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ABOUT SOLUTION OF THE DAMPED OSCILLATIONS

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Vibrations, by the physical nature, are diverse. There are mechanical, electromagnetic, electromechanical, chemical, thermodynamic vibration there. Solutions of differential equation of these vibrations were obtained by different mathematical methods. We obtain analytical solutions of Lienard's differential equations by the method of partial sampling discretization of nonlinear differential equations. Graphic of solution, which describes the considering process and corresponds to damped oscillations has been plotted.

Keywords: vibrations, nonlinear differential equations, method of partial sampling discretization, damped wave process, the dissipative term of the equation.

1. Introduction

Second order differential equations are encountered in many applications. In relatively simple cases, these equations turn out to linear, and with constant coefficients, i.e. they have the form

$$A\frac{d^2x}{dt^2}+B\frac{dx}{dt}+Cx=F(t). \qquad (*)$$

Two well-known examples are a linear spring and electrical circuit. In the case of the spring x is an offset, A – cargo weight, $-B\frac{dx}{dt}$ – resistance of the medium, $-Cx$ – the restoring force spring and $F(t)$ – quantity the «force». In the case of the electrical circuit x is a current, A – induction, B – resistance, $1/C$ – capacitance and $\int F(t)dt$ represents the electromotive force. Particularly important case is when F has the oscillatory character, and then the oscillatory «response» is searched for, i.e. oscillating solution of the equation.

Since then, we have second-order equation, that $A\neq 0$, and, dividing it by A, we obtain

$$\frac{d^2x}{dt^2}+f\frac{dx}{dt}+g\,x=e(t). \qquad (**)$$

Middle quantity corresponds to the energy dissipation, and therefore in the general case we call it as a dissipative term. As it is known it characterizes the deviation from the law of energy conservation.

Usually equations of type (*), (**) appear because they permit a solution «in closed form», and thereby all the properties of their solutions can be easily studied. For this reason, physicists and engineers are eager to simplify the task they are dealing with, that it is described by linear equations with constant coefficients. However on many reasons it cannot be done according to the substantially nonlinear problems [1].

Quite general types of equations are

$$\frac{d^2x}{dt^2}+f(x)\frac{dx}{dt}+g(x)=e(t), \qquad \frac{d^2x}{dt^2}+f\left(x,\frac{dx}{dt}\right)\frac{dx}{dt}+g(x)=e(t).$$

Also the following partial cases have to be considered:

$$\frac{d^2x}{dt^2}+f(x)\frac{dx}{dt}+g(x)=0, \qquad \frac{d^2x}{dt^2}+f\left(x,\frac{dx}{dt}\right)\frac{dx}{dt}+g(x)=0.$$

French physicist A. Lienard, in his important but little-known article [2] investigated in detail the quite general equation with dissipative middle quantity

$$\frac{d^2x}{dt^2} + f(x)\frac{dx}{dt} + g(x) = 0,$$

so called Lienard's equation.

In the present paper Lienard's equation is solved by method of partial discretization of nonlinear differential equations. The method essence consists of partial discretization of nonlinear term of the equation and we obtain the solution in the class of generalized functions.

2. Formulation of the problem

Consider Lienard's nonlinear differential equation in the following form

$$\frac{d^2x}{dt^2} + \alpha x^2 \frac{dx}{dt} + \beta x = 0, \tag{1}$$

under the initial conditions

$$\mathrm{t} = 0: \; x = x_0, \quad \dot{x} = \dot{x}_0. \tag{2}$$

3. Solution of the problem

Transform equation (1)

$$\frac{d^2x}{dt^2} + \frac{\alpha}{3}\frac{d(x^3)}{dt} + \beta x = 0, \tag{3}$$

where $\alpha, \;\; \beta$ – the real numbers.

Integrating equation twice (3) by t, we obtain

$$\frac{dx}{dt} + \frac{\alpha}{3}x^3 + \beta \int x dt + A = 0, \tag{4}$$

$$x + \frac{\alpha}{3}\int x^3 dt + \beta \int\int x\, dt\, dt + At + B = 0. \tag{5}$$

Discretizing the second, third terms of the equation (5)

$$x + \frac{\alpha}{3}\int \frac{1}{2}\sum (t_k + t_{k+1})\left[x_k^3\, \delta(t - t_k)dt - x_{k+1}^3\, \delta(t - t_{k+1})dt\right] +$$
$$+ \beta \iint \frac{1}{2}\sum (t_k + t_{k+1})\left[x_k\, \delta(t - t_k)dt\, dt - x_{k+1}\, \delta(t - t_{k+1})dt\, dt\right] + At + B = 0, \tag{6}$$

and after integration, we have

$$x + \frac{\alpha}{6}\sum (t_k + t_{k+1})\left[x_k^3 H(t - t_k) - x_{k+1}^3 H(t - t_{k+1})\right] +$$
$$+ \frac{\beta}{2}\sum (t_k + t_{k+1})\left[x_k (t - t_k) H(t - t_k) - x_{k+1}(t - t_{k+1}) H(t - t_{k+1})\right] + At + B = 0. \tag{7}$$

where $\delta(\xi)$ – Dirac's delta function, $H(\xi)$ – Heaviside's unit function.

Taking advantage of initial conditions (2), we find the constant integration A and B

$$A = \frac{\alpha}{3}x_0^3, \qquad B = -x_0. \tag{8}$$

From the equation (7) we obtain solving equation for each point t_k. Determine for point t_1 $(t_1 < t < t_2)$, $t = \tau_1 = \dfrac{t_1 + t_2}{2}$

$$\frac{\alpha}{6}(t_1 + t_2)x_1^3 + \left[1 + \frac{\beta}{2}(t_1 + t_2)(\tau_1 - t_1)\right]x_1 + A\tau_1 + B = 0, \tag{9}$$

for point t_2 $(t_2 < t < t_3)$, $t = \tau_2 = \dfrac{t_2 + t_3}{2}$

$$\frac{\alpha}{6}(t_3-t_1)x_2^3+\left[1+\frac{\beta}{2}(t_3-t_1)(\tau_2-t_2)\right]x_2+\frac{\alpha}{6}(t_1+t_2)x_1^3+\frac{\beta}{2}(t_1+t_2)x_1(\tau_2-t_1)+A\tau_2+B=0, \quad (10)$$

using method of mathematical induction of solving equation we determine for point t_k, where $k=\overline{3,n}$, $(t_k < t < t_{k+1})$, $t=\tau_k = \dfrac{t_k + t_{k+1}}{2}$

$$\begin{aligned}&\frac{\alpha}{6}(t_{k+1}-t_{k-1})x_k^3+\left[1+\frac{\beta}{2}(t_{k+1}-t_{k-1})(\tau_k-t_k)\right]x_k+\frac{\alpha}{6}(t_{k-1}+t_k)x_{k-1}^3+\frac{\alpha}{6}\sum_{i=1}^{k-2}(t_i+t_{i+1})\left[x_i^3-x_{i+1}^3\right]+\\&+\frac{\beta}{2}(t_{k-1}+t_k)x_{k-1}(\tau_k-t_{k-1})+\frac{\beta}{2}\sum_{i=1}^{k-2}(t_i+t_{i+1})\left[x_i(\tau_k-t_i)-x_{i+1}(\tau_k-t_{i+1})\right]+A\tau_k+B=0.\end{aligned} \quad (11)$$

Designated through

$$a_1=\frac{\alpha}{6}(t_1+t_2), \qquad b_1=1+\frac{\beta}{2}(t_1+t_2)(\tau_1-t_1), \qquad c_1=A\tau_1+B \quad (12)$$

for point t_1 we can lead equation (9) to cubed equation

$$a_1x_1^3+b_1x_1+c_1=0. \quad (13)$$

Divide the equation (13) to the senior coefficient a_1. Then it will be assumed

$$x_1^3+p_1x_1+q_1=0, \quad (14)$$

where $p_1=\dfrac{b_1}{a_1}$, $q_1=\dfrac{c_1}{a_1}$.

The equation such as (14) is trinomial cubed equation, where there is not a term with unknown quantity in the second power. According to Cardano's formula trinomial cubed equation

$$x^3+p\ x\ +q\ =0 \quad (15)$$

is solved by leading them to squared equation.

For this purpose we will seek for the solution of the equation (15) in the following form

$$x=y-\frac{p}{3y}, \quad (16)$$

where y – a new variable term. By substitution (16) equation (15) is led to the form

$$y^3-\frac{p^3}{27y^3}+q=0. \quad (17)$$

Multiplying the equation (17) by y^3, we obtain the squared equation concerning y^3:

$$y^6+q\,y^3-\frac{p^3}{27}=0. \quad (18)$$

Solution of the equation (18) has the following form:

$$y_{1,2}^3=-\frac{q}{2}\pm\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}.$$

Therefore, $y_1=\sqrt[3]{-\dfrac{q}{2}+\sqrt{\dfrac{q^2}{4}+\dfrac{p^3}{27}}}$, $y_2=\sqrt[3]{-\dfrac{q}{2}-\sqrt{\dfrac{q^2}{4}+\dfrac{p^3}{27}}}$.

According to (16), it is followed that the equation (15) has two solutions:

$$x_1=y_1-\frac{p}{3y_1}, \qquad x_2=y_2-\frac{p}{3y_2}. \quad (19)$$

In detail these solutions are given in such way:

$$x_1 = \sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}} - \frac{p}{3\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}}, \tag{20}$$

$$x_2 = \sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}} - \frac{p}{3\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}}. \tag{21}$$

In spite of apparent differences, solutions (20) and (21) are coincided. Thus, the root of the equation is an unique and it equals to

$$x = x_1 = x_2 = y_1 + y_2,$$

and for the solution of the equation (15) we obtain the formula

$$x = \sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}. \tag{22}$$

The solution of the equation (14) for point t_1 $(t_1 < t < t_2)$, $t = \tau_1 = \frac{t_1 + t_2}{2}$ we write, using signs (12):

$$x(t_1) = \sqrt[3]{-\frac{q_1}{2}+\sqrt{\frac{q_1^2}{4}+\frac{p_1^3}{27}}} + \sqrt[3]{-\frac{q_1}{2}-\sqrt{\frac{q_1^2}{4}+\frac{p_1^3}{27}}}, \tag{23}$$

where $q_1 = \frac{c_1}{a_1} = \frac{A\tau_1 + B}{\frac{\alpha}{6}(t_1 + t_2)}$, $\quad p_1 = \frac{b_1}{a_1} = \frac{1 + \frac{\beta}{2}(t_1 + t_2)(\tau_1 - t_1)}{\frac{\alpha}{6}(t_1 + t_2)}$. (24)

Determine coefficient for point t_2 $(t_2 < t < t_3)$, $t = \tau_2 = \frac{t_2 + t_3}{2}$

$$q_2 = \frac{c_2}{a_2} = \frac{A\tau_2 + B + \frac{\alpha}{6}(t_1 + t_2)x_1^3 + \frac{\beta}{2}(t_1 + t_2)x_1(\tau_2 - t_1)}{\frac{\alpha}{6}(t_3 - t_1)}, \qquad p_2 = \frac{b_2}{a_2} = \frac{1 + \frac{\beta}{2}(t_3 - t_1)(\tau_2 - t_2)}{\frac{\alpha}{6}(t_3 - t_1)}. \tag{25}$$

Using method of mathematical induction we determine coefficient for point t_k, where $k = \overline{3,n}$, $(t_k < t < t_{k+1})$, $t = \tau_k = \frac{t_k + t_{k+1}}{2}$

$$q_k = \frac{c_k}{a_k} = \frac{1}{\frac{\alpha}{6}(t_{k+1} - t_{k-1})}\Big[A\tau_k + B + \frac{\alpha}{6}(t_{k-1} + t_k)x_{k-1}^3 + + \frac{\alpha}{6}\sum_{i=1}^{k-2}(t_i + t_{i+1})\left[x_i^3 - x_{i+1}^3\right] +$$
$$+ \frac{\beta}{2}(t_{k-1} + t_k)x_{k-1}(\tau_k - t_{k-1}) + \frac{\beta}{2}\sum_{i=1}^{k-2}(t_i + t_{i+1})\left[x_i(\tau_k - t_i) - x_{i+1}(\tau_k - t_{i+1})\right]\Big], \tag{26}$$

$$p_k = \frac{b_k}{a_k} = \frac{1 + \frac{\beta}{2}(t_{k+1} - t_{k-1})(\tau_k - t_k)}{\frac{\alpha}{6}(t_{k+1} - t_{k-1})}.$$

Thus, solution of the equation (11) for point t_k, where $k = \overline{3,n}$:

$$x(t_k) = \sqrt[3]{-\frac{q_k}{2} + \sqrt{\frac{q_k^2}{4} + \frac{p_k^3}{27}}} + \sqrt[3]{-\frac{q_k}{2} - \sqrt{\frac{q_k^2}{4} + \frac{p_k^3}{27}}}, \quad (27)$$

4. Results of calculation

Substituting discovered coefficient (26) into the root of equation (27) we find all terms $x(t_k)$ when $k = \overline{1,n}$, which given in Fig.1. Graphic was drawn by using program MathCAD under calculated data $\alpha = 6,\ \beta = 2$.

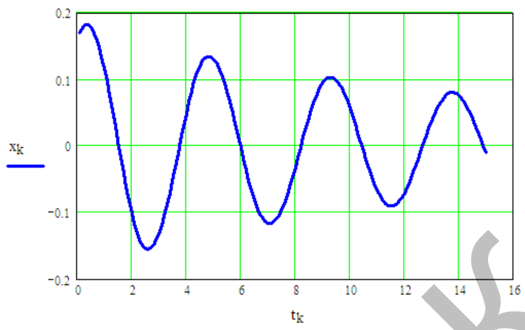

Fig.1. Graphic of dependence $x_k = x(t_k)$.

5. Conclusion

The graphic shows that the equation (1) describes a damped wave process. Solutions of forced oscillations can also be obtained by sampling method of partial discretization nonlinear differential equations. If we disassemble the vibrations, they are very diverse in their physical nature [3]. Such vibrations may be caused:

1) mechanical vibrations, such as the vibrations of the pendulum , the bridge, the ship in the wake, the strings;

2) electromagnetic waves, such as the vibrations in the resonant circuit , resonant cavity , waveguide, radio-waves and etc.;

3) electromechanical oscillations, such as the vibration of the phone membrane ;

4) chemical fluctuations, for example, fluctuations in the concentration of the reactants at periodic chemical reactions ;

5) the thermodynamic fluctuations, for example, temperature fluctuations .

Taking into account the importance of these physical processes, construction of analytical solutions of such problems is highly relevant. In this paper the problem was solved by method of partial discretization of nonlinear differential equations by A.N.Tyurehodzhaev. Also using this method we obtained analytical solutions and graphics of problems on deformation of a circular elastic, flexible plate under a variable load [4], which confirmed that using the partial sampling method of discretization we can solve the nonlinear differential equation of higher order with variable coefficients.

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