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Summation of some infinite series by the methods of Hypergeometric functions and partial fractions

In this article, we obtain the summations of some infinite series by partial fraction method and by using certain hypergeometric summation theorems of positive and negative unit arguments, Riemann Zeta functions, polygamma functions, lower case beta functions of one-variable and other associated functions. We also obtain some hypergeometric summation theorems for:

$${}_8F_7\left[\tfrac{9}{2},\tfrac{3}{2},\tfrac{3}{2},\tfrac{3}{2},\tfrac{3}{2},3,3,1;\tfrac{7}{2},\tfrac{7}{2},\tfrac{7}{2},\tfrac{7}{2},\tfrac{1}{2},2,2;1\right],\ {}_5F_4\left[\tfrac{5}{3},\tfrac{4}{3},\tfrac{4}{3},\tfrac{1}{3},\tfrac{1}{3};\tfrac{2}{3},1,2,2;1\right],\ {}_5F_4\left[\tfrac{9}{4},\tfrac{5}{2},\tfrac{3}{2},\tfrac{1}{2},\tfrac{1}{2};\tfrac{5}{4},2,3,3;1\right]$$

$${}_5F_4\left[\tfrac{13}{8},\tfrac{5}{4},\tfrac{5}{4},\tfrac{1}{4},\tfrac{1}{4};\tfrac{5}{8},2,2,1;1\right],\quad {}_5F_4\left[\tfrac{1}{2},\tfrac{1}{2},\tfrac{5}{2},\tfrac{5}{2},1;\tfrac{3}{2},\tfrac{3}{2},\tfrac{7}{2},\tfrac{7}{2};-1\right],\ {}_4F_3\left[\tfrac{3}{2},\tfrac{3}{2},1,1;\tfrac{5}{2},\tfrac{5}{2},2;1\right],$$

$${}_4F_3\left[\tfrac{2}{3},\tfrac{1}{3},1,1;\tfrac{7}{3},\tfrac{5}{3},2;1\right],\ {}_4F_3\left[\tfrac{7}{6},\tfrac{5}{6},1,1;\tfrac{13}{6},\tfrac{11}{6},2;1\right]\ \text{and}\ {}_4F_3\left[1,1,1,1;3,3,3;-1\right].$$

Keywords: Riemann Zeta functions, Polygamma functions, Dougall's theorem, Bernoulli polynomials, Catalan's constant.

Introduction and preliminaries

In this paper, we shall use the following standard notations:

$$\mathbb{N} := \{1,2,3,\cdots\};\quad \mathbb{N}_0 := \mathbb{N}\bigcup\{0\};\quad \mathbb{Z}_0^- := \mathbb{Z}^-\bigcup\{0\} = \{0,-1,-2,-3,\cdots\}.$$

The symbols $\mathbb{C}$, $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}^+$ and $\mathbb{R}^-$ denote the sets of complex numbers, real numbers, natural numbers, integers, positive and negative real numbers respectively.

The classical Pochhammer symbol $(\alpha)_p$ $(\alpha, p \in \mathbb{C})$ is defined by ([1; 22, Eq.(1), p.32, Q.N.(8) and Q.N.(9)], see also [2; 23, Eq.(22) and Eq.(23)]).

A natural generalization of the Gaussian hypergeometric series ${}_2F_1[\alpha,\beta;\gamma;z]$ is accomplished by introducing any arbitrary number of numerator and denominator parameters [2; 42, Eq.(1)].

The Riemann Zeta function $\zeta(z)$([3; 19, 4; 1037]) is defined as:

$$\zeta(z) = \sum_{k=1}^{\infty}\frac{1}{k^z};\quad \mathfrak{Re}(z) > 1,$$

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k^z} = (2^{1-z}-1)\zeta(z);\qquad \mathfrak{Re}(z) > 0.,$$

The Catalan constant is defined as:

$$\mathbf{G} = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2} = {}_3F_2\left[\begin{array}{c} 1,\ \frac{1}{2},\ \frac{1}{2}; \\ \\ \frac{3}{2},\ \frac{3}{2}; \end{array} -1\right] = 0.9159655942...$$

The logarithmic derivative of the Gamma function also known as psi function or Digamma function ([1; 10, Eq.(1)], [5; 24, Eq.(2)], [6; 12, Eq.(1)]), is defined as:

$$\psi(z) = \frac{d}{dz}\ell n\{\Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)};\quad z \neq 0,-1,-2,-3,...$$

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$$\psi(z) = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \frac{z}{n(z+n)}; \quad z \neq 0, -1, -2, -3, ...,$$

$$\psi(z) = -\gamma - \sum_{n=0}^{\infty} \left\{ \frac{1}{(z+n)} - \frac{1}{(n+1)} \right\}; \quad z \neq 0, -1, -2, -3, ...,$$

where γ is Euler-Mascheroni constant and $\gamma \cong 0.5772156649015328606065 12....$

$$\psi(1) = -\gamma, \quad \psi\left(\frac{2}{3}\right) = -\gamma + \frac{\pi\sqrt{3}}{6} - \frac{3}{2}\ell n\ 3, \quad \psi\left(\frac{3}{2}\right) = 2 - 2\ \ell n\ 2 - \gamma, \tag{1}$$

$$\psi\left(\frac{5}{6}\right) = -\gamma + \frac{\pi\sqrt{3}}{2} - \frac{3}{2}\ell n\ 3 - 2\ell n\ 2, \quad \psi\left(\frac{7}{6}\right) = 6 - \gamma - \frac{\pi\sqrt{3}}{2} - \frac{3}{2}\ \ell n\ 3 - 2\ \ell n\ 2. \tag{2}$$

$$\psi^{(1)}\left(\frac{3}{2}\right) = \frac{\pi^2}{2} - 4, \quad \psi^{(1)}\left(\frac{5}{2}\right) = \frac{\pi^2}{2} - 4.4,$$

$$\psi^{(2)}\left(\frac{3}{2}\right) = -\frac{14\pi^3}{25.79436} + 16, \quad \psi^{(2)}\left(\frac{5}{2}\right) = -\frac{14\pi^3}{25.79436} + \frac{448}{27}.$$

The polygamma function $\psi^{(n)}(z)$ ([5; 33, Eq.(52), Eq.(53), p.34, Eq.(58)], see also ([7; 260, Eq.(6.4.10), Eq.(6.4.4)], [8; 45, Eq.(9)], [3; 15]), is defined as:

$$\psi^{(n)}(z) = \frac{d^{n+1}}{d\ z^{n+1}}\ell n\,(\Gamma(z)) = \frac{d^n}{d\ z^n}\psi(z); \quad n \in \mathbb{N}_0, \ z \neq 0, -1, -2, ...$$

$$\psi^{(n)}(z) = (-1)^{n+1}\ n! \sum_{k=0}^{\infty} \frac{1}{(z+k)^{n+1}}; \quad n \in \mathbb{N}, \ z \neq 0, -1, -2, ...$$

Lower case beta function of one variable:

$$\beta(z) = \frac{1}{2}\left[\psi\left(\frac{z+1}{2}\right) - \psi\left(\frac{z}{2}\right)\right] = \frac{G(z)}{2}, \ z \neq 0, -1, -2, -3, ...$$

$$\beta(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(z+k)} = \frac{1}{z}\ {}_2F_1\left[\begin{array}{c} 1, \ z; \\ \\ 1+z; \end{array} -1\right], \quad z \neq 0, -1, -2, -3, ...$$

$$\beta^{(n)}(z) = \frac{d^n}{dz^n}\beta(z) = (-1)^n\ n! \sum_{k=0}^{\infty} \frac{(-1)^k}{(z+k)^{n+1}}; \qquad -z \in \mathbb{N}_0.$$

$$\beta(1) = \ell n\ 2, \ \beta^{(1)}(1) = -\frac{\pi^2}{12}, \quad \beta(2) = 1 - \ell n\ 2, \ \beta^{(1)}(2) = \frac{\pi^2}{12} - 1, \tag{3}$$

$$\beta\left(\frac{1}{2}\right) = \frac{\pi}{2}, \ \beta^{(1)}\left(\frac{1}{2}\right) = -4\mathbf{G}, \ \beta\left(\frac{3}{2}\right) = \frac{4-\pi}{2}, \ \beta^{(1)}\left(\frac{3}{2}\right) = 4\mathbf{G} - 4, \tag{4}$$

$$\beta\left(\frac{5}{2}\right) = \frac{\pi}{2} - \frac{4}{3}, \ \beta^{(1)}\left(\frac{5}{2}\right) = -4\mathbf{G} + \frac{32}{9}, \quad \beta^{(2)}(1) = \frac{3\pi^3}{51.58872}, \quad \beta^{(2)}(2) = 2 - \frac{3\pi^3}{51.58872}. \tag{5}$$

Some hypergeometric summation theorems in terms of Digamma $\psi(b)$, trigamma $\psi^{(1)}(b)$, tetragamma $\psi^{(2)}(b)$ functions and derivatives of lower case beta function of one-variable are given below ... [9; 489, Entry (7.3.6.(9))]

$${}_2F_1\left[\begin{array}{c} 1, \quad a; \\ \\ a+1; \end{array} -1\right] = a\beta(a); \quad 1+a \in \mathbb{C}\backslash\mathbb{Z}_0^-. \tag{6}$$

See ref. [9; 536, Entry (7.4.4.(33))]

$${}_3F_2\left[\begin{array}{c} 1, \quad a, \quad b; \\ \\ 1+a, \ 1+b; \end{array} 1\right] = \frac{ab}{(b-a)}\left[\psi(b) - \psi(a)\right], \tag{7}$$

where $1+a, \ 1+b \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $b \neq a$.

See ref. [9; 536, Entry (7.4.4.(34))]

$$ {}_3F_2\left[\begin{array}{c} 1, \quad b, \quad b; \\ \\ b+1, \ b+1; \end{array} 1 \right] = b^2 \ \psi^{(1)}(b), \tag{8}$$

where $1+b \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $b=a$.

See ref. [9; 546, Entry (7.4.5.(5))]

$$ {}_3F_2\left[\begin{array}{c} 1, \quad a, \quad a; \\ \\ a+1, \ a+1; \end{array} -1 \right] = -a^2 \ \beta^{(1)}(a), \tag{9}$$

where $1+a \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $b=a$.

See ref. [9; 554, Entry (7.5.3.(3))]

$$ {}_4F_3\left[\begin{array}{c} 1, \quad a, \quad b, \quad c; \\ \\ 1+a, 1+b, 1+c; \end{array} 1 \right] = -abc\left[\frac{\psi(a)}{(b-a)(c-a)} + \frac{\psi(b)}{(a-b)(c-b)} + \frac{\psi(c)}{(a-c)(b-c)}\right], \tag{10}$$

where $1+a, \ 1+b, \ 1+c \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $a \neq b, b \neq c, a \neq c$.

See ref. [9; 554, Entry (7.5.3.(5))]

$$ {}_4F_3\left[\begin{array}{c} 1, \quad b, \quad b, \quad b; \\ \\ b+1, \ b+1, \ b+1; \end{array} 1 \right] = \frac{-b^3}{2} \ \psi^{(2)}(b), \tag{11}$$

where $1+b \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $a=b=c$.

See ref. [9; 561, Entry (7.5.4.(5))]

$$ {}_4F_3\left[\begin{array}{c} 1, \quad a, \quad a, \quad a; \\ \\ a+1, \ a+1, \ a+1; \end{array} -1 \right] = \frac{a^3}{2} \ \beta^{(2)}(a), \tag{12}$$

where $1+a \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $a=b=c$.

Gauss' classical summation theorem [1; 49, Th.(18)] in terms of Gamma function is given by:

$$ {}_2F_1\left[\begin{array}{c} \alpha, \quad \beta; \\ \\ \gamma; \end{array} 1 \right] = \frac{\Gamma(\gamma)\ \Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}, \tag{13}$$

where $\mathfrak{Re}(\gamma-\alpha-\beta)>0$ and $\gamma \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Dougall's theorem ([10; 71, Eq.(2.2.10), p.147, Entry(3.5.2)],[11],[9; 564, Entry(7.6.2(3))], [12; 56, Eq.(2.3.4.5), p.244, Entry(III.12)]), see also [13; 27, Eq.(4.4(1))] in terms of Gamma function is given as:

$$ {}_5F_4\left[\begin{array}{c} a, \quad 1+\frac{a}{2}, \quad b, \quad c, \quad d; \\ \\ \frac{a}{2}, 1+a-b, 1+a-c, 1+a-d; \end{array} 1 \right] = \frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-b-c-d)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)}, \tag{14}$$

provided $\mathfrak{Re}(a-b-c-d)>-1$ and $\frac{a}{2}, \ 1+a-b, \ 1+a-c, \ 1+a-d \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

The present article is organized as follows. In section 2, we have shown that the difference of two divergent series may be convergent. In section 3, we have obtained the summation of some infinite series whose general terms are rational functions of n, by using some summation theorems of positive and negative unit arguments and section 4 is related to the hypergeometrical representations of the involved infinite series.

The difference of two divergent series

Consider the two positive terms infinite series $\sum_{n=0}^{\infty} \frac{1}{(3+2n)}$ and $\sum_{n=0}^{\infty} \frac{1}{(5+2n)}$, which are divergent in nature by using the comparison test.

Taking the difference of the above two series, we get

$$\sum_{n=0}^{\infty} \frac{1}{(3+2n)} - \sum_{n=0}^{\infty} \frac{1}{(5+2n)} = \sum_{n=0}^{\infty} \frac{2}{(3+2n)(5+2n)}. \tag{15}$$

The right hand side of equation (15) is convergent by using the Raabe's higher ratio test.
In terms of hypergeometric function, the equation (15) can be written as

$$\frac{1}{3}\sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}\right)_n}{\left(\frac{5}{2}\right)_n} - \frac{1}{5} \sum_{n=0}^{\infty} \frac{\left(\frac{5}{2}\right)_n}{\left(\frac{7}{2}\right)_n} = \frac{2}{15} \sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}\right)_n}{\left(\frac{7}{2}\right)_n},$$

$$\frac{1}{3}\ {}_2F_1\left[\begin{array}{c} \frac{3}{2},\ 1; \\ \frac{5}{2}; \end{array} 1\right] - \frac{1}{5}\ {}_2F_1\left[\begin{array}{c} \frac{5}{2},\ 1; \\ \frac{7}{2}; \end{array} 1\right] = \frac{2}{15}\ {}_2F_1\left[\begin{array}{c} \frac{3}{2},\ 1; \\ \frac{7}{2}; \end{array} 1\right]. \tag{16}$$

Since both the Gauss' series having the positive unit argument on left hand side of equation (16) are divergent. On using Gauss' classical summation theorem (13) on right hand side of equation (16), we get

$$\frac{1}{3}\ {}_2F_1\left[\begin{array}{c} \frac{3}{2},\ 1; \\ \frac{5}{2}; \end{array} 1\right] - \frac{1}{5}\ {}_2F_1\left[\begin{array}{c} \frac{5}{2},\ 1; \\ \frac{7}{2}; \end{array} 1\right] = \frac{2}{15}\ \frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{7}{2}-\frac{3}{2}-1\right)}{\Gamma\left(\frac{7}{2}-\frac{3}{2}\right)\Gamma\left(\frac{7}{2}-1\right)},$$

$$\frac{1}{3}\ {}_2F_1\left[\begin{array}{c} \frac{3}{2},\ 1; \\ \frac{5}{2}; \end{array} 1\right] - \frac{1}{5}\ {}_2F_1\left[\begin{array}{c} \frac{5}{2},\ 1; \\ \frac{7}{2}; \end{array} 1\right] = \frac{1}{3}, \tag{17}$$

which is convergent.

Multiplying both sides of equation (17) by $\frac{3}{16}$, for application point of view in next section, we get the difference of two divergent Gauss' series having the positive unit argument may be convergent

$$\frac{1}{16}\ {}_2F_1\left[\begin{array}{c} \frac{3}{2},\ 1; \\ \frac{5}{2}; \end{array} 1\right] - \frac{3}{80}\ {}_2F_1\left[\begin{array}{c} \frac{5}{2},\ 1; \\ \frac{7}{2}; \end{array} 1\right] = \frac{1}{16}. \tag{18}$$

Summation of some infinite series

The following summation formulas of some infinite series are derived:

$$\sum_{n=0}^{\infty} \frac{(4n^4+32n^3+87n^2+92n+28)}{(64n^6+768n^5+3792n^4+9856n^3+14220n^2+10800n+3375)} = \frac{5}{27} - \frac{\pi^2}{64}. \tag{19}$$

$$\sum_{n=0}^{\infty} \frac{(27n^3+36n^2+15n+2)\left\{\left(\frac{1}{3}\right)_n\right\}^4}{(n!)^4\ (1+n)^2} = \frac{27}{4\left[\Gamma\left(\frac{2}{3}\right)\right]^3}. \tag{20}$$

$$\sum_{n=0}^{\infty} \frac{(32n^4+120n^3+156n^2+82n+15)\ \left\{\left(\frac{1}{2}\right)_n\right\}^4}{(n!)^4\ (n^5+7n^4+19n^3+25n^2+16n+4)} = \frac{128}{3\pi^2}. \tag{21}$$

$$\sum_{n=0}^{\infty} \frac{(128n^3+144n^2+48n+5)\left\{\left(\frac{1}{4}\right)_n\right\}^4}{(n!)^4\ (n^2+2n+1)} = \frac{32\sqrt{2}}{3\sqrt{\pi}\left[\Gamma\left(\frac{3}{4}\right)\right]^2}. \tag{22}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(16n^4+96n^3+184n^2+120n+25)} = \frac{\mathbf{G}}{8} - \frac{11}{144}. \tag{23}$$

$$\sum_{n=0}^{\infty} \frac{1}{(4n^3+16n^2+21n+9)} = 4 - 2\ell n\ 2 - \frac{\pi^2}{4}. \tag{24}$$

$$\sum_{n=0}^{\infty} \frac{1}{(81n^4+270n^3+315n^2+150n+24)} = \frac{1}{6} + \frac{\pi}{12\sqrt{3}} - \frac{1}{4}\ell n\ 3. \tag{25}$$

$$\sum_{n=0}^{\infty} \frac{1}{(36n^3+108n^2+107n+35)} = \ell n\ 12 + \ell n\ \sqrt{3} - 3. \tag{26}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n^6+9n^5+33n^4+63n^3+66n^2+36n+8)} = 10 - 12\ \ell n\ 2 - \frac{3}{2}\ \zeta(3). \tag{27}$$

Proof of the result (19):

On factorizing the general term of equation (19) and making use of partial fractions, we have

$$\frac{(4n^4+32n^3+87n^2+92n+28)}{(64n^6+768n^5+3792n^4+9856n^3+14220n^2+10800n+3375)} =$$

$$= \frac{\frac{3}{16}}{(3+2n)} + \frac{\frac{-1}{16}}{(3+2n)^2} + \frac{\frac{-1}{4}}{(3+2n)^3} + \frac{\frac{-3}{16}}{(5+2n)} + \frac{\frac{-1}{16}}{(5+2n)^2} + \frac{\frac{1}{4}}{(5+2n)^3}. \tag{28}$$

Now taking summation on both sides of equation (28) and n varying from 0 to ∞, we get

$$\sum_{n=0}^{\infty} \frac{(4n^4+32n^3+87n^2+92n+28)}{(64n^6+768n^5+3792n^4+9856n^3+14220n^2+10800n+3375)} =$$

$$= \sum_{n=0}^{\infty} \frac{\frac{3}{16}}{(3+2n)} + + \frac{\frac{-1}{16}}{(3+2n)^2} + \frac{\frac{-1}{4}}{(3+2n)^3} + \frac{\frac{-3}{16}}{(5+2n)} + \frac{\frac{-1}{16}}{(5+2n)^2} + \frac{\frac{1}{4}}{(5+2n)^3} =$$

$$= \frac{1}{16} \sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}\right)_n}{\left(\frac{5}{2}\right)_n} - \frac{1}{144} \sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}\right)_n \left(\frac{3}{2}\right)_n}{\left(\frac{5}{2}\right)_n \left(\frac{5}{2}\right)_n} - \frac{1}{108} \sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}\right)_n \left(\frac{3}{2}\right)_n \left(\frac{3}{2}\right)_n}{\left(\frac{5}{2}\right)_n \left(\frac{5}{2}\right)_n \left(\frac{5}{2}\right)_n} -$$

$$- \frac{3}{80} \sum_{n=0}^{\infty} \frac{\left(\frac{5}{2}\right)_n}{\left(\frac{7}{2}\right)_n} - \frac{1}{400} \sum_{n=0}^{\infty} \frac{\left(\frac{5}{2}\right)_n \left(\frac{5}{2}\right)_n}{\left(\frac{7}{2}\right)_n \left(\frac{7}{2}\right)_n} + \frac{1}{500} \sum_{n=0}^{\infty} \frac{\left(\frac{5}{2}\right)_n \left(\frac{5}{2}\right)_n \left(\frac{5}{2}\right)_n}{\left(\frac{7}{2}\right)_n \left(\frac{7}{2}\right)_n \left(\frac{7}{2}\right)_n}.$$

Using the definition of generalized hypergeometric function, we get

$$\sum_{n=0}^{\infty} \frac{(4n^4+32n^3+87n^2+92n+28)}{(64n^6+768n^5+3792n^4+9856n^3+14220n^2+10800n+3375)} =$$

$$= \frac{1}{16}\ {}_2F_1\left[\begin{matrix} \frac{3}{2},\ 1; \\ \frac{5}{2}; \end{matrix}\ 1\right] - \frac{1}{144}\ {}_3F_2\left[\begin{matrix} \frac{3}{2},\ \frac{3}{2},\ 1; \\ \frac{5}{2},\ \frac{5}{2}; \end{matrix}\ 1\right] - \frac{1}{108}\ {}_4F_3\left[\begin{matrix} \frac{3}{2},\ \frac{3}{2},\ \frac{3}{2},\ 1; \\ \frac{5}{2},\ \frac{5}{2},\ \frac{5}{2}; \end{matrix}\ 1\right] -$$

$$- \frac{3}{80}\ {}_2F_1\left[\begin{matrix} \frac{5}{2},\ 1; \\ \frac{7}{2}; \end{matrix}\ 1\right] - \frac{1}{400}\ {}_3F_2\left[\begin{matrix} \frac{5}{2},\ \frac{5}{2},\ 1; \\ \frac{7}{2},\ \frac{7}{2}; \end{matrix}\ 1\right] + \frac{1}{500}\ {}_4F_3\left[\begin{matrix} \frac{5}{2},\ \frac{5}{2},\ \frac{5}{2},\ 1; \\ \frac{7}{2},\ \frac{7}{2},\ \frac{7}{2}; \end{matrix}\ 1\right].$$

Using summation theorems (8), (11) and the result (18), we get

$$\sum_{n=0}^{\infty} \frac{(4n^4+32n^3+87n^2+92n+28)}{(64n^6+768n^5+3792n^4+9856n^3+14220n^2+10800n+3375)} =$$

$$= \frac{1}{16} - \frac{1}{64}\ \psi^{(1)}\left(\frac{3}{2}\right) - \frac{1}{64}\ \psi^{(1)}\left(\frac{5}{2}\right) + \frac{1}{64}\ \psi^{(2)}\left(\frac{3}{2}\right) - \frac{1}{64}\ \psi^{(2)}\left(\frac{5}{2}\right) =$$

$$= \frac{1}{16} - \frac{1}{64}\ \left(\frac{\pi^2}{2} - 4\right) - \frac{1}{64}\ \left(\frac{\pi^2}{2} - \frac{40}{9}\right) + \frac{1}{64}\ \left(\frac{-14\pi^3}{25.79436} + 16\right) - \frac{1}{64}\ \left(\frac{-14\pi^3}{25.79436} + \frac{448}{27}\right).$$

On simplifying further, we arrive at the result (19).

Proof of the results (20) to (22):

The proof of the results (20) and (22) can be obtained in an analogous manner by following the same steps as in the proof of the result (19) and making use of the summation theorem (14).

Proof of the result (23):

The proof of the result (23) can be obtained by following the same procedure as in the proof of the result (19) and making use of the summation theorems (6), (9) and using the equations (4) and (5). So we omit the details here.

Proof of the result (24):

On factorizing the general term of equation (24) and making use of partial fractions, we have

$$\frac{1}{(4n^3+16n^2+21n+9)} = \frac{1}{(1+n)} + \frac{-2}{(3+2n)} + \frac{-2}{(3+2n)^2}. \tag{29}$$

Now taking summation on both sides of equation (29) and n varying from 0 to ∞, we get

$$\sum_{n=0}^{\infty} \frac{1}{(4n^3+16n^2+21n+9)} = \sum_{n=0}^{\infty} \left\{ \frac{1}{(1+n)} + \frac{-2}{(3+2n)} \right\} - 2\sum_{n=0}^{\infty} \frac{1}{(3+2n)^2} =$$

$$= \sum_{n=0}^{\infty} \frac{1}{(1+n)\ (3+2n)} - 2\sum_{n=0}^{\infty} \frac{1}{(3+2n)^2} =$$

$$= \frac{1}{3}\sum_{n=0}^{\infty} \frac{(1)_n\ \left(\frac{3}{2}\right)_n}{(2)_n\ \left(\frac{5}{2}\right)_n} - \frac{2}{9}\sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}\right)_n\ \left(\frac{3}{2}\right)_n}{\left(\frac{5}{2}\right)_n\ \left(\frac{5}{2}\right)_n}.$$

Using the definition of generalized hypergeometric function, we get

$$\sum_{n=0}^{\infty} \frac{1}{(4n^3+16n^2+21n+9)} = \frac{1}{3}\ {}_3F_2\left[\begin{matrix} \frac{3}{2},\ 1,\ 1; \\ \frac{5}{2},\ 2; \end{matrix}\ 1\right] - \frac{2}{9}\ {}_3F_2\left[\begin{matrix} \frac{3}{2},\ \frac{3}{2},\ 1; \\ \frac{5}{2},\ \frac{5}{2}; \end{matrix}\ 1\right].$$

Using summation theorems (7) and (8), we get

$$\sum_{n=0}^{\infty} \frac{1}{(4n^3+16n^2+21n+9)} = \psi\left(\frac{3}{2}\right) - \psi(1) - \frac{1}{2}\ \psi^{(1)}\left(\frac{3}{2}\right).$$

On simplifying further, we arrive at the result (24).

Proof of the result (25):

The proof of the result (25) can be obtained in an analogous manner by following the same steps as in the proof of the result (19) and (24) and making use of Gauss' classical summation theorem (13), the summation theorem (7) and using the equation (1). So, we omit the details here.

Proof of the result (26):

The proof of the result (26) can be obtained by following the same procedure as in the proof of the result (19) and (24) and making use of the summation theorem (10) and using the equations (1) and (2). So, we omit the details here.

Proof of the result (27):

Similarly for the proof of the result (27), we make use of the summation theorems (6), (9), (12) and the equations (3) and (5). So, we omit the details here.

Representation of infinite series (19) to (27) in Hypergeometric forms

The following hypergeometric representation formulas are derived:

$${}_8F_7\left[\begin{matrix} \frac{9}{2},\ \frac{3}{2},\ \frac{3}{2},\ \frac{3}{2},\ \frac{3}{2},\ 3,\ 3,\ 1; \\ \frac{7}{2},\ \frac{7}{2},\ \frac{7}{2},\ \frac{7}{2},\ \frac{1}{2},\ 2,\ 2; \end{matrix}\ 1\right] = \frac{625}{28} - \frac{3375\pi^2}{1792}. \tag{30}$$

$$ {}_5F_4\left[\begin{array}{c} \frac{5}{3},\ \frac{4}{3},\ \frac{4}{3},\ \frac{1}{3},\ \frac{1}{3}; \\ \\ \frac{2}{3},\ 1,\ 2,\ 2; \end{array} 1\right] = \frac{27}{8\left[\Gamma\left(\frac{2}{3}\right)\right]^3}. \tag{31} $$

$$ {}_5F_4\left[\begin{array}{c} \frac{9}{4},\ \frac{5}{2},\ \frac{3}{2},\ \frac{1}{2},\ \frac{1}{2}; \\ \\ \frac{5}{4},\ 2,\ 3,\ 3; \end{array} 1\right] = \frac{512}{45\pi^2}. \tag{32} $$

$$ {}_5F_4\left[\begin{array}{c} \frac{13}{8},\ \frac{5}{4},\ \frac{5}{4},\ \frac{1}{4},\ \frac{1}{4}; \\ \\ 2,\ 2,\ 1,\ \frac{5}{8}; \end{array} 1\right] = \frac{32\sqrt{2}}{15\sqrt{\pi}\left[\Gamma\left(\frac{3}{4}\right)\right]^2}. \tag{33} $$

$$ {}_5F_4\left[\begin{array}{c} 1,\ \frac{1}{2},\ \frac{1}{2},\ \frac{5}{2},\ \frac{5}{2}; \\ \\ \frac{3}{2},\ \frac{3}{2},\ \frac{7}{2},\ \frac{7}{2}; \end{array} -1\right] = \frac{25\ \mathbf{G}}{8} - \frac{275}{144}. \tag{34} $$

$$ {}_4F_3\left[\begin{array}{c} \frac{3}{2},\ \frac{3}{2},\ 1,\ 1; \\ \\ \frac{5}{2},\ \frac{5}{2},\ 2; \end{array} 1\right] = 36 - 18\ell n\ 2 - \frac{9\pi^2}{4}. \tag{35} $$

$$ {}_4F_3\left[\begin{array}{c} \frac{2}{3},\ \frac{1}{3},\ 1,\ 1; \\ \\ \frac{7}{3},\ \frac{5}{3},\ 2; \end{array} 1\right] = 4 - 6\ell n\ 3 + \frac{2\pi}{\sqrt{3}}. \tag{36} $$

$$ {}_4F_3\left[\begin{array}{c} \frac{7}{6},\ \frac{5}{6},\ 1,\ 1; \\ \\ \frac{13}{6},\ \frac{11}{6},\ 2; \end{array} 1\right] = 35\ell n\ 12 + 35\ell n\ \sqrt{3} - 105. \tag{37} $$

$$ {}_4F_3\left[\begin{array}{c} 1,\ 1,\ 1,\ 1; \\ \\ 3,\ 3,\ 3; \end{array} -1\right] = 96\ \ell n\ 2 - 80 + 12\ \zeta(3). \tag{38} $$

Proof of the result (30):

$$ \sum_{n=0}^{\infty} \frac{(4n^4 + 32n^3 + 87n^2 + 92n + 28)}{(64n^6 + 768n^5 + 3792n^4 + 9856n^3 + 14220n^2 + 10800n + 3375)} = $$

$$ = \sum_{n=0}^{\infty} \frac{(7+2n)\ (1+2n)\ (2+n)^2}{(3+2n)^3\ (5+2n)^3} = \frac{28}{3375} \sum_{n=0}^{\infty} \frac{\left(\frac{9}{2}\right)_n\ \left(\frac{3}{2}\right)_n\ \left(\frac{3}{2}\right)_n\ \left(\frac{3}{2}\right)_n\ \left(\frac{3}{2}\right)_n\ (3)_n\ (3)_n}{\left(\frac{7}{2}\right)_n\ \left(\frac{1}{2}\right)_n\ \left(\frac{7}{2}\right)_n\ \left(\frac{7}{2}\right)_n\ \left(\frac{7}{2}\right)_n\ (2)_n\ (2)_n}. $$

Using the definition of generalized hypergeometric function, we get

$$ \sum_{n=0}^{\infty} \frac{(4n^4 + 32n^3 + 87n^2 + 92n + 28)}{(64n^6 + 768n^5 + 3792n^4 + 9856n^3 + 14220n^2 + 10800n + 3375)} = $$

$$ = \frac{28}{3375}\ {}_8F_7\left[\begin{array}{c} \frac{9}{2},\ \frac{3}{2},\ \frac{3}{2},\ \frac{3}{2},\ \frac{3}{2},\ 3,\ 3,\ 1; \\ \\ \frac{7}{2},\ \frac{7}{2},\ \frac{7}{2},\ \frac{7}{2},\ \frac{1}{2},\ 2,\ 2; \end{array} 1\right]. \tag{39} $$

Using equation (19) in equation (39), we arrive at the result (30).

Proof of the results (31) to (38):

The proof of the results (31) to (38) can be obtained in an analogous manner by following the same steps as in the proof of the above result (30). So we omit the details here.

Conclusion

In this paper, we have obtained the summation of some infinite series by using some summation theorems of positive and negative unit arguments, Riemann Zeta functions, polygamma functions, lower case beta functions of one-variable and other associated functions. We have also obtained some new hypergeometric summation theorems, which are not found in the literature. We conclude this paper with the remark that the summation of various other infinite series can be derived in an analogous manner. Moreover, the results deduced above are expected to lead to some potential applications in several fields of Applied Mathematics, Statistics and Engineering Sciences.

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Гипергеометриялық функциялар мен жартылай бөлшек әдістерімен кейбір шексіз серияларды жинақтау

Мақалада кейбір шексіз қатарлардың жартылай бөлшек әдісімен оң және теріс сингулярлық дәлелдерді, Риманның Зета функцияларын, полигамма функцияларын, кіші регистрдегі бір айнымалының бета функцияларын және басқа да байланысты функцияларды жинақтаудың кейбір гипергеометриялық теоремалары жинақталған. Сондай-ақ кейбір гипергеометриялық жиынтық теоремалар алынған:

$${}_8F_7\left[\ \tfrac{9}{2},\tfrac{3}{2},\tfrac{3}{2},\tfrac{3}{2},\tfrac{3}{2},3,3,1;\tfrac{7}{2},\tfrac{7}{2},\tfrac{7}{2},\tfrac{7}{2},\tfrac{1}{2},2,2;1\ \right],\ {}_5F_4\left[\ \tfrac{5}{3},\tfrac{4}{3},\tfrac{4}{3},\tfrac{1}{3},\tfrac{1}{3};\tfrac{2}{3},1,2,2;1\ \right],\ {}_5F_4\left[\ \tfrac{9}{4},\tfrac{5}{2},\tfrac{3}{2},\tfrac{1}{2},\tfrac{1}{2};\tfrac{5}{4},2,3,3;1\ \right]$$

$${}_5F_4\left[\ \tfrac{13}{8},\tfrac{5}{4},\tfrac{5}{4},\tfrac{1}{4},\tfrac{1}{4};\tfrac{5}{8},2,2,1;1\ \right],\quad {}_5F_4\left[\ \tfrac{1}{2},\tfrac{1}{2},\tfrac{5}{2},\tfrac{5}{2},1;\tfrac{3}{2},\tfrac{3}{2},\tfrac{7}{2},\tfrac{7}{2};-1\ \right],\quad {}_4F_3\left[\ \tfrac{3}{2},\tfrac{3}{2},1,1;\tfrac{5}{2},\tfrac{5}{2},2;1\ \right],$$

$${}_4F_3\left[\ \tfrac{2}{3},\tfrac{1}{3},1,1;\tfrac{7}{3},\tfrac{5}{3},2;1\ \right],\ {}_4F_3\left[\ \tfrac{7}{6},\tfrac{5}{6},1,1;\tfrac{13}{6},\tfrac{11}{6},2;1\ \right]\ \text{and}\ \ {}_4F_3\left[\ 1,1,1,1;3,3,3;-1\ \right].$$

Кілт сөздер: Риманның Зета функциялары, полигамма функциялары, Дугалл теоремасы, Бернулли көпмүшелері, Каталан константасы.

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Суммирование некоторых бесконечных рядов методами гипергеометрических функций и частных дробей

В статье получено суммирование некоторых бесконечных рядов методом частичных дробей и с помощью некоторых гипергеометрических теорем суммирования положительных и отрицательных единичных аргументов, дзета-функций Римана, полигамма-функций, бета-функций одной переменной в нижнем регистре и других связанных функций. Кроме того, авторами получены некоторые гипергеометрические теоремы суммирования для:

$$ {}_8F_7\left[\ \tfrac{9}{2},\tfrac{3}{2},\tfrac{3}{2},\tfrac{3}{2},\tfrac{3}{2},3,3,1;\tfrac{7}{2},\tfrac{7}{2},\tfrac{7}{2},\tfrac{7}{2},\tfrac{1}{2},2,2;1\ \right],\ {}_5F_4\left[\ \tfrac{5}{3},\tfrac{4}{3},\tfrac{4}{3},\tfrac{1}{3},\tfrac{1}{3};\tfrac{2}{3},1,2,2;1\ \right],\ {}_5F_4\left[\ \tfrac{9}{4},\tfrac{5}{2},\tfrac{3}{2},\tfrac{1}{2},\tfrac{1}{2};\tfrac{5}{4},2,3,3;1\ \right] $$

$$ {}_5F_4\left[\ \tfrac{13}{8},\tfrac{5}{4},\tfrac{5}{4},\tfrac{1}{4},\tfrac{1}{4};\tfrac{5}{8},2,2,1;1\ \right],\quad {}_5F_4\left[\ \tfrac{1}{2},\tfrac{1}{2},\tfrac{5}{2},\tfrac{5}{2},1;\tfrac{3}{2},\tfrac{3}{2},\tfrac{7}{2},\tfrac{7}{2};-1\ \right],\quad {}_4F_3\left[\ \tfrac{3}{2},\tfrac{3}{2},1,1;\tfrac{5}{2},\tfrac{5}{2},2;1\ \right], $$

$$ {}_4F_3\left[\ \tfrac{2}{3},\tfrac{1}{3},1,1;\tfrac{7}{3},\tfrac{5}{3},2;1\ \right],\ {}_4F_3\left[\ \tfrac{7}{6},\tfrac{5}{6},1,1;\tfrac{13}{6},\tfrac{11}{6},2;1\ \right]\ \text{и}\ \ {}_4F_3\left[\ 1,1,1,1;3,3,3;-1\ \right]. $$

Ключевые слова: дзета-функции Римана, полигамма-функции, теорема Дугалла, многочлены Бернулли, константа Каталана.