t, \tau) d\tau = \lim_{t \rightarrow 0} \frac{1}{\Gamma(\beta)} \cdot \left[t \cdot \gamma\left(\beta, \frac{t^{2\omega-1}}{4}\right) + \frac{t^{2\omega}}{4} \cdot \Gamma\left(\beta - 1, \frac{t^{2\omega-1}}{4}\right) \right] =$$

$$= \lim_{t \rightarrow 0} \left[t \cdot \frac{\gamma\left(\beta, \frac{t^{2\omega-1}}{4}\right)}{\Gamma(\beta)} + \frac{t^2}{4} \cdot \frac{\Gamma\left(\beta-1, \frac{t^{2\omega-1}}{4}\right)}{\Gamma(\beta)} \right] = 0. \tag{13}$$

Based on (13) [7], we obtain the following statement about the solvability of the integral equation (11): *if $\tilde{F}_0(t) \in C(0, \infty)$, then the integral equation (11) has a unique continuous solution for any values λ .*

In the case $k = 1$, $\bar{x}(t) = t^\omega$, $\omega \in R$, the integral equation (7) is determined by the expression

$$\begin{aligned} \tilde{\mu}_1(t) - \lambda \cdot \int_0^t \tilde{K}_1(t, \tau) \cdot \tilde{\mu}_1(\tau) \cdot d\tau &= \tilde{F}_1(t), \\ \tilde{\mu}_1(t) = \frac{\partial u}{\partial x} \Big|_{x=t^\omega}, \quad \tilde{F}_1(t) &= \frac{\partial \bar{F}(x, t)}{\partial x} \Big|_{x=t^\omega}, \\ \tilde{K}_1(t, \tau) = \frac{\partial Q(x, t - \tau)}{\partial x} \Big|_{x=t^\omega} &= \frac{1}{\Gamma(\beta) \cdot 2^{2\beta-1}} \cdot \frac{t^{(2\beta-1)\omega}}{(t - \tau)^\beta} \cdot \exp\left(-\frac{t^{2\omega}}{4(t - \tau)}\right). \end{aligned} \tag{14}$$

The kernel $\tilde{K}_1(t, \tau)$ has a singularity of order β , where $0 < \beta < 1$ [7], therefore, we formulate the following statement about the solvability of integral equation (14): *if $\tilde{F}_1(t) \in C(0, \infty)$, then the integral equation (14) has a unique continuous solution for any values λ .*

For $k = 2$, $\bar{x}(t) = t^\omega$, $\omega \in R$, the integral equation (7) is written as follows

$$\begin{aligned} \tilde{\mu}_2(t) - \lambda \cdot \int_0^t \tilde{K}_2(t, \tau) \cdot \tilde{\mu}_2(\tau) \cdot d\tau &= \tilde{F}_2(t), \\ \tilde{\mu}_2(t) = \frac{\partial^2 u}{\partial x^2} \Big|_{x=t^\omega}, \quad \tilde{K}_2(t, \tau) = \frac{\partial^2 Q(x, t - \tau)}{\partial x^2} \Big|_{x=t^\omega}, \quad \tilde{F}_2(t) &= \frac{\partial^2 \bar{F}(x, t)}{\partial x^2} \Big|_{x=t^\omega}. \end{aligned} \tag{15}$$

Taking into account (9) and considering that $x = \bar{x}(t) = t^\omega$ we find the kernel $\tilde{K}_2(t, \tau)$ of the integral equation (15) in the form

$$\begin{aligned} \tilde{K}_2(t, \tau) = \frac{\partial^2 Q(x, t - \tau)}{\partial x^2} \Big|_{x=t^\omega} &= \frac{1}{\Gamma(\beta) \cdot 2^{2\beta-1}} \cdot \left[\frac{(2\beta-1) \cdot t^{(2\beta-2)\omega}}{(t - \tau)^\beta} \cdot \exp\left(-\frac{t^{2\omega}}{4(t - \tau)}\right) - \right. \\ &\left. - \frac{t^{2\beta\omega}}{2(t - \tau)^{\beta+1}} \cdot \exp\left(-\frac{t^{2\omega}}{4(t - \tau)}\right) \right]. \end{aligned}$$

We calculate the integral

$$\int_0^t \tilde{K}_2(t, \tau) d\tau = \frac{2\beta-1}{2 \cdot \Gamma(\beta)} \cdot \Gamma\left(\beta-1, \frac{t^{2\omega-1}}{4}\right) - \frac{1}{\Gamma(\beta)} \cdot \Gamma\left(\beta, \frac{t^{2\omega-1}}{4}\right). \tag{16}$$

Making the limiting transition as $t \rightarrow 0$ in the relation (16), we find that

$$\begin{aligned}
 \lim_{t \rightarrow 0} \int_0^t \tilde{K}_2(t, \tau) d\tau &= \lim_{t \rightarrow 0} \left[\frac{2\beta - 1}{2} \cdot \frac{\Gamma\left(\beta - 1, \frac{t^{2\omega-1}}{4}\right)}{\Gamma(\beta)} - \frac{\Gamma\left(\beta, \frac{t^{2\omega-1}}{4}\right)}{\Gamma(\beta)} \right] = \\
 &= \lim_{t \rightarrow 0} \left[\frac{2\beta - 1}{2(\beta - 1)} \cdot \frac{\Gamma\left(\beta - 1, \frac{t^{2\omega-1}}{4}\right)}{\Gamma(\beta - 1)} - \frac{\Gamma\left(\beta, \frac{t^{2\omega-1}}{4}\right)}{\Gamma(\beta)} \right]. \quad (17)
 \end{aligned}$$

By performing (17) calculations for different values of the parameter ω , we receive

$$\lim_{t \rightarrow 0} \int_0^t \tilde{K}_2(t, \tau) d\tau = \begin{cases} 0, & \omega < \frac{1}{2}, \\ \frac{2\beta - 1}{2 \cdot \Gamma(\beta)} \cdot \Gamma\left(\beta - 1, \frac{1}{4}\right) - \frac{1}{\Gamma(\beta)} \cdot \Gamma\left(\beta, \frac{1}{4}\right), & \omega = \frac{1}{2}, \\ \frac{1}{2(\beta - 1)}, & \omega > \frac{1}{2}. \end{cases}$$

Thus, the kernel $K_2(t, \tau)$ of the integral equation (15) has the property $\lim_{t \rightarrow 0} \int_0^t \tilde{K}_2(t, \tau) d\tau = 0$

only in the case $\omega < \frac{1}{2}$, as we were convinced by the direct calculation, so we formulate a statement similar to the previous: *if $F_2(t) \in C(0, \infty)$ and $\omega < \frac{1}{2}$, then the integral equation (15) has a unique continuous solution for any values λ .*

Since $\lim_{t \rightarrow 0} \int_0^t \tilde{K}_2(t, \tau) d\tau \neq 0$ in the case $\omega \geq \frac{1}{2}$, then the integral equation (15) is a singular integral equation of Volterra. The class of such integral equations is extremely wide and diverse and cannot be unambiguously classified. As a rule, each particular singular integral equation requires a specific study. Methods for studying the solvability and spectral problems of singular integral equations of Volterra are presented in [7].

Based on the statements, we formulate the following theorem.

Theorem 2. *If the function $F(t) \in C(0, \infty)$, $\bar{x}(t) = t^\omega$, where $\omega \in \mathbb{R}$, then for $k = 0$ and $k = 1$ with any values of ω , and for $k = 2$ with $\omega < \frac{1}{2}$, the integral equation (7) with (8) has a unique continuous solution for any values λ .*

5. Solvability of the boundary value problem

Since according to Theorem 1 the integral equation (7), to which the boundary value problem (1) - (3) is reduced, has a unique continuous solution, if $F(t) \in C(0, \infty)$, then we have the following solvability theorem for the stated boundary value problem in the case of a fixed point of load.

Theorem 3. If $f \in C_{x,t}^{k-1,1}(\Omega)$, where $\Omega = \{(x,t) : x \in (0,\infty); t \in (0,\infty)\}$, then the boundary value problem (1) – (3) for essentially loaded heat equation has a unique solution $u(x,t) \in C_{x,t}^{k,1}(\Omega)$ in the form (4) when $\bar{x}(t) = x_0$, $x_0 \in R_+$ for each fixed value $k = 0, 1, 2, \dots$ [6].

In the case when the load point moves with a variable speed: $\bar{x}(t) = t^\omega$, $\omega \in R$, from Theorem 2 the following theorem on the solvability of the boundary value problem (1) – (3) is obtained.

Theorem 4. If $f \in C_{x,t}^{k-1,1}(\Omega)$, where $\Omega = \{(x,t) : x \in (0,\infty); t \in (0,\infty)\}$, then the boundary value problem (1) – (3) for essentially loaded heat equation has a unique solution $u(x,t) \in C_{x,t}^{k,1}(\Omega)$ in the form (4) when $\bar{x}(t) = t^\omega$, where $\omega \in R$, for $k = 0$ and $k = 1$ with any values ω , and for $k = 2$ with $\omega < \frac{1}{2}$.

Conclusion

Free-burning arcs are the most intense and high-temperature sources of heat radiation. With an increase in the magnetic field strength and the arc current, the speed of its movement increases and the heat transfer of the arc column with the environment is intensified. This leads to a change in the position of the current-voltage characteristics of the electric arc discharge.

For the studied boundary-value problem (1) - (3), the solvability questions are defined and the solutions are obtained in the form (4) when the order of the derivative in the loaded summand $k = 0, 1, 2, \dots$, that is, is less than, is equal to and greater than the order of the differential part of the heat equation, at a fixed point of load $\bar{x}(t) = x_0$, $x_0 \in R_+$, and in the case when the load point moves with variable speed $\bar{x}(t) = t^\omega$, where $\omega \in R$.

From solution (4) it is clear that the temperature of the arc column is very non-uniform in length and in diameter. Note that the temperature along the cross section of the arc column is also unevenly distributed. It has a maximum on the axis of the column and goes down to its periphery.

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