

## Research Article

# On the Heat and Wave Equations with the Sturm-Liouville Operator in Quantum Calculus

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In this paper, we explore a generalised solution of the Cauchy problems for the  $q$ -heat and  $q$ -wave equations which are generated by Jackson's and the  $q$ -Sturm-Liouville operators with respect to  $t$  and  $x$ , respectively. For this, we use a new method, where a crucial tool is used to represent functions in the Fourier series expansions in a Hilbert space on quantum calculus. We show that these solutions can be represented by explicit formulas generated by the  $q$ -Mittag-Leffler function. Moreover, we prove the unique existence and stability of the weak solutions.

## 1. Introduction

In the last decade, the theory of quantum groups and  $q$ -deformed algebras have been the subject of intense investigation. Many physical applications have been investigated on the basis of the  $q$ -deformation of the Heisenberg algebra (see [1, 2]). For instance, the  $q$ -deformed Schrödinger equations have been proposed in [3, 4], and applications to the study of  $q$ -deformed version of the hydrogen atom and of the quantum harmonic oscillator have been presented (see [5]). Fractional calculus and the  $q$ -deformed Lie algebras are closely related. A new class of fractional  $q$ -deformed Lie algebras is proposed, which for the first time allows a smooth transition between the different Lie algebras (see [6]).

The origin of the  $q$ -difference calculus can be traced back to the works by Jackson (see [7, 8]) and Carmichael (see [9]) from the beginning of the twentieth century, while basic definitions

and properties can be found, e.g., in the monographs [10, 11] and the PhD thesis [12]. Recently, the fractional  $q$ -difference calculus has been proposed by Al-salam (see [13]) and Agarwal (see [14]). We can also mention papers [15, 16], where the authors investigated the explicit solutions to linear fractional  $q$ -differential equations with the  $q$ -fractional derivative, and in [17], the  $q$ -analogue nonhomogeneous wave equations were studied.

A motivation behind this work is to state some new results about the  $q$ -heat and  $q$ -wave equations associated to the  $q$ -Sturm-Liouville operator (see (10)). We attempt to extend the heat representation theory studied in some cases (see [18–20], etc.). We define a generalised solution of the Cauchy problem for these equations generated by the  $q$ -Mittag-Leffler function and the  $q$ -associated functions of a biorthogonal system (see (13)). We investigate the well-posedness of the Cauchy problem for the  $q$ -heat and  $q$ -wave equations for

operators with a discrete nonnegative spectrum acting on  $L^2_q[0, 1]$ . In particular, we prove both unique existence and stability of the corresponding the generalised solution.

The paper is organized as follows: the main results are presented and proved in Section 3 and Section 4. In order to not disturb these presentations, we include in Section 2 some necessary Preliminaries.

### 2. Preliminaries

In this section, we recall some notations and basic facts in  $q$ -calculus. We will always assume that  $0 < q < 1$ . The  $q$ -real number  $[\alpha]_q$  is defined by

$$[\alpha]_q = \frac{1 - q^\alpha}{1 - q}.$$

The  $q$ -shifted factorial is defined by

$$(a; q)_n = \begin{cases} 1, & n = 0, \\ (1 - a)(1 - aq) \cdots (1 - aq^{n-1}), & n \in \mathbb{N}. \end{cases}$$

Moreover, their natural expansions to the reals are

$$(a - b)_q^\alpha = a^\alpha \frac{(b/a; q)_\infty}{(q^\alpha b/a; q)_\infty}, (a; q)_\alpha = \frac{(a; q)_\infty}{(aq^\alpha; q)_\infty}, (a; q)_\infty = \prod_{i=0}^\infty (1 - aq^i). \tag{1}$$

The Jackson's  $q$ -difference operator  $D_q f(x)$  is (see, [8, 12] Section 2.1)]

$$D_q f(x) = \frac{f(x) - f(qx)}{x(1 - q)}. \tag{2}$$

The  $q$ -derivative  $D_q$  of a product of the functions  $f$  and  $g$  as defined by

$$D_q(fg)(x) = f(qx)D_q(g)(x) + D_q(f)(x)g(x). \tag{3}$$

As given in [10], two  $q$ -analogues of the exponential functions are defined by

$$e_q^x = \frac{1}{((1 - q)x; q)_\infty}, E_q^x = (- (1 - q)x; q)_\infty. \tag{4}$$

Moreover, we have that

$$D_q e_q^x = e_q^x, D_q E_q^{-x} = E_q^{-qx}, e_q^x E_q^{-x} = 1. \tag{5}$$

Due to the various types of  $q$ -differences introduced in quantum calculus, trigonometric functions have various  $q$ -analogues (see, [21] Section 2 [10], Section 10 and [12], Section 2.12). The following definition of cosine and sine will be useful in this investigation (see [20]):

$$\cos(z; q^2) = \sum_{k=0}^\infty \frac{(-1)^k q^{k^2} z^{2k}}{[2k]_q!}, \sin(z; q^2) = \sum_{k=0}^\infty \frac{(-1)^k q^{k(k+1)} z^{2k+1}}{[2k+1]_q!}, \tag{6}$$

where the  $q$ -analogue of the binomial coefficients  $[n]_q!$  is defined by

$$[n]_q! = \begin{cases} 1, & \text{if } n = 0, \\ [1]_q \times [2]_q \times \cdots \times [n]_q, & \text{if } n \in \mathbb{N}. \end{cases}$$

The  $q$ -integral (or Jackson's integral) is defined by (see [8])

$$\int_0^x f(t) d_q t = (1 - q)x \sum_{m=0}^\infty q^m f(xq^m), \tag{7}$$

and a more general form is given by

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x,$$

for  $0 < a < b$ .

The  $q$ -version of integration by parts reads

$$\int_a^b f(x) D_q g(x) d_q x = [fg]_a^b - \int_a^b g(qx) D_q f(x) d_q x, \tag{8}$$

and if  $f \equiv 1$ , then we get that

$$\int_a^b D_q g(x) d_q x = g(b) - g(a). \tag{9}$$

*The  $q$ -Sturm-Liouville Problem.* Let  $L^2_q[0, 1]$  be the space of all real-valued functions defined on  $[0, 1]$  such that

$$\|f\|_{L^2_q[0,1]} := \left( \int_0^1 |f(x)|^2 d_q x \right)^{1/2} < \infty.$$

The space  $L^2_q[0, 1]$  is a separable Hilbert space with the inner product:

$$(f, g) := \int_0^1 f(x)g(x) d_q x, f, g \in L^2_q[0, 1].$$

Now, we shortly describe the study introduced by Annaby and Mansour in of a basic  $q$ -Sturm-Liouville eigenvalue problem in a Hilbert space (see [21], Chapter 3). In particular, they investigated the basic  $q$ -Sturm-Liouville equation:

$$-\frac{1}{q} D_{q^{-1}} D_q y(x) + v(x)y(x) = \lambda y(x), (0 \leq x \leq 1; \lambda \in \mathbb{C}),$$

where  $v(\cdot)$  is defined on  $[0, 1]$  and continuous at zero. Let  $C^2_{q,0}[0, 1]$  denotes the space of all functions  $y(\cdot)$  such that  $y$

and  $D_q y$  are continuous at zero. If  $v \equiv 0$ , then we get the operator  $\mathcal{L}$  in the following form:

$$\mathcal{L} := \begin{cases} -\frac{1}{q} D_{q^{-1}} D_q y(x) = \lambda y(x), \\ y(0) = y(1) = 0, \end{cases} \quad (10)$$

for  $0 \leq x \leq 1$  and  $\lambda \in \mathbb{R}$ . The operator  $\mathcal{L}$  is self adjoint on  $C_{q,0}^2[0, 1] \cap L_q^2[0, 1]$  (see [21], Theorem 3.4.). A fundamental set of solutions of (10) are  $\cos(\sqrt{\lambda}; q^2)$  and  $\sin(\sqrt{\lambda}; q^2)/\sqrt{\lambda}$ . Moreover, the eigenvalues  $\{\lambda_k\}_{k=1}^\infty$  are the zeros of  $\sin(\sqrt{\lambda_k}; q^2)$ , where

$$\lambda_k = (1 - q)^{-2} q^{-2k+2\mu_k^{-1/2}}, \quad k = 0, 1, \dots, \quad (11)$$

and  $\sum_{k=1}^\infty \mu_k < \infty$ ,  $0 \leq \mu_k \leq 1$ , and

$$\lambda_0 := (1 - q)^{-2} q \leq \lambda_k, \quad k = 1, 2, 3, \dots. \quad (12)$$

Additionally, the corresponding set of eigenfunctions  $\{\sin(\sqrt{\lambda_k}; q^2)/\sqrt{\lambda_k}\}_{k=1}^\infty$  is an orthogonal basis in  $L_q^2(0, 1)$ . Thus, we can identify  $f \in L_q^2[0, 1]$  with its Fourier series:

$$f(x) := \sum_{k=1}^\infty \langle f, \phi_k \rangle \phi_k(x),$$

where

$$\phi_k(x) = \frac{\sin(\sqrt{\lambda_k} x; q^2)}{\sqrt{\lambda_k}}. \quad (13)$$

*The Sobolev Space Associated with  $\mathcal{L}$ .* The next step is to recall the essential elements of the Fourier analysis presented in [22–24], as well as its applications to the spectral properties of  $\mathcal{L}$ . The space  $C_{\mathcal{L}}^\infty[0, 1] := \bigcap_{m=1}^\infty \text{Dom}(\mathcal{L}^m)$  is called the space of test functions for  $\mathcal{L}$ , where

$$\text{Dom}(\mathcal{L}^m) := \left\{ f \in L_q^2[0, 1] : \mathcal{L}^j f \in \text{Dom}(\mathcal{L}), j = 0, 1, 2, \dots, m-1 \right\}.$$

For  $g \in C_{\mathcal{L}}^\infty[0, 1]$ , we introduce the Fréchet topology of  $C_{\mathcal{L}}^\infty[0, 1]$  by the family of norms:

$$\|g\|_{C_{\mathcal{L}}^m[0,1]} := \max_{i \leq m} \|\mathcal{L}^i g\|_{L_q^2[0,1]}.$$

The space of  $\mathcal{L}$ -distributions  $\mathcal{D}'_{\mathcal{L}}[0, 1] := L(C_{\mathcal{L}}^\infty[0, 1], \mathbb{R})$  is the space of all linear continuous functionals on  $C_{\mathcal{L}}^\infty[0, 1]$ .

Thus, for  $s \in \mathbb{R}$ , we can also define the Sobolev spaces  $W_{q,\mathcal{L}}^s$  associated to  $\mathcal{L}$  in the following form:

$$W_{q,\mathcal{L}}^s := \left\{ f \in \mathcal{D}'_{\mathcal{L}}[0, 1] : \mathcal{L}^{s/2} f \in L_q^2[0, 1] \right\},$$

with the norm  $\|f\|_{W_{q,\mathcal{L}}^s} := \|\mathcal{L}^{s/2} f\|_{L_q^2[0,1]}$ .

For  $m \in \mathbb{N}_0$ , we introduce the space  $C_q^m([0, 1]; W_{q,\mathcal{L}}^s[0, 1])$  defined by the norms

$$\|u\|_{C_q^m([0,T]; W_{q,\mathcal{L}}^s[0,1])} := \sum_{n=0}^m \max_{0 \leq t \leq T} \|D_{q,t}^n u(t, \cdot)\|_{W_{q,\mathcal{L}}^s[0,1]}, \quad 0 < T < \infty,$$

where the  $q$ -partial differential operator  $D_{q,t} u(t, x)$  with respect to  $t$  has the following form:

$$D_{q,t} u(t, x) = \frac{u(t, x) - u(qt, x)}{(1 - q)t}.$$

Notation: the symbol  $M \lesssim K$  means that there exists  $\gamma > 0$  such that  $M \leq \gamma K$ , where  $\gamma$  is a constant.

### 3. The $q$ -Heat Equation

We start with a study of the following Cauchy problem:

$$D_{q,t} u(t, x) + \mathcal{L} u(t, x) = f(t, x), \quad x \in [0, 1], t > 0, \quad (14)$$

with the initial condition

$$u(0, x) = \varphi(x), \quad x \in [0, 1]. \quad (15)$$

We say a generalised solution of the problem (14)-(15) is a function  $u(t, x)$  such that they satisfy equation (14) and condition (15).

**Theorem 1.** *We assume that  $0 < T < \infty$ . Let  $\varphi \in W_{q,\mathcal{L}}^2[0, 1]$  and  $f \in C([0, T]; W_{q,\mathcal{L}}^2[0, 1])$ . Then, there exists the generalised solution of  $u$  to problem (14)-(15), and*

$$u \in C_q^1([0, T]; L_q^2[0, 1]) \cap C([0, T]; W_{q,\mathcal{L}}^2[0, 1]). \quad (16)$$

Moreover, this solution can be written in the following explicit form

$$u(t, x) = \sum_{k \in \mathbb{N}} \left[ e_q^{-\lambda_k t} \varphi_k + e_q^{-\lambda_k t} \int_0^t E_q^{\lambda_k q s} f_k(s, \cdot) d_q s \right] \phi_k(x). \quad (17)$$

*Proof. Existence.* Since the system of eigenfunctions  $\{\phi_k\}_{k=1}^\infty$  is a basis in  $L_q^2[0, 1]$  (see (11)), we seek for a function  $u(t, x)$  in the form

$$u(t, x) = \sum_{k \in \mathbb{N}} u_k(t) \phi_k(x), \quad (18)$$

for each fixed  $0 < t < T < \infty$ . The coefficients will then be given by the Fourier coefficients formula  $u_k(t) = \langle u(t, \cdot), \phi_k \rangle$ .

We can similarly expand the source function,

$$f(t, x) = \sum_{k \in \mathbb{N}} f_k(t) \phi_k(x), f_k(t) = \langle f(t, \cdot), \phi_k \rangle. \quad (19)$$

From (11) and (18), we have that

$$\mathcal{L}\phi_k(x) = \lambda_k \phi_k(x), k \in \mathbb{N}.$$

Hence,

$$\mathcal{L}u(t, x) = \sum_{k \in \mathbb{N}_0} u_k(t) \lambda_k \phi_k(x), \quad (20)$$

and

$$D_{q,t}u(t, x) = \sum_{k \in \mathbb{N}} D_q u_k(t) \phi_k(x). \quad (21)$$

Substituting (20) and (21) into the equation (14), we find that

$$\sum_{k \in \mathbb{N}} [D_q u_k(t) + \lambda_k u_k(t)] \phi_k(x) = \sum_{k \in \mathbb{N}} f_k(t) \phi_k(x). \quad (22)$$

But then, due to the completeness,

$$D_q u_k(t) + \lambda_k u_k(t) = f_k(t), k \in \mathbb{N}, \quad (23)$$

which are ODEs for the coefficients  $u_k(t)$  of the series (18). Using the integrating factor  $E_q^{\lambda_k q t}$  and (2) and (3), we can rewrite the ODE as

$$\begin{aligned} E_q^{\lambda_k q t} f_k(t) &= E_q^{\lambda_k q t} D_q u_k(t) + E_q^{\lambda_k q t} \lambda_k u_k(t) \\ &= E_q^{\lambda_k q t} D_q u_k(t) + D_q [E_q^{\lambda_k t}] u_k(t) \\ &= D_q [E_q^{\lambda_k t} u_k(t)]. \end{aligned} \quad (24)$$

Form (3), (5), and (24), we get that

$$\int_0^t D_q [E_q^{\lambda_k t} u_k(t)] d_q s = \int_0^t E_q^{\lambda_k q s} f_k(t) f_k(s) d_q s,$$

so that

$$E_q^{\lambda_k t} u_k(t) = u_k(0) + \int_0^t E_q^{\lambda_k q t} f_k(s) d_q s,$$

which, in its turn, implies that

$$u_k(t) = \frac{u_k(0)}{E_q^{\lambda_k t}} + \frac{1}{E_q^{\lambda_k t}} \int_0^t E_q^{\lambda_k q s} f_k(s) d_q s,$$

and we conclude that

$$u_k(t) = e_q^{-\lambda_k t} u_k(0) + e_q^{-\lambda_k t} \int_0^t E_q^{\lambda_k q s} f_k(s) d_q s.$$

But the initial conditions (16) and (22) imply that  $u_k(0) = \varphi_k$ . Thus,

$$u_k(t) = e_q^{-\lambda_k t} \varphi_k + e_q^{-\lambda_k t} \int_0^t E_q^{\lambda_k q s} f_k(s, \cdot) d_q s. \quad (25)$$

Therefore, the solution  $u(t, x)$  can be written in the series form as

$$u(t, x) = \sum_{k \in \mathbb{N}} \left[ e_q^{-\lambda_k t} \varphi_k + e_q^{-\lambda_k t} \int_0^t E_q^{\lambda_k q s} f_k(s, \cdot) d_q s \right] \phi_k(x),$$

so, also (17) is proved.

Convergence. From (1), (4), and (5), we have that

$$e_q^{-x} = \frac{1}{(-1-q)x; q}_\infty \leq \frac{1}{1+(1-q)x} \leq 1, E_q^{qx} \leq E_q^x,$$

for  $x \in [0, 1]$ . Hence, using for  $0 < t < T < \infty$ , (5), (23), and (25), we get that

$$\begin{aligned} |u_k(t)| &\stackrel{(25)}{\leq} e_q^{-\lambda_k t} |\varphi_k| + \int_0^t \frac{E_q^{\lambda_k q s}}{e_q^{\lambda_k t}} |f_k(s)| d_q s \leq |\langle \varphi, \phi_k \rangle| \\ &\quad + \int_0^t |\langle f(s, \cdot), \phi_k \rangle| d_q s \leq |\langle \varphi, \phi_k \rangle| \\ &\quad + T \max_{0 \leq s \leq T} |\langle f(s, \cdot), \phi_k \rangle| \leq \max \{1, T\} \\ &\quad \cdot [|\langle \varphi, \phi_k \rangle| + \max_{0 \leq s \leq T} |\langle f(s, \cdot), \phi_k \rangle|] \leq |\langle \varphi, \phi_k \rangle| \\ &\quad + \max_{0 \leq s \leq T} |\langle f(s, \cdot), \phi_k \rangle|, \end{aligned} \quad (26)$$

and

$$\begin{aligned} |D_q u_k(t)| &\stackrel{(23)}{\leq} \lambda_k |u_k(t)| + |f_k(t)| \stackrel{(26)}{\leq} |\langle \lambda_k \varphi, \phi_k \rangle| \\ &\quad + |\langle \lambda_k f_k(t, \cdot), \phi_k \rangle| + \lambda_k^{-1} |\langle \lambda_k f_k(t, \cdot), \phi_k \rangle| \leq |\langle \lambda_k \varphi, \phi_k \rangle| \\ &\quad + (1 + \lambda_0) |\langle \lambda_k f_k(t, \cdot), \phi_k \rangle| \leq |\langle \mathcal{L}\varphi, \phi_k \rangle| \\ &\quad + \max_{0 \leq t \leq T} |\langle \mathcal{L}f_k(t, \cdot), \phi_k \rangle|. \end{aligned} \quad (27)$$

Hence,

$$\begin{aligned} |\mathcal{L}u(t, \cdot)| &= |\langle \lambda_k u_k(t), \phi_k \rangle| \stackrel{(26)}{\leq} |\langle \lambda_k \varphi, \phi_k \rangle| \\ &\quad + \max_{0 \leq s \leq T} |\langle \lambda_k f(s, \cdot), \phi_k \rangle| = |\langle \mathcal{L}\varphi, \phi_k \rangle| \\ &\quad + \max_{0 \leq s \leq T} |\langle \mathcal{L}f(s, \cdot), \phi_k \rangle|. \end{aligned} \quad (28)$$

Since  $\varphi \in W_{q, \mathcal{L}}^2$ ,  $f \in C([0, 1]; W_{q, \mathcal{L}}^2)$ , and, hence, by using the Plancherel identity and (27) and (28), we can conclude

that

$$\begin{aligned} \|u(t, \cdot)\|_{L^2_q[0,1]}^2 &= \sum_{k \in \mathbb{N}} |u_k(t)|^2 \stackrel{(3.13)}{\lesssim} \sum_{k \in \mathbb{N}} |\langle \varphi, \phi_k \rangle|^2 + \max_{0 \leq s \leq T} \sum_{k \in \mathbb{N}} |\langle f(s, \cdot), \phi_k \rangle|^2 \\ &= \|\varphi\|_{W^2_{q,\mathcal{L}}[0,1]}^2 + \|f\|_{C([0,T];W^2_{q,\mathcal{L}}[0,1])}^2 < \infty, \end{aligned}$$

and

$$\begin{aligned} \|D_q u(t, \cdot)\|_{L^2_q[0,1]}^2 &= \sum_{k \in \mathbb{N}} |D_q u_k(t, \cdot)|^2 \stackrel{(27)}{\lesssim} \sum_{k \in \mathbb{N}} |\langle \mathcal{L}\varphi, \phi_k \rangle|^2 \\ &\quad + \max_{0 \leq s \leq T} \sum_{k \in \mathbb{N}} |\langle \mathcal{L}f(s, \cdot), \phi_k \rangle|^2 \\ &= \|\varphi\|_{W^2_{q,\mathcal{L}}[0,1]}^2 + \|f\|_{C([0,T];W^2_{q,\mathcal{L}}[0,1])}^2 \\ &< \infty, \end{aligned}$$

and

$$\|\mathcal{L}u(t, \cdot)\|_{L^2_q[0,1]}^2 \stackrel{(28)}{\lesssim} \|\varphi\|_{W^2_{q,\mathcal{L}}[0,1]}^2 + \|f\|_{C([0,T];W^2_{q,\mathcal{L}}[0,1])}^2 < \infty,$$

which mean that  $u \in C^1_q([0, T]; L^2_q[0, 1]) \cap C([0, T]; W^2_{q,\mathcal{L}}[0, 1])$ .

*Uniqueness.* It only remains to prove the uniqueness of the solution. We assume the opposite; namely, that there exist functions  $u(t, x)$  and  $v(t, x)$ , which are two different solutions of problem (14)-(15). Let  $0 < t < T < \infty$ . Then, we have that

$$\begin{cases} D_{q,t}u(t, x) + \mathcal{L}u(t, x) = f(t, x), & 0 < x < 1, \\ u(0, x) = \varphi(x), & 0 \leq x \leq 1, \\ D_{q,t}v(t, x) + \mathcal{L}v(t, x) = f(t, x), & 0 < x < 1, \\ v(0, x) = \varphi(x), & 0 \leq x \leq 1. \end{cases}$$

We define  $W(t, x) = u(t, x) - v(t, x)$ . Then, the function  $W(t, x)$  is a solution of the following problem

$$\begin{cases} D_{q,t}w(t, x) + \mathcal{L}w(t, x) = 0, & 0 < x < 1, \\ w(0, x) = 0, & 0 \leq x \leq 1. \end{cases}$$

From (18), it follows that  $W(t, x) \equiv 0$ , that is,  $u(x, t) \equiv v(x, t)$ , and this contradiction to our assumption proves the uniqueness of the solution. The proof is complete.

#### 4. The $q$ -Wave Equation

In this section, we will seek for a generalised function  $u(t, x)$ , which satisfies the following  $q$ -wave equation

$$D^2_{q,t}u(t, x) + \mathcal{L}u(t, x) = f(t, x), \quad 0 < x < 1, \quad (29)$$

for  $0 < t < T < \infty$  with the initial conditions

$$u(0, x) = \psi(x), D_{q,t}u(0, x) = \eta(x), \quad 0 < x \leq 1. \quad (30)$$

**Theorem 2.** We assume that  $0 < T < \infty$ . Let  $\psi, \eta \in W^2_{q,\mathcal{L}}[0, 1]$  and  $f \in C^1_q([0, T]; W^2_{q,\mathcal{L}}[0, 1])$ . Then, there exists the generalised solution of problem (29)-(30):

$$u \in C^2_q([0, 1]; L^2_q[0, T]) \cap C([0, T]; W^2_{q,\mathcal{L}}[0, 1]).$$

Moreover, this solution can be written in the following explicit form:

$$\begin{aligned} u(t, x) &= \sum_{k \in \mathbb{N}_0} \left( \psi_k e_{2,1}(-\lambda_k t^2; q) + t \eta_k e_{2,2}(\lambda_k t^2; q) \right. \\ &\quad \left. - \frac{1}{\lambda_k} f_k(0) e_{2,1}(-\lambda_k t^2; q) - \frac{1}{\lambda_k} \int_0^t e_{2,1} \right. \\ &\quad \left. \cdot (\lambda_k(t - q^3 s)_q^2; q) D_{q,s} f_k(s) d_q s \right) \phi_k, \end{aligned} \quad (31)$$

where the  $q$ -Mittag-Leffler function  $e_{\alpha,\beta}(\lambda_k(t - q^2 s)_q^\alpha; q)$  is given by (see [25] and [26], Section 7):

$$e_{\alpha,\beta}(\lambda_k(t - qs)_q^\alpha; q) = \sum_{m=0}^{\infty} \frac{\lambda_k^m (t - qs)_q^{m\alpha}}{\Gamma_q(m\alpha + \beta)}, \quad (32)$$

for  $\alpha, \beta \in \mathbb{R}$  and  $0 < s \leq t < \infty$ , where the gamma function  $\Gamma_q(x)$  is defined by

$$\Gamma_q(x) = \frac{(q, q)_q^\infty}{(q^x, q)_q^\infty} (1 - q)^{1-x}, \quad \Gamma_q(n + 1) = [n]_q!, \quad n \in \mathbb{N}. \quad (33)$$

*Proof. Existence.* By repeating the arguments in the proof of Theorem 1, we have the Cauchy type problem:

$$D^2_q u_k(t) + \lambda_k u_k(t) = f_k(t), \quad k \in \mathbb{N}_0, \quad (34)$$

with the initial conditions

$$u_k(0) = \psi_k, D_q u_k(0) = \eta_k, \quad k \in \mathbb{N}_0, \quad (35)$$

where  $f_k(t) = \langle f(t, \cdot), \phi_k \rangle$ ,  $\psi_k = \langle \psi(\cdot), \phi_k \rangle$  and  $\eta_k = \langle \eta(\cdot), \phi_k \rangle$ .

Then, the solution to this Cauchy type in problem (29)-(30) is given (see [25], Example 6)

$$\begin{aligned} u_k(t) &= \psi_k e_{2,1}(-\lambda_k t^2; q) + t \eta_k e_{2,2}(-\lambda_k t^2; q) \\ &\quad + \int_0^t (t - qs) e_{2,2}(-\lambda_k(t - q^2 s)_q^2; q) f_k(s) d_q s. \end{aligned} \quad (36)$$

By using (2) and we find that

$$\begin{aligned}
 & D_{q,s} \left[ e_{2,1} \left( -\lambda_k (t - q^2 s)^2; q \right) \right] \\
 &= - \sum_{k \in \mathbb{N}} \frac{(-\lambda_k)^m}{\Gamma_q(2m+1)} [2m]_q (t - q^3 s)^{2m-1}_q \\
 &= \lambda_k (t - qs) \sum_{k \in \mathbb{N}} \frac{(-\lambda_k)^{m-1}}{\Gamma_q(2m)} (t - q^2 s)^{2m-2}_q \\
 &= \lambda_k (t - qs) e_{2,2} \left( \lambda_k (t - q^2 s)^2; q \right).
 \end{aligned} \tag{37}$$

By applying (8) and using (36) and (37), we get that

$$\begin{aligned}
 u_k(t) &= \psi_k e_{2,1}(-\lambda_k t^2; q) + t \eta_k e_{2,2}(-\lambda_k t^2; q) \\
 &+ \frac{1}{\lambda_k} \int_0^t D_{q,s} \left[ e_{2,1} \left( -\lambda_k (t - q^2 s)^2; q \right) \right] f_k(s) d_q s \\
 &= \psi_k e_{2,1}(-\lambda_k t^2; q) + t \eta_k e_{2,2}(-\lambda_k t^2; q) \\
 &+ \frac{1}{\lambda_k} f_k(t) e_{2,1} \left( -\lambda_k (t - q^2)^2; q \right) \\
 &- \frac{1}{\lambda_k} f_k(0) e_{2,1}(-\lambda_k t^2; q) \\
 &- \frac{1}{\lambda_k} \int_0^t e_{2,1} \left( -\lambda_k (t - q^3 s)^2; q \right) D_{q,s} f_k(s) d_q s.
 \end{aligned} \tag{38}$$

Since  $e_{2,1}(-\lambda_k t^2(q^2; q)_2) \equiv 0$  (see [21], Theorem 7.12), by using (18) and (38), it follows that solution exists and can be written as

$$\begin{aligned}
 u(t, x) &= \sum_{k \in \mathbb{N}} \left( \psi_k e_{2,1}(-\lambda_k t^2; q) + t \eta_k e_{2,2}(\lambda_k t^2; q) \right. \\
 &- \frac{1}{\lambda_k} f_k(0) e_{2,1}(-\lambda_k t^2; q) - \frac{1}{\lambda_k} \int_0^t e_{2,1} \\
 &\cdot \left. \left( -\lambda_k (t - q^3 s)^2; q \right) D_{q,s} f_k(s) d_q s \right) \phi_k,
 \end{aligned}$$

i.e., on the explicit form (34).

Convergence. Firstly, using the results in [27], Lemma 6 and in [17], Lemma 1 for the  $q$ -trigonometric functions in (6), we see that  $e_{2,2}(-\lambda_k t^2; q)$  and  $e_{2,1}(-\lambda_k t^2; q)$  are also bounded with  $t > 0$ . Then, forms (4), (12), and (32) follow that

$$\begin{aligned}
 |e_{2,2}(-\lambda_k t^2; q)| &= \left| \sum_{m=0}^{\infty} \frac{(-\lambda_k)^m t^{2m}}{[2m+1]_q!} \right| \\
 &\leq \sum_{m=0}^{\infty} \frac{\lambda_k^m T^{2m}}{[2m+1]_q!} \\
 &= \frac{\sin(\sqrt{\lambda_k} T; q^2)}{2T \sqrt{\lambda_k}} \\
 &\leq \frac{C_{1,q}}{2T \sqrt{\lambda_k}},
 \end{aligned} \tag{39}$$

and

$$\begin{aligned}
 |e_{2,1}(-\lambda_k t^2; q)| &= \left| \sum_{m=0}^{\infty} \frac{(-1)^m (T \sqrt{\lambda_k})^{2m}}{[2m]_q!} \right| \\
 &\leq \cos(\sqrt{\lambda_k} T; q^2) \leq C_{2,q},
 \end{aligned} \tag{40}$$

where  $C_{1,q}, C_{2,q}$  are any constant which only depends on  $q$ . Next, by using (38), (39), and (40), we obtain that

$$\begin{aligned}
 |u_k(t)| &\stackrel{(40)}{\lesssim} |\langle \psi, \phi_k \rangle| + |\langle \eta, \phi_k \rangle| + \frac{|f(0, \cdot), \phi_k|}{\lambda_k} \\
 &+ \frac{1}{\lambda_k} \int_0^t |\langle D_q f(s, \cdot), \phi_k \rangle| d_q s \leq |\langle \psi, \phi_k \rangle| + |\langle \eta, \phi_k \rangle| \\
 &+ \frac{|f(0, \cdot), \phi_k|}{\lambda_k} + \frac{T}{\lambda_k} \max_{0 \leq s \leq T} |\langle D_q f(s, \cdot), \phi_k \rangle| \leq C_{\lambda_0} \\
 &\cdot \left[ |\langle \psi, \phi_k \rangle| + |\langle \eta, \phi_k \rangle| + \sum_{m=0}^1 \max_{0 \leq t \leq T} |\langle D_q^m f(t, \cdot), \phi_k \rangle| \right],
 \end{aligned} \tag{41}$$

where  $C_{\lambda_0} := \max \{1, (1/\lambda_0), (T/\lambda_0)\}$ .

Therefore, by using (7), (8), (30), (34), (33), and (41), we have that

$$\begin{aligned}
 |D_q u_k(t)| &= \left| -D_q u_k(0) + \int_0^t D_q^2 u_k(s) d_q s \right| \stackrel{(30)(34)}{\leq} |\langle \eta, \phi_k \rangle| \\
 &+ \int_0^t |f(s, \cdot), \phi_k| d_q s \\
 &+ \lambda_k \int_0^t |u_k(s)| d_q s \stackrel{(41)}{\lesssim} \frac{1}{\lambda_k} |\langle \lambda_k \eta, \phi_k \rangle| \\
 &+ \frac{T}{\lambda_k} \max_{0 \leq s \leq T} |\langle \lambda_k f(s, \cdot), \phi_k \rangle| + TC_{\lambda_0} \\
 &\cdot \left[ |\langle \lambda_k \psi, \phi_k \rangle| + |\langle \lambda_k \eta, \phi_k \rangle| + \sum_{m=0}^1 \max_{0 \leq t \leq T} |\langle \lambda_k D_q^m f(t, \cdot), \phi_k \rangle| \right] \\
 &\lesssim |\langle \mathcal{L} \psi, \phi_k \rangle| + |\langle \mathcal{L} \eta, \phi_k \rangle| + \sum_{m=0}^1 \max_{0 \leq t \leq T} |\langle \mathcal{L} D_q^m f(t, \cdot), \phi_k \rangle|,
 \end{aligned} \tag{42}$$

and

$$\begin{aligned}
 |D_a^2 u_k(t)| &\stackrel{(34)}{\lesssim} \lambda_k |u_k(t)| + |f_k(t)| \stackrel{(41)}{\lesssim} |\langle \lambda_k \psi, \phi_k \rangle| + |\langle \lambda_k \eta, \phi_k \rangle| \\
 &+ \sum_{m=0}^1 \max_{0 \leq t \leq T} |\langle \lambda_k D_q^m f(t, \cdot), \phi_k \rangle| = |\langle \mathcal{L} \psi, \phi_k \rangle| + |\langle \mathcal{L} \eta, \phi_k \rangle| \\
 &+ \sum_{m=0}^1 \max_{0 \leq t \leq T} |\langle \mathcal{L} D_q^m f(t, \cdot), \phi_k \rangle|.
 \end{aligned} \tag{43}$$

Thus,

$$\begin{aligned} \|u(t)\|_{L^2_q[0,1]}^2 &= \sum_{k \in \mathbb{N}} |u_k(t)|^2 \stackrel{(41)}{\leq} \sum_{k \in \mathbb{N}} |\langle \psi, \phi_k \rangle|^2 + \sum_{k \in \mathbb{N}} |\langle \eta, \phi_k \rangle|^2 \\ &\quad + \sum_{m=0}^1 \max_{0 \leq t \leq T} \sum_{k \in \mathbb{N}} \left| \langle D_q^m f(t, \cdot), \phi_k \rangle \right|^2 \\ &= \|\eta\|_{W^2_{q,\mathcal{L}}[0,1]}^2 + \|\psi\|_{W^2_{q,\mathcal{L}}[0,1]}^2 \\ &\quad + \|f\|_{C^1_q([0,T];W^2_{q,\mathcal{L}}[0,1])}^2 < \infty, \end{aligned}$$

$$\begin{aligned} \|D_q u(t)\|_{L^2_q[0,1]}^2 &= \sum_{k \in \mathbb{N}} |D_q u_k(t)|^2 \leq \sum_{k \in \mathbb{N}} |\langle \mathcal{L}\psi, \phi_k \rangle|^2 \\ &\quad + \sum_{k \in \mathbb{N}} |\langle \mathcal{L}\eta, \phi_k \rangle|^2 \\ &\quad + \sum_{m=0}^1 \max_{0 \leq t \leq T} \left| \sum_{k \in \mathbb{N}} \langle \mathcal{L}D_q^m f(t, \cdot), \phi_k \rangle \right|^2 \\ &\leq \|\eta\|_{W^2_{q,\mathcal{L}}[0,1]}^2 + \|\psi\|_{W^2_{q,\mathcal{L}}[0,1]}^2 \\ &\quad + \|f\|_{C^1_q([0,T];W^2_{q,\mathcal{L}}[0,1])}^2 < \infty, \end{aligned}$$

and

$$\begin{aligned} \|D_q^2 u(t)\|_{L^2_q[0,1]}^2 &= \sum_{k \in \mathbb{N}} |D_q^2 u_k(t)|^2 \leq \|\eta\|_{W^2_{q,\mathcal{L}}[0,1]}^2 \\ &\quad + \|\psi\|_{W^2_{q,\mathcal{L}}[0,1]}^2 + \|f\|_{C^1_q([0,T];W^2_{q,\mathcal{L}}[0,1])}^2 < \infty, \end{aligned}$$

and

$$\begin{aligned} \|\mathcal{L}u(t)\|_H^2 &= \sum_{k \in I} |\langle \mathcal{L}u(t), \phi_k \rangle_H|^2 \\ &= \sum_{k \in I} [\lambda_k |u_k(t)|]^2 \leq \|\eta\|_{W^2_{q,\mathcal{L}}[0,1]}^2 \\ &\quad + \|\psi\|_{W^2_{q,\mathcal{L}}[0,1]}^2 \\ &\quad + \|f\|_{C^1_q([0,T];W^2_{q,\mathcal{L}}[0,1])}^2, \end{aligned}$$

which is means that  $u \in C^2_q([0, 1]; L^2_q[0, T]) \cap C_q([0, T]; W^2_{q,\mathcal{L}}[0, 1])$ .

Uniqueness. This part can be proved completely similar as the proof of Theorem 1.. So we omit the details.

### Data Availability

Data supporting this manuscript are available from Scopus, Web of Science, and Google Scholar.

### Conflicts of Interest

The authors declare that they have no conflicts of interest.

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