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Interpolation Formula for Functions of Several Variables

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In this paper, we study the approximate reconstruction problem for functions of several variables from the Sobolev and Korobov spaces, $W_p^\alpha[0,1]^n$ and $E^\alpha[0,1]^n$, respectively, using its values at some nodes. In contrast to the well-known papers [1–10], in this paper the interpolation formula is written out in explicit form for an arbitrary dimension n.

To the multi-index $\mu=(\mu_1,\dots,\mu_n)\in\mathbb{Z}^n$ we assign $\nu(\mu)=(\nu_1,\dots,\nu_n)\in\mathbb{N}^n$, so that for $|\mu_i|>0$ we have

$$2^{\nu_i(\mu)-3}\le|\mu_i|<2^{\nu_i(\mu)-2},\qquad i=1,\dots,n,$$

while for $|\mu_i|=0$ we set $\nu_i(\mu)=0$.

We define the functional $P_{2^m}(f\,;\mu)$ as follows:

$$P_{2^m}(f\,;\mu)=\sum_{\substack{k_1+\dots+k_n=m\\ k_j\ge\nu_j}}\frac{1}{2^m}\sum_{r_1=0}^{2^{k_1}-1}\dots\sum_{r_n=0}^{2^{k_n}-1}(-1)^{\sum_{j=1}^{n-1}(r_j+\operatorname{sgn}(k_j-\nu_j))}$$
$$\times f\biggl(\frac{r_1}{2^{k_1}},\dots,\frac{r_n}{2^{k_n}}\biggr)e^{2\pi i\sum_{i=1}^n\mu_i r_i/2^{k_i}}.$$

This functional is a finite linear combination of the values of the function f, and the coefficients of the linear combination take values $1/2^m$ or $-1/2^m$.

We define the set

$$\Delta_m=\bigcup_{\substack{k_1+\dots+k_n=m\\ k_j\ge2}}Q_k,$$

where $Q_k=\{r\in\mathbb{Z}^n\colon|r_i|<2^{k_i-3},\ i=1,\dots,n\}$.

Theorem 1. *Let $m\in\mathbb{N}$, and let $f(x)=\sum_{\mu\in\Delta_m}\hat f(x)e^{2\pi i\mu x}$. Then the functional $P_{2^m}(f\,;\mu)$ reconstructs the exact values of the coefficients of the polynomial, i.e.,*

$$\hat f(\mu)=P_{2^m}(f\,;\mu).$$

By $E_{\Delta_m}(f)_{E^\alpha[0,1]^n}$ and $E_{\Delta_m}(f)_{W_p^\alpha[0,1]^n}$ we denote the best approximations of a function f from the corresponding classes by trigonometric polynomials with harmonics from the set Δ_m.

Theorem 2. *Suppose that $m,n\in\mathbb{N}$. If $\alpha>1$ and $f\in E^\alpha[0,1]^n$, then*

$$|\hat f(\mu)-P_{2^m}(f\,;\mu)|\le C\frac{(m-\sum_{j=1}^n\nu_j(\mu))^{n-1}}{2^{m\alpha}}E_{\Delta_m}(f)_{E^\alpha[0,1]^n}.$$

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Theorem 3. *Suppose that* $1 < p \le \infty$, $r = \min(p, 2)$, $m, n \in \mathbb{N}$, $\alpha > 1/r$, *and* $f \in E^\alpha[0,1]^n$. *Then*

$$|\hat{f}(\mu) - P_{2^m}(f\,;\,\mu)| \le C \frac{(m - \sum_{j=1}^n \nu_j(\mu))^{\frac{n-1}{r}}}{2^{m\alpha}} E_{\Delta_m}(f)_{W_p^\alpha[0,1]^n}.$$

Suppose that $m \in \mathbb{N}$, $k \in \mathbb{Z}_+^n$ and

$$Q_k = \{r \in \mathbb{Z}^n \colon |r_j| \le 2^{k_j - 1},\ j = 1, 2, \dots, n\} \setminus \{(\pm[2^{k_1-1}], \dots, \pm[2^{k_n-1}])\}$$

is an integer parallelepiped with excluded vertices. The set

$$G_m = \bigcup_{\substack{k_1+\dots+k_n=m+1\\ k\in\mathbb{Z}_+^n}} Q_k,$$

is called a *staircase hyperbolic cross of order* m.

For a function $f \in C[0,1]^n$, we define the operator

$$\begin{aligned} F_{2^m}(f\,;\,x) &= \sum_{\mu\in G_m} P_{2^m}(f\,;\,\mu) e^{2\pi i\mu x} \\ &= \sum_{\mu\in G_m} \sum_{\substack{k_1+\dots+k_n=m\\ k_j\ge\nu_j(\mu)}} \frac{1}{2^m} \sum_{r_1=0}^{2^{k_1}-1} \cdots \sum_{r_n=0}^{2^{k_n}-1} (-1)^{\sum_{j=1}^{n-1}(r_j+\operatorname{sgn}(k_j-\nu_j))} \\ &\qquad \times f\left(\frac{r_1}{2^{k_1}}, \dots, \frac{r_n}{2^{k_n}}\right) e^{2\pi i\mu(r/2^k+x)}. \end{aligned} \tag{1}$$

This is the reconstruction operator for the function f from its values at $M = O(m^{(n-1)}2^m)$ nodes.

Theorem 4. *Suppose that* $2 \le q \le \infty$, $q' = q/(q-1)$, $\alpha > 1$, $f \in E^\alpha[0,1]^n$, *and* G_m *is a staircase hyperbolic cross of order* m. *Then*

$$\|f - F_{2^m}(f\,;\,x)\|_{L_q} \le C_{q,\alpha} \frac{m^{\frac{n-1}{q'}}}{2^{m(\alpha-\frac{1}{q'})}} E_{\Delta_m}(f)_{E^\alpha[0,1]^n}.$$

Corollary 1. *Suppose that* M *is the number of nodes in the interpolation formula* (1), *with* $2 \le q \le \infty$, $q' = q/(q-1)$, *and* $\alpha > 1$. *Then*

$$\sup_{\|f\|_{W_p^\alpha}=1} \|f - F_{2^m}(f\,;\,\cdot)\|_{L_q} \sim \frac{(\ln M)^{(n-1)\alpha}}{M^{(\alpha-\frac{1}{q'})}}.$$

Theorem 5. *Suppose that* $1 < p \le 2 \le q \le \infty$, $f \in W_p^\alpha[0,1]^n$, $\alpha > 1/p$, $1/r = 1/p - 1/q$, *and* G_m *is a staircase hyperbolic cross. Then the following estimate holds*:

$$\|f - F_{2^m}(f\,;\,x)\|_{L_q} \le C_{q,p,\alpha} \frac{m^{\frac{n-1}{r}}}{2^{m(\alpha-\frac{1}{r})}} E_{\Delta_m}(f)_{W_p^\alpha[0,1]^n}.$$

Corollary 2. *Suppose that* M *is the number of nodes in the interpolation formula* (1), $\alpha > 1/p$, $1 < p \le 2 \le q \le \infty$, *and* $1/r = 1/p - 1/q$. *Then*

$$\sup_{\|f\|_{W_p^\alpha}=1} \|f - F_{2^m}(f\,;\,\cdot)\|_{L_q} \sim \frac{(\ln M)^{(n-1)\alpha}}{M^{(\alpha-\frac{1}{r})}}.$$

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