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On the second boundary value problem for the equation of heat conduction in an unbounded plane angle

In the article, the second homogeneous boundary value problem is considered in an infinite angular domain. Solution of the problem is reduced to solving the singular Volterra integral equations of the second kind with kernel whose norm is equal to unity. By the method of Carleman-Vekua, solving the integral equation is reduced to solving the inhomogeneous equation of Abel. The theorem on the existence of a non-trivial solution of the second homogeneous boundary value problem in a non-cylindrical domain is proved. The solution of the given problem is obtained in an explicit form.

Key words: singular Volterra integral equation, Abel equation, non-cylindrical domain, non-trivial solution.

The need to study boundary value problems of heat conduction (diffusion) in the domain with moving boundaries is dictated by numerous practical applications in modeling the processes of electrocontact apparatuses in a related field of designing the plasma torches, the creation of new technologies, production of crystals, laser technology and other industries. Mathematical modeling these processes allows to carry out the optimal choice of parameters and operating modes of technological equipment and maximize economic and ecological benefits. Actuality of studies of parabolic boundary value problems in non-cylindrical domains is conditioned by this aspect.

The complexity in finding analytical solutions of heat conduction problems (diffusion) in domains with moving boundaries is determined by the fact that classical methods of differential equations of mathematical physics are not applicable to this type of problems directly. Staying within these methods, the solutions can not be reconciled with the movement of the domain boundary. The last proposition is equally characteristic for boundary value problems of non-stationary and stationary transfer with diverse boundary conditions on the lines.

Constructive methods for solving thermal problems for parabolic equations based on using thermal potentials and the reducing the initial boundary value problems to integral equations were developed by E.I.Kim [1].

To find the analytical solutions to these classes of transfer problems the special techniques or modification of known approaches are needed. Presentation of the results accumulated in the field of analytical theory of heat conductivity of solids is given in Refs [2, 3].

The most of researchers [4, 5] consider such problems in-mainly in non-cylindrical domains without singular point. In paper [6] it is established the asymptotically exponential convergence of solutions to the solution of the elliptic problem defined in the spatial domain, independent of time. In [7] some inverse problems for the heat conduction equation in non-cylindrical domain are studied. In [8] space-time Brownian motion and the heat conduction equation in non-cylindrical domains are studied. In [9] for the study of the problem the authors go on to a weak formulation and then prove the existence and uniqueness of the solution.

In spite of numerous studies conducted by many authors for boundary value problems of heat conduction in non-cylindrical degenerating domains to date efficient numerical algorithms for their solution do not exist, mainly because of the lack of the mathematical theory of these problems. This has determined the theme for this work, its actuality and content.

1 Statement of the problem

We consider the boundary value problem of heat conduction in the degenerating domain (domain with a moving boundary).

In the domain $G=\{(x;t):\quad t>0,\quad 0<x<t\}$ it is required to find a solution the heat conduction equation

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}, \tag{1}$$

satisfying the boundary conditions:

$$\left.\frac{\partial u}{\partial x}\right|_{x=0}=0, \tag{2}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=t}=0. \tag{3}$$

2 Reduction of the problem to an integral equation

We are looking for solution of the boundary problem (1)–(2) as the sum of the heat potentials of the simple layer:

$$\begin{gathered}u(x,t)=\frac{1}{2a\sqrt{\pi}}\int_0^t\frac{1}{\sqrt{t-\tau}}\exp\left\{-\frac{x^2}{4a^2(t-\tau)}\right\}\nu(\tau)\,d\tau+\\+\frac{1}{2a\sqrt{\pi}}\int_0^t\frac{1}{(t-\tau)^{1/2}}\exp\left\{-\frac{(x-\tau)^2}{4a^2(t-\tau)}\right\}\varphi(\tau)\,d\tau.\end{gathered} \tag{4}$$

It is known that function (4) satisfies the equation (1) for any $\nu(t)$ and $\varphi(t)$.

Since the:

$$\frac{\partial u}{\partial x}=-\frac{1}{4a^3\sqrt{\pi}}\int_0^t\frac{x}{(t-\tau)^{3/2}}\exp\left\{-\frac{x^2}{4a^2(t-\tau)}\right\}v(\tau)\,d\tau-$$

$$-\frac{1}{4a^3\sqrt{\pi}}\int_0^t\frac{x-\tau}{(t-\tau)^{3/2}}\exp\left\{-\frac{(x-\tau)^2}{4a^2(t-\tau)}\right\}\varphi(\tau)\,d\tau$$

then from the properties of heat potentials, we have:

$$\left.\frac{\partial u}{\partial x}\right|_{x=0}=-\frac{v(t)}{2a^2}+\frac{1}{4a^3\sqrt{\pi}}\int_0^t\frac{\tau}{(t-\tau)^{3/2}}\exp\left\{-\frac{\tau^2}{4a^2(t-\tau)}\right\}\varphi(\tau)\,d\tau. \tag{5}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=t}=-\frac{1}{4a^3\sqrt{\pi}}\int_0^t\frac{t}{(t-\tau)^{3/2}}\exp\left\{-\frac{t^2}{4a^2(t-\tau)}\right\}v(\tau)\,d\tau+$$
$$+\frac{\varphi(t)}{2a^2}-\frac{1}{4a^3\sqrt{\pi}}\int_0^t\frac{1}{(t-\tau)^{1/2}}\exp\left\{-\frac{t-\tau}{4a^2}\right\}\varphi(\tau)\,d\tau. \tag{6}$$

Using conditions (2)–(3) and the properties of heat potentials, we have the following system of integral equations for the unknown densities $v(t)$ and $\varphi(t)$:

$$\begin{cases}-\dfrac{v(t)}{2a^2}+\dfrac{1}{4a^3\sqrt{\pi}}\displaystyle\int_0^t\frac{\tau}{(t-\tau)^{3/2}}\exp\left\{-\frac{\tau^2}{4a^2(t-\tau)}\right\}\varphi(\tau)\,d\tau=0;\\ -\dfrac{1}{4a^3\sqrt{\pi}}\displaystyle\int_0^t\frac{t}{(t-\tau)^{3/2}}\exp\left\{-\frac{t^2}{4a^2(t-\tau)}\right\}v(\tau)\,d\tau+\\ +\dfrac{\varphi(t)}{2a^2}-\dfrac{1}{4a^3\sqrt{\pi}}\displaystyle\int_0^t\frac{1}{(t-\tau)^{1/2}}\exp\left\{-\frac{t-\tau}{4a^2}\right\}\varphi(\tau)\,d\tau=0.\end{cases} \tag{7}$$

We express from the first equation of system (7) function $v(t)$:

$$v(t)=\frac{1}{2a\sqrt{\pi}}\int_0^t\frac{\tau}{(t-\tau)^{3/2}}\exp\left\{-\frac{\tau^2}{4a^2(t-\tau)}\right\}\varphi(\tau)\,d\tau. \tag{8}$$

We substitute (8) into the second equation of system (7):

$$-\frac{1}{8a^4\pi}\int_0^t\frac{t}{(t-\tau)^{3/2}}\exp\left\{-\frac{t^2}{4a^2(t-\tau)}\right\}\int_0^\tau\frac{\theta}{(\tau-\theta)^{3/2}}\exp\left\{-\frac{\theta^2}{4a^2(\tau-\theta)}\right\}\varphi(\theta)\,d\theta\,d\tau+$$
$$+\frac{\varphi(t)}{2a^2}-\frac{1}{4a^3\sqrt{\pi}}\int_0^t\frac{1}{(t-\tau)^{1/2}}\exp\left\{-\frac{t-\tau}{4a^2}\right\}\varphi(\tau)\,d\tau=0 \tag{9}$$

We introduce the following notation:

$$J(t)=\int_0^t\frac{t}{(t-\tau)^{3/2}}\exp\left\{-\frac{t^2}{4a^2(t-\tau)}\right\}\int_0^\tau\frac{\theta}{(\tau-\theta)^{3/2}}\exp\left\{-\frac{\theta^2}{4a^2(\tau-\theta)}\right\}\varphi(\theta)\,d\theta\,d\tau.$$

The integral $J(t)$ has the property commute, in the sense of Dirichlet formula, if $J(t)\in M_\beta(h)=\left\{\varphi(t):\ \lim\limits_{t\to 0}\dfrac{\varphi(t)}{t^\beta}=h=const,\quad h\neq 0\right\},\quad \beta>-1$. Then we change the order of integration and equation (9) can be rewritten as

$$\varphi(t)-\frac{1}{2a\sqrt{\pi}}\int_0^t\frac{1}{(t-\tau)^{1/2}}\exp\left\{-\frac{t-\tau}{4a^2}\right\}\varphi(\tau)\,d\tau-\frac{1}{4a^2\pi}\int_0^t\varphi(\theta)\,I(t,\theta)\,d\theta=0, \tag{10}$$

where

$$I(t,\theta)=\int_{\theta}^{t}\frac{t\theta}{(t-\tau)^{3/2}(\tau-\theta)^{3/2}}\exp\left\{-\frac{t^{2}}{4a^{2}(t-\tau)}-\frac{\tau^{2}}{4a^{2}(t-\tau)}\right\}d\tau. \qquad (11)$$

For integral (11) we make the substitution of the form:

$$z=\sqrt{\frac{t-\tau}{\tau-\theta}}.$$

Then

$$\tau=\frac{t+z^{2}\theta}{1+z^{2}};\quad t-\tau=\frac{z^{2}(t-\theta)}{1+z^{2}};\quad \tau-\theta=\frac{t-\theta}{1+z^{2}};\quad d\tau=\frac{-2z(t-\theta)}{\left(1+z^{2}\right)^{2}}dz.$$

After substituting the integral $I(t,\theta)$ takes the form

$$I(t,\theta)=\frac{2t\theta}{(t-\theta)^{2}}\exp\left\{-\frac{t^{2}+\theta^{2}}{4a^{2}(t-\theta)}\right\}\int_{0}^{\infty}\left(\frac{1}{z^{2}}+1\right)\exp\left\{-\frac{t^{2}}{4a^{2}(t-\theta)z^{2}}-\frac{\theta^{2}z^{2}}{4a^{2}(t-\theta)}\right\}dz.$$

Using the known equality

$$\int_{0}^{\infty}\exp\left\{-\mu x^{2}-\frac{\eta}{x^{2}}\right\}dx=\frac{1}{2}\frac{\sqrt{\pi}}{\sqrt{\eta}}\exp\left\{-2\sqrt{\mu\eta}\right\},$$

we reach the result (for the first integral we have previously introduced a replacement $x=\frac{1}{z}$):

$$I(t,\theta)=\frac{2a\sqrt{\pi}\,(t+\theta)}{(t-\theta)^{3/2}}\exp\left\{-\frac{(t+\theta)^{2}}{4a^{2}(t-\theta)}\right\}. \qquad (12)$$

Expression (12) we substitute into equation (10):

$$\varphi(t)-\frac{1}{2a\sqrt{\pi}}\int_{0}^{t}\frac{1}{(t-\tau)^{1/2}}\exp\left\{-\frac{t-\tau}{4a^{2}}\right\}\varphi(\tau)\,d\tau-$$

$$-\frac{1}{2a\sqrt{\pi}}\int_{0}^{t}\frac{t+\tau}{(t-\tau)^{3/2}}\exp\left\{-\frac{(t+\tau)^{2}}{4a^{2}(t-\tau)}\right\}\varphi(\tau)\,d\tau=0.$$

Introducing the notation

$$K(t,\tau)=\frac{1}{2a\sqrt{\pi}}\left\{\frac{1}{(t-\tau)^{1/2}}\exp\left(-\frac{t-\tau}{4a^{2}}\right)+\frac{t+\tau}{(t-\tau)^{3/2}}\exp\left(-\frac{(t+\tau)^{2}}{4a^{2}(t-\tau)}\right)\right\}, \qquad (13)$$

we obtain

$$\varphi(t)-\int_{0}^{t}K(t,\tau)\,\varphi(\tau)\,d\tau=0. \qquad (14)$$

We note that the kernel $\mathrm{K}(t,\tau)$ has the following properties:

1) $K(t,\tau)\geq 0$ and continuously at $0<\tau\leq t\leq 1$;

2) $\lim\limits_{t\to t_0}\int_{t_0}^{t}K(t,\tau)\,d\tau=0,\ t_0\geq\varepsilon>0$;

3) $\lim\limits_{t\to 0}\int_{0}^{t}K(t,\tau)\,d\tau=1,\ \lim\limits_{t\to\infty}\int_{0}^{t}K(t,\tau)\,d\tau=1.$

Properties 1) and 2) are obvious. We prove property 3) for the kernel (13), that is, we show that

$$\lim_{t\to 0}\int_{0}^{t}\frac{1}{2a\sqrt{\pi}}\left\{\frac{t+\tau}{(t-\tau)^{3/2}}\exp\left(-\frac{(t+\tau)^{2}}{4a^{2}(t-\tau)}\right)+\frac{1}{(t-\tau)^{1/2}}\exp\left(-\frac{t-\tau}{4a^{2}}\right)\right\}d\tau=1.$$

We make the substitution:

$$x=\sqrt{t-\tau}.$$

Then we obtain

$$\int_0^t K(t,\tau)\,d\tau = \frac{2}{\sqrt{\pi}} e^{\frac{2t}{a^2}} \int_0^{\sqrt{t}} \exp\left\{-\left(\frac{t}{ax}+\frac{x}{2a}\right)^2\right\}\left(\frac{t}{a\,x^2}-\frac{1}{2a}\right)dx + \frac{1}{a\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-\frac{x^2}{4a^2}}\,dx =$$

$$= \left\| z = \frac{t}{ax}+\frac{x}{2a}; \quad \xi = \frac{x}{2a} \right\| = e^{\frac{2t}{a^2}}\, erfc\left(\frac{3\sqrt{t}}{2a}\right) + erf\left(\frac{\sqrt{t}}{2a}\right).$$

Hear

$$erf\left(t\right) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-z^2}\,dz; \quad erfc\left(t\right) = \frac{2}{\sqrt{\pi}}\int_t^\infty e^{-z^2}\,dz.$$

That means

$$\lim_{t\to 0}\int_0^t K(t,\tau)\,d\tau = 1, \ \lim_{t\to\infty}\int_0^t K(t,\tau)\,d\tau = 1.$$

3 Investigating the integral equation

Feature of the investigated equation consists in property 3) of the kernel $K(t,\tau)$ and expressed in the fact that the corresponding inhomogeneous equation can not be solved by the method of successive approximations. Equations of this type were first considered in the works of S.N. Kharin, in which the asymptotics of integrals of potential type was studied and approximate solutions of some applied problems are constructed [10, 11]. He proposed and justified the method in which the solution of the integral equation is represented in the form of an asymptotic expansion in half-integer powers of the variable t. And later, integral equation (14) has been the subject of research by many authors. In the general case the integral equations whose kernels have the property 3), (such equations are called by us Volterra integral equations with «incompressible» kernel) are considered in [12].

It should be noted that to this kind of singular integral equations also boundary value problems for spectrally loaded parabolic equations are reduced when the load line moves by law $x = \alpha(t)$ [13, 14].

We consider homogeneous equation (14):

$$\varphi(t) - \frac{1}{2a\sqrt{\pi}}\int_0^t\left[\frac{t+\tau}{(t-\tau)^{3/2}}\exp\left\{-\frac{(t+\tau)^2}{4a^2(t-\tau)}\right\} + \frac{1}{(t-\tau)^{1/2}}\exp\left\{-\frac{t-\tau}{4a^2}\right\}\right]\varphi(\tau)\,d\tau = 0, \tag{15}$$

$(t>0)$

Using the relations:

$$t+\tau = 2t-(t-\tau), \quad \frac{(t+\tau)^2}{4a^2\left(t-\tau\right)} = \frac{t\tau}{a^2\left(t-\tau\right)} + \frac{t-\tau}{4a^2},$$

we obtain

$$\varphi(t) - \int_0^t \frac{1}{2a\sqrt{\pi}}\left\{\frac{2t}{(t-\tau)^{3/2}}\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\} - \frac{1}{(t-\tau)^{1/2}}\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\} + \frac{1}{(t-\tau)^{1/2}}\right\}\times$$
$$\times\exp\left\{-\frac{t-\tau}{4a^2}\right\}\varphi(\tau)\,d\tau = 0. \tag{16}$$

It is known that if the solution of the integral equation

$$y(x) + \int_a^x K\left(x,t\right) y(t)\,dt = f(x),$$

is given by formula $y(x) = f(x) + \int_a^x R\left(x,t\right) f(t)\,dt,$ then the solution of the equation [15; 183]

$$y(x) + \int_a^x K\left(x,t\right)\, e^{\alpha(x-t)}\, y(t)\,dt = f(x);$$

has the form

$$y(x)=f(x)+\int_a^x R(x,t)\,e^{\alpha(x-t)}f(t)\,dt.$$

Therefore it is sufficient to find a solution of «simplified» equation

$$\varphi(t)-\int_0^t k(t,\tau)\varphi(\tau)\,d\tau=0, \qquad (17)$$

where

$$k(t,\tau)=\frac{1}{2a\sqrt{\pi}}\left\{\frac{2t}{(t-\tau)^{3/2}}\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}+\frac{1}{(t-\tau)^{1/2}}\left(1-\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}\right)\right\}.$$

4 Solving the characteristic equation

To investigate complete equation (17) we distinguish its characteristic part, namely:

$$\varphi(t)-\int_0^t k_o(t,\tau)\varphi(\tau)\,d\tau=f_1(t), \qquad (18)$$

where

$$k_o(t,\tau)=\frac{t}{a\sqrt{\pi}(t-\tau)^{3/2}}\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\};$$

$$f_1(t)=\int_0^t k_h(t,\tau)\,\varphi(\tau)\,d\tau, \qquad (19)$$

where

$$k_h(t,\tau)=\frac{1}{2a\sqrt{\pi}(t-\tau)^{1/2}}\left(1-\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}\right).$$

Equation (18) is characteristic equation for (16), as:

$$\lim_{t\to0}\int_0^t k_o(t,\tau)\,d\tau=1;\ \lim_{t\to0}\int_0^t k_h(t,\tau)\,d\tau=0.$$

Indeed

$$\lim_{t\to0}\int_0^t k_o(t,\tau)\,d\tau=\frac{1}{a\sqrt{\pi}}\lim_{t\to0}\int_0^t\frac{t}{(t-\tau)^{3/2}}\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}d\tau=\left\|z=\frac{t}{a\sqrt{t-\tau}}\right\|=$$

$$=\frac{2}{\sqrt{\pi}}\lim_{t\to0}\int_{\frac{\sqrt{t}}{a}}^{\infty}\exp\left\{-\left(z^2-\frac{t}{a^2}\right)\right\}dz=\lim_{t\to0}\left\{e^{\frac{t}{a^2}}\cdot erfs\left(\frac{\sqrt{t}}{a}\right)\right\}=1.$$

The validity of equality

$$\lim_{t\to0}\int_0^t k_h(t,\tau)\,d\tau=0,$$

follows from the estimate

$$k_h(t,\tau)=\frac{1}{2a\sqrt{\pi}(t-\tau)^{1/2}}\left(1-\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}\right)\le\frac{1}{2a\sqrt{\pi}(t-\tau)^{1/2}}\cdot\frac{t\tau}{a^2(t-\tau)}=\frac{t\tau}{2a^3\sqrt{\pi}(t-\tau)^{3/2}},$$

i.e., the function $k_h(t,\tau)$ has a weak singularity.

Assuming that the right side of equation (18) is known, we find its solution, i.e. solution of the characteristic equation (18).

Similarly, [16; 174], we reduce the integral equation (18) to an equation with a difference kernel. To do this, we will make in it replacements:

$$t=\frac{1}{y},\quad \tau=\frac{1}{x};\ \psi(y)=\frac{1}{\sqrt{y}}\varphi\left(\frac{1}{y}\right);\quad f_2(y)=\frac{1}{\sqrt{y}}f_1\left(\frac{1}{y}\right). \qquad (20)$$

Then we obtain the equation of the form

$$\psi(y)-\int_{y}^{\infty}\frac{1}{a\sqrt{\pi}(x-y)^{3/2}}\exp\left\{-\frac{1}{a^2(x-y)}\right\}\psi(x)\,dx=f_2(y).\quad (y>0) \tag{21}$$

The solution of equation (21) can be found by an operational method [16] or by reducing it to the Riemann boundary value problem [16]. The index of boundary value problem in this case is equal to 1 and the function $\psi(y)=C\quad (C-const)$ is a solution of the homogeneous equation

$$\psi(y)-\int_{y}^{\infty}\frac{1}{a\sqrt{\pi}(x-y)^{3/2}}\exp\left\{-\frac{1}{a^2(x-y)}\right\}\psi(x)\,dx=0,$$

corresponding (21).

The solution of inhomogeneous equation (21) has the form

$$\psi(y)=f_2(y)+\int_{y}^{\infty}r_{-}(y-x)\,f_2(x)\,dx+C,\quad \left(C-const\right) \tag{22}$$

where

$$r_{-}(y)=\frac{1}{a\sqrt{\pi}(-y)^{3/2}}\sum_{n=1}^{\infty}n\cdot\exp\left\{-\frac{n^2}{a^2(-y)}\right\}.$$

Making reverse substitution (20) to (22), we obtain the solution of inhomogeneous equation (18):

$$\varphi(t)=f_1(t)+\int_{0}^{t}r(t,\tau)\,f_1(\tau)\,d\tau+\frac{C}{\sqrt{t}}, \tag{23}$$

where

$$r(t,\tau)=\frac{t}{a\sqrt{\pi}(t-\tau)^{3/2}}\sum_{n=1}^{\infty}n\cdot\exp\left\{-n^2\frac{t\tau}{a^2(t-\tau)}\right\}. \tag{24}$$

We will obtain an estimate for the resolvent. Since

$$|\theta|^{-3/2}\sum_{n=1}^{\infty}n\cdot\exp\left\{-\frac{n^2}{a^2|\theta|}\right\}\le\frac{a^2}{2\sqrt{|\theta|}}\int_{1}^{\infty}\exp\left\{-\frac{y^2}{a^2|\theta|}\right\}d\left(\frac{y^2}{a^2|\theta|}\right)=\frac{a^2}{2\sqrt{|\theta|}}\exp\left\{-\frac{1}{a^2|\theta|}\right\},$$

then

$$|r(t,\tau)|\le\frac{a}{2\sqrt{\pi}}\cdot\frac{1}{\tau\sqrt{t-\tau}}\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}.$$

Then a necessary condition for the function $f_1(t)$ is

$$|f_1(t)|\le M\cdot t^{\varepsilon};\quad \varepsilon>0.$$

5 Reducing the initial «simplified» equation to Abel equation

We will now proceed to solving equation (17), i.e. «simplified» version of initial equation (15).

Using the formula for the solution of characteristic equation (23), taking into account relations (19) for the function $f_1(t)$, we obtain

$$\varphi(t)=\int_{0}^{t}\frac{1}{2a\sqrt{\pi(t-\tau)}}\left(1-\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}\right)\varphi(\tau)\,d\tau+$$

$$+\int_{0}^{t}r(t,\tau)\left(f(\tau)+\int_{0}^{\tau}\frac{1}{2a\sqrt{\pi(\tau-\tau_1)}}\left(1-\exp\left\{-\frac{\tau\tau_1}{a^2(\tau-\tau_1)}\right\}\right)\varphi(\tau_1)\,d\tau_1\right)d\tau+\frac{C}{\sqrt{t}}.$$

Changing the order of integration in the right-hand side of obtained equation and interchanging the roles of τ and τ_1 we have

$$\varphi(t)=\int_0^t\left\{\frac{1}{2a\sqrt{\pi(t-\tau)}}\left(1-\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}\right)+\int_\tau^t r(t,\tau_1)\frac{1}{2a\sqrt{\pi(\tau_1-\tau)}}\times\right.$$
$$\left.\times\left(1-\exp\left\{-\frac{\tau_1\tau}{a^2(\tau_1-\tau)}\right\}\right)d\tau_1\right\}\varphi(\tau)\,d\tau+\frac{C}{\sqrt{t}} \qquad (25)$$

Calculating the inner integral into (25) and taking into account formula (24), after simple transformations we obtain [17]

$$\varphi(t)-\frac{1}{2a\sqrt{\pi}}\int_0^t\frac{\varphi(\tau)}{\sqrt{t-\tau}}d\tau=\frac{C}{\sqrt{t}}. \qquad (26)$$

Thus, initial «simplified» integral equation (17) was reduced to equation (26) that is Abel integral equation of the second kind.

6 Solving the Abel equation

The solution of the Abel equation of the second kind [15; 117]

$$y(x)+\lambda\int_a^x\frac{y(t)}{\sqrt{x-t}}dt=f(x),$$

has the form

$$y(x)=F(x)+\pi\lambda^2\int_a^x\exp\left[\pi\lambda^2(x-t)\right]F(t)\,dt,$$

where

$$F(x)=f(x)-\lambda\int_a^x\frac{f(t)}{\sqrt{x-t}}dt.$$

Therefore, the solution of equation (26) can be written as

$$\varphi(t)=F(t)+\frac{1}{4a^2}\int_0^t\exp\left[\frac{t-\tau}{4a^2}\right]F(\tau)d\tau,$$

where

$$F(t)=C\cdot\left\{\frac{1}{\sqrt{t}}+\frac{1}{2a\sqrt{\pi}}\int_0^t\frac{d\tau}{\sqrt{\tau(t-\tau)}}\right\}=C\cdot\left\{\frac{1}{\sqrt{t}}+\frac{\sqrt{\pi}}{2a}\right\}.$$

Then

$$\varphi(t)=C\cdot\left\{\frac{1}{\sqrt{t}}+\frac{\sqrt{\pi}}{2a}+\frac{1}{4a^2}\int_0^t\exp\left[\frac{t-\tau}{4a^2}\right]\cdot\left(\frac{1}{\sqrt{\tau}}+\frac{\sqrt{\pi}}{2a}\right)d\tau\right\}=$$
$$=C\cdot\left\{\frac{1}{\sqrt{t}}+\frac{\sqrt{\pi}}{2a}\right\}+\frac{C}{4a^2}\exp\left[\frac{t}{4a^2}\right]\left\{\int_0^t\exp\left[-\frac{\tau}{4a^2}\right]\frac{d\tau}{\sqrt{\tau}}+\frac{\sqrt{\pi}}{2a}\int_0^t\exp\left[-\frac{\tau}{4a^2}\right]d\tau\right\}$$

After simplifications, we obtain

$$\varphi(t)=C\cdot\left\{\frac{1}{\sqrt{t}}+\frac{\sqrt{\pi}}{2a}e^{\frac{t}{4a^2}}erf\left(\frac{\sqrt{t}}{2a}\right)+\frac{\sqrt{\pi}}{2a}e^{\frac{t}{4a^2}}\right\}. \qquad (27)$$

(27) is the solution of Abel equation (26), i.e. the solution of «simplified» equation (17).

We note that after multiplying equality (27) by $\exp\left(-\frac{t}{4a^2}\right)$, we obtain the solution of original equation (15) (in virtue of the foregoing)

$$\varphi(t)=C\cdot\left\{\frac{1}{\sqrt{t}}e^{-\frac{t}{4a^2}}+\frac{\sqrt{\pi}}{2a}erf\left(\frac{\sqrt{t}}{2a}\right)+\frac{\sqrt{\pi}}{2a}\right\}. \qquad (28)$$

Thus, the eigenfunction of equation (15) has the form $\varphi_0(t) = C \cdot \left\{ \frac{1}{\sqrt{t}} e^{-\frac{t}{4a^2}} + \frac{\sqrt{\pi}}{2a} erf\left(\frac{\sqrt{t}}{2a}\right) + \frac{\sqrt{\pi}}{2a} \right\}.$

7 The solution of the initial boundary value problem

Thus, we have found the solution of second problem (1)–(3) for homogeneous equation of heat conduction in the degenerating domain $G = \{(x;t):\ \ t > 0,\ \ 0 < x < t\}$ with homogeneous boundary conditions. We write it in an explicit form.

$$u(x,t) = \frac{1}{2a\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{t-\tau}} \exp\left\{-\frac{x^2}{4a^2(t-\tau)}\right\} \nu(\tau)\,d\tau +$$

$$+ \frac{1}{2a\sqrt{\pi}} \int_0^t \frac{1}{(t-\tau)^{1/2}} \exp\left\{-\frac{(x-\tau)^2}{4a^2(t-\tau)}\right\} \varphi(\tau)\,d\tau,$$

where

$$\nu(t) = \frac{1}{2a\sqrt{\pi}} \int_0^t \frac{\tau}{(t-\tau)^{3/2}} \exp\left\{-\frac{\tau^2}{4a^2(t-\tau)}\right\} \varphi(\tau)\,d\tau$$

and the function $\varphi(t)$ is determined by formula (28).

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М.Т.Дженалиев, В.К.Калантаров, М.Т.Космакова, М.Ы.Рамазанов

Шектелмеген жазық бұрыштағы жылуөткізгіштік теңдеуі үшін екінші шеттік есеп жайында

Мақалада шектелмеген бұрыштық облыстағы біртекті екінші шеттік есеп қарастырылды. Есептің шешімі, нормасы бірге тең, ерекше интегралды Вольтерра теңдеуінің шешіміне келтірілді. Карлеман-Векуа әдісі арқылы интегралды теңдеудің шешімі біртексіз Абель теңдеуінің шешіміне келеді. Цилиндрлік емес облыста біртекті екінші шеттік есептің 0-дік емес шешімінің бар болуы туралы теорема дәлелденген. Қойылған есептің шешімі айқын түрде алынған.

М.Т.Дженалиев, В.К.Калантаров, М.Т.Космакова, М.И.Рамазанов

О второй краевой задаче для уравнения теплопроводности в неограниченном плоском углу

В статье рассмотрена вторая однородная краевая задача в неограниченной угловой области. Решение задачи редуцируется к решению особого интегрального уравнения Вольтерра второго рода с ядром, норма которого равна единице. Методом Карлемана-Векуа решение интегрального уравнения сводится к решению неоднородного уравнения Абеля. Доказана теорема о существовании нетривиального решения второй однородной краевой задачи в нецилиндрической области. Решение поставленной задачи получено в явном виде.

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