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Numerical solutions of source identification problems for telegraph-parabolic equations

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This paper presents a numerical study of source identification problems for one-dimensional telegraph-parabolic equations subject to Dirichlet and Neumann boundary conditions. In these inverse problems, the unknown source terms are assumed to be space-dependent, which introduces both analytical and computational challenges. The study begins by discretizing the considered problems using the finite difference method – first in space and subsequently in time – resulting in a system of discrete equations. Stability results for the solutions of the resulting finite difference schemes are established to ensure the reliability of the numerical approach. A numerical algorithm is proposed for solving the discrete inverse problems. The algorithm begins by eliminating the unknown source terms, which transforms the original discretized problem into a new nonlocal problem with unknown initial data. To approximate this initial data, an iterative procedure based on fixed-point iterations is constructed. Once the transformed nonlocal problem is solved, the solution of the main finite difference scheme and approximations of the unknown source term are recovered. Numerical results for two test problems are presented to illustrate the proposed method in practice. The findings confirm the accuracy of the approach in solving space-dependent inverse source problems.

Keywords: source identification problem, inverse problem, mixed-type differential equation, telegraph-parabolic equation, finite difference scheme, numerical algorithm, nonlocal problems, fixed-point iterations.

2020 Mathematics Subject Classification: 65M06, 35M10, 35R30.

Introduction

Partial differential equations with unknown source terms are widely used in the mathematical modelling of real-world phenomena in various applied fields (see, e.g., [1] and the references therein). A problem involving a differential equation with a time- and/or space-dependent source term is referred to as a source identification problem (SIP). These types of inverse problems have been extensively studied in the literature (see, e.g., [2–4] and the references therein).

In recent years, the analysis of SIPs for mixed-type differential equations, as well as the development and investigation of numerical methods for their solution, has attracted significant attention (see, e.g., [5, 6] for parabolic-elliptic, [7–9] for elliptic-hyperbolic, and [10, 11] for telegraph-parabolic SIPs). By mixed-type, we mean that the differential equation is of one type in one part of the domain and of a different type in another part. For instance, consider a physical system initially modelled by the heat equation. At a certain moment in time, due to an instantaneous change in the system, the governing model transitions to the wave equation with a damping term. In such cases, the resulting differential equations are referred to as telegraph-parabolic equations.

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Consider the following abstract formulation for telegraph-parabolic equations with an unknown space-dependent source term p:

$$\begin{cases} w''(t) + \alpha w'(t) + Aw(t) = p + f(t), & t \in (0,1), \\ w'(t) + Aw(t) = p + g(t), & t \in (-1,0), \\ w(0^+) = w(0^-),\ w'(0^+) = w'(0^-), & \\ w(-1) = \varphi,\ w(\lambda) = \psi,\ \lambda \in (-1,1], & \end{cases} \tag{1}$$

where the problem is posed in a Hilbert space H with a self-adjoint positive definite (SAPD) operator A satisfying $A \geq \delta I$, for some $\delta > \frac{\alpha^2}{4}$ and $\alpha > 0$. Here, $\varphi,\ \psi \in D(A)$ and the functions $f(t)$ and $g(t)$ are assumed to be continuously differentiable on $[0,1]$ and $[-1,0]$, respectively. The existence, uniqueness, and stability of solutions of the problem (1) in the space $C(H)$ of continuous H-valued functions $w(t)$ defined on the interval $[-1,1]$, equipped with the norm

$$\|w\|_{C(H)} = \max_{t\in[-1,1]} \|w(t)\|_H$$

are established in [10].

For the approximate solution of the abstract problem (1), the following stable difference scheme (DS) of first-order accuracy is constructed in [11]:

$$\begin{cases} \frac{w_{k+1}-2w_k+w_{k-1}}{\tau^2} + \alpha \frac{w_{k+1}-w_k}{\tau} + Aw_{k+1} = p + f_k, & 1 \leq k \leq N-1, \\ \frac{w_k - w_{k-1}}{\tau} + Aw_k = p + g_k, & -N+1 \leq k \leq 0, \\ \frac{w_1 - w_0}{\tau} = p - Aw_0 + g_0, & \\ w_{-N} = \varphi, \quad w_\ell = \psi, & \end{cases} \tag{2}$$

where $\tau = \frac{1}{N}$ is sufficiently small positive number, $t_k = k\tau,\ -N \leq k \leq N,\ \ell = \lceil \frac{\lambda}{\tau} \rceil,\ f_k = f(t_k)$, $1 \leq k \leq N-1$ and $g_k = g(t_k),\ -N+1 \leq k \leq 0$.

The unique solvability of the DS (2) and the stability estimates for its solution were established in [11]. However, the abstract results for the DS (2), presented in [11], require further investigation from an implementation perspective. In the present paper, we consider the application of the aforementioned abstract results to two SIPs for one-dimensional telegraph-parabolic equations with Dirichlet and Neumann boundary conditions. We provide a complete discretization of the considered problems and propose a numerical algorithm for solving the resulting DSs. Numerical examples are presented to illustrate the proposed numerical procedure.

1 *SIPs for one-dimensional telegraph-parabolic equations*

In this section, we consider two SIPs for one-dimensional telegraph-parabolic equations: one with Dirichlet boundary conditions and the other with Neumann boundary conditions. Since the discretization procedures for the considered problems are very similar, we describe the approach for both SIPs simultaneously.

First, consider the following SIP for one-dimensional telegraph-parabolic equations

$$\begin{cases} w_{tt}(t,x) + \alpha\ w_t(t,x) - \big(a(x)\ w_x(t,x)\big)_x = p(x) + f(t,x), & x \in (0,1),\ t \in (0,1), \\ w_t(t,x) - \big(a(x)\ w_x(t,x)\big)_x = p(x) + g(t,x), & x \in (0,1),\ t \in (-1,0), \\ w(0^+,x) = w(0^-,x),\ w_t(0^+,x) = w_t(0^-,x), & x \in [0,1], \\ w(-1,x) = \varphi(x),\ w(1,x) = \psi(x), & x \in [0,1], \\ w(t,0) = w(t,1) = 0, & t \in [-1,1] \end{cases} \tag{3}$$

with homogeneous Dirichlet boundary conditions. Here and throughout the paper, $p(x)$ denotes the unknown source term, $a(x) \geq a > 0$, $\varphi(x)$, $\psi(x)$, $f(t,x)$, and $g(t,x)$ are given sufficiently smooth functions, and α is a positive constant. SIP (3) can be reduced to the abstract problem (1) in a Hilbert space $H = L_2(0,1)$ with a SAPD operator $A = A^x$ defined by the formula

$$A^x w(x) = -\big(a(x)w_x(x)\big)_x \tag{4}$$

with domain $D(A^x) = \big\{w(x) \ : \ w(x),\ w_x(x),\ \big(a(x)w_x\big)_x \in L_2[0,1],\ w(0) = w(1) = 0\big\}$.

Second, consider the SIP for one-dimensional telegraph-parabolic equations with Neumann boundary conditions

$$\begin{cases} w_{tt}(t,x) + \alpha w_t(t,x) - \big(a(x)w_x(t,x)\big)_x + \delta w(t,x) = p(x) + f(t,x), & x \in (0,1),\ t \in (0,1), \\ w_t(t,x) - \big(a(x)w_x(t,x)\big)_x + \delta w(t,x) = p(x) + g(t,x), & x \in (0,1),\ t \in (-1,0), \\ w(0^+,x) = w(0^-,x),\ w_t(0^+,x) = w_t(0^-,x), & x \in [0,1], \\ w(-1,x) = \varphi(x),\ w(1,x) = \psi(x), & x \in [0,1], \\ w_x(t,0) = w_x(t,1) = 0, & t \in [-1,1], \end{cases} \tag{5}$$

where δ is a positive constant. SIP (5) can be reduced to the abstract problem (1) in a Hilbert space $H = L_2(0,1)$ with a SAPD operator $A = A^x$ defined by the formula

$$A^x w(x) = -\big(a(x)w_x(x)\big)_x + \delta w \tag{6}$$

with domain $D(A^x) = \big\{w(x) \ : \ w(x),\ w_x(x),\ \big(a(x)w_x\big)_x \in L_2(0,1),\ w_x(0) = w_x(1) = 0\big\}$.

By means of the abstract result from [10], both problems (3) and (5) have a unique smooth solution $\big\{w(t,x), p(x)\big\}$ for given smooth data satisfying all compatibility conditions.

We start the discretization of SIPs (3) and (5) by defining the grid space $[0,1]_h = \big\{x \ \big| \ x_m = m\,h,\ 0 \leq m \leq M,\ \ M\,h = 1\big\}$.

Let us introduce the Hilbert space $L_{2h} = L_2\left([0,1]_h\right)$ of grid functions $\varphi^h(x) = \{\varphi^m\}_0^M$ defined on $[0,1]_h$ and equipped with the norm $\left\|\varphi^h\right\|_{L_{2h}} = \left(\sum\limits_{x\in[0,1]_h} \left|\varphi^h(x)\right|^2 h\right)^{1/2}$. To the differential operator A^x, defined by formula (4), we associate the difference operator A_h^x, given by the formula

$$A_h^x \varphi^h(x) = \Big\{ -\big(a(x)\varphi_{\bar{x}}^m(x)\big)_x \Big\}_1^{M-1},$$

which acts in the space of grid functions $\varphi^h(x) = \{\varphi^m\}_0^M$ satisfying boundary conditions $\varphi_0 = \varphi_M = 0$. Here and throughout the paper,

$$\varphi_{\bar{x}}^m = \left\{\frac{\varphi^m - \varphi^{m-1}}{h}\right\}_1^M \quad \text{and} \quad \varphi_x^m = \left\{\frac{\varphi^{m+1} - \varphi^m}{h}\right\}_0^{M-1}.$$

Similarly, to the differential operator A^x, defined by formula (6), we assign the corresponding difference operator A_h^x, given by the formula

$$A_h^x \varphi^h(x) = \Big\{ -\big(a(x)\varphi_{\bar{x}}^m(x)\big)_x + \delta\varphi^m(x) \Big\}_1^{M-1},$$

acting in the space of grid functions $\varphi^h(x) = \{\varphi^m\}_0^M$, subject to the boundary conditions $\varphi_0 = \varphi_1$ and $\varphi_M = \varphi_{M-1}$.

Note that in both cases, A_h^x corresponds to the second-order accuracy centered difference approximation of the respective differential operator A^x, incorporating Dirichlet and Neumann boundary conditions, respectively. Moreover, A_h^x is a SAPD operator in L_{2h} in both cases.

With the help of the corresponding operator A_h^x, the first step of the discretization of both SIPs (3) and (5) leads to the following problem:

$$\begin{cases} \frac{d^2 w^h(t,x)}{dt^2} + \alpha \frac{dw^h(t,x)}{dt} + A_h^x w^h(t,x) = p^h(x) + f^h(t,x), & t \in (0,1), \\ \frac{dw^h(t,x)}{dt} + A_h^x w^h(t,x) = p^h(x) + g^h(t,x), & t \in (-1,0), \\ w^h(0^+,x) = w^h(0^-,x), \quad \frac{dw^h(0^+,x)}{dt} = \frac{dw^h(0^-,x)}{dt}, & \\ w^h(-1,x) = \varphi^h(x), \quad w^h(1,x) = \psi^h(x), & \end{cases} \tag{7}$$

where $x \in [0,1]_h$.

Now, in the second step of the discretization process, we define $\tau = \frac{1}{N},\ t_k = k\tau,\ -N \le k \le N$ and replace problem (7) with DS (2)

$$\begin{cases} \frac{w_{k+1}^h(x) - 2w_k^h(x) + w_{k-1}^h(x)}{\tau^2} + \alpha \frac{w_{k+1}^h(x) - w_k^h(x)}{\tau} + A_h^x w_{k+1}^h(x) = p^h(x) + f_k^h(x), & 1 \le k \le N-1, \\ \frac{w_k^h(x) - w_{k-1}^h(x)}{\tau} + A_h^x w_k^h(x) = p^h(x) + g_k^h(x), & -N+1 \le k \le 0, \\ \frac{w_1^h(x) - w_0^h(x)}{\tau} = p^h(x) - A_h^x w_0^h(x) + g_0^h(x), & \\ w_{-N}^h(x) = \varphi^h(x),\ w_N^h(x) = \psi^h(x), & \end{cases} \tag{8}$$

where $x \in [0,1]_h$, $f_k^h(x) = f^h(t_k,x),\ 1 \le k \le N-1$ and $g_k^h(x) = g^h(t_k,x),\ -N+1 \le k \le 0$. Then, the following theorem follows readily from the abstract result stated in Theorem 1.

Theorem 1. The solution of DS (8) satisfies the following stability estimate

$$\max_{-N \le k \le N} \left\| w_k^h \right\|_{L_{2h}} + \left\| (A_h^x)^{-1} p^h \right\|_{L_{2h}}$$
$$\le M(\delta,\alpha) \left[\max_{1 \le k \le N-1} \left\| f_k^h \right\|_{L_{2h}} + \max_{-N+1 \le k \le 0} \left\| g_k^h \right\|_{L_{2h}} + \left\| \varphi^h \right\|_{L_{2h}} + \left\| \psi^h \right\|_{L_{2h}} \right].$$

Here, $M(\delta,\alpha)$ is independent of τ, h, $\varphi^h(x)$, $\psi^h(x)$, $f_k^h(x)$ and $g_k^h(x)$.

2 Numerical algorithm

In this section, we propose a numerical algorithm to solve the difference scheme (8). The approach relies on a suitable substitution that eliminates the unknown source term p^h. Let us denote

$$w_k^h(x) = v_k^h(x) + (A_h^x)^{-1} p^h(x),\ x \in [0,1]_h,\ -N \le k \le N.$$

Then, the scheme (8) results in the following auxiliary DS

$$\begin{cases} \frac{v_{k+1}^h(x) - 2v_k^h(x) + v_{k-1}^h(x)}{\tau^2} + \alpha \frac{v_{k+1}^h(x) - v_k^h(x)}{\tau} + A_h^x v_{k+1}^h(x) = f_k^h(x), & 1 \le k \le N-1, \\ \frac{v_k^h(x) - v_{k-1}^h(x)}{\tau} + A_h^x v_k^h(x) = g_k^h(x), & -N+1 \le k \le 0, \\ \frac{v_1^h(x) - v_0^h(x)}{\tau} = -A_h^x v_0^h(x) + g_0^h(x), & \\ v_{-N}^h(x) = v_N^h(x) + \varphi^h(x) - \psi^h(x), & \end{cases} \tag{9}$$

where $x \in [0,1]_h$. Note that the scheme (9) no longer involves the unknown source p^h. However, it exhibits a non-local nature due to the coupling between v_{-N}^h and v_N^h, and therefore it cannot be solved using a standard time-marching approach.

We attempt to solve the non-local difference problem (9) iteratively. Let $\{v_k^h(x;\theta)\}$ be the solution of the following scheme

$$\begin{cases} \frac{v_{k+1}^h(x;\theta)-2v_k^h(x;\theta)+v_{k-1}^h(x;\theta)}{\tau^2} + \alpha\frac{v_{k+1}^h(x;\theta)-v_k^h(x;\theta)}{\tau} + A_h^x v_{k+1}^h(x;\theta) = f_k^h(x), & 1 \le k \le N-1, \\ \frac{v_k^h(x;\theta)-v_{k-1}^h(x;\theta)}{\tau} + A_h^x v_k^h(x;\theta) = g_k^h(x), & -N+1 \le k \le 0, \\ \frac{v_1^h(x;\theta)-v_0^h(x;\theta)}{\tau} = -A_h^x v_0^h(x;\theta) + g_0^h(x), & \\ v_{-N}^h(x) = \theta^h(x), & \end{cases} \tag{10}$$

where $x \in [0,1]_h$. For $\{v_k^h(x;\theta)\}$ to be a solution of the scheme (9), the initial vector $\theta = \theta^h(x)$, where $x \in [0,1]_h$, must satisfy the following condition

$$\theta = v_N^h(x;\theta) + \varphi^h(x) - \psi^h(x), \quad x \in [0,1]_h.$$

We can then construct an iterative procedure, such as fixed point iterations, to approximate the initial vector θ. Taking all of the above into account, the following algorithm can be used to solve the difference scheme (8).

1. To approximate the initial vector θ iteratively, we use the following formula:

$$\theta^{m+1} = v_N^h(x;\theta^m) + \varphi^h(x) - \psi^h(x),\ x \in [0,1]_h,\ m = 0,1,2,\ldots$$

   At each iteration step, the difference scheme (10) must be solved to compute $v_N^h(x;\theta^m)$.
2. Next, we approximate the source $p(x)$ using the formula

$$p^h(x) = A_h^x\left(\varphi^h(x) - \theta\right),\ x \in [0,1]_h,$$

   where θ is the initial vector, approximated in the first step.
3. Finally, we obtain the solution of the difference scheme (8) using the formula:

$$w_k^h(x) = v_k^h(x) + \varphi^h(x) - \theta, \quad x \in [0,1]_h, \quad -N+1 \le k \le N-1.$$

   Here, $v_k^h(x)$ is the solution of the difference scheme (10) with the initial vector θ obtained from the iterative procedure.

3 Numerical example

First, we consider the following initial-boundary value problem

$$\begin{cases} w_{tt} + 2w_t - w_{xx} = p(x) + f(t,x), & x \in (0,1),\ t \in (0,1), \\ w_t - w_{xx} = p(x) + g(t,x), & x \in (0,1),\ t \in (-1,0), \\ w(0^+,x) = w(0^-,x),\ w_t(0^+,x) = w_t(0^-,x), & x \in [0,1], \\ w(-1,x) = e^1 \sin \pi x,\ w(1,x) = e^{-1} \sin \pi x, & x \in [0,1], \\ w(t,0) = 0,\ w(t,1) = 0, & t \in [-1,1], \end{cases} \tag{11}$$

where $f(t,x) = g(t,x) = \left((\pi^2-1)e^{-t} - 1\right) \sin \pi x$. The analytical solution of problem (11) is

$$w(t,x) = e^{-t} \sin \pi x, \quad x \in [0,1],\ t \in [-1,1]$$

with the source term $p(x) = \sin \pi x,\ \ x \in (0, 1)$.

Second, we consider the initial-boundary value problem

$$\begin{cases} w_{tt} + 2w_t - w_{xx} + 3w = p(x) + f(t,x), & x \in (0,1),\ t \in (0,1), \\ w_t - w_{xx} + 3w = p(x) + g(t,x), & x \in (0,1),\ t \in (-1,0), \\ w(0^+,x) = w(0^-,x),\ w_t(0^+,\ x) = w_t(0^-,\ x), & x \in [0,1], \\ w(-1,x) = e^1 \cos \pi x,\ w(1,x) = e^{-1} \cos \pi x, & x \in [0,1], \\ w_x(t,0) = w_x(t,1) = 0, & t \in [-1,1], \end{cases} \tag{12}$$

where $f(t,x) = g(t,x) = \left((\pi^2+2)e^{-t} - 1\right) \cos \pi x$. The analytical solution of problem (12) is

$$w(t,x) = e^{-t} \cos \pi x, \quad x \in [0,1],\ t \in [-1,1]$$

with the source term $p(x) = \cos \pi x,\ \ x \in (0, 1)$.

The numerical solutions for SIPs (11) and (12) are computed using the first-order accuracy DS and the aforementioned numerical procedure for various values of $M = N$. To evaluate the accuracy of the method, we compute the error between the analytical and numerical solutions using the following formulas:

$$E_p = \max_n |p(x_n) - p_n|, \quad E_w = \max_{k,n} \left| w(t_k, x_n) - w_n^k \right|.$$

Here, w_n^k and p_n denote the corresponding numerical approximations of the exact solution $\{w(t,x), p(x)\}$ at the grid points $t = t_k$ and $x = x_n$. Figure 1 shows the errors between the exact and numerical solutions of problems (11) and (12) for different values of τ, confirming the first-order convergence of the proposed method. Since we have taken $M = N$, which implies $h = \tau$, and the error of the method is $O(\tau + h^2)$, observe only the temporal errors here, i.e., the first-order convergence of the method.

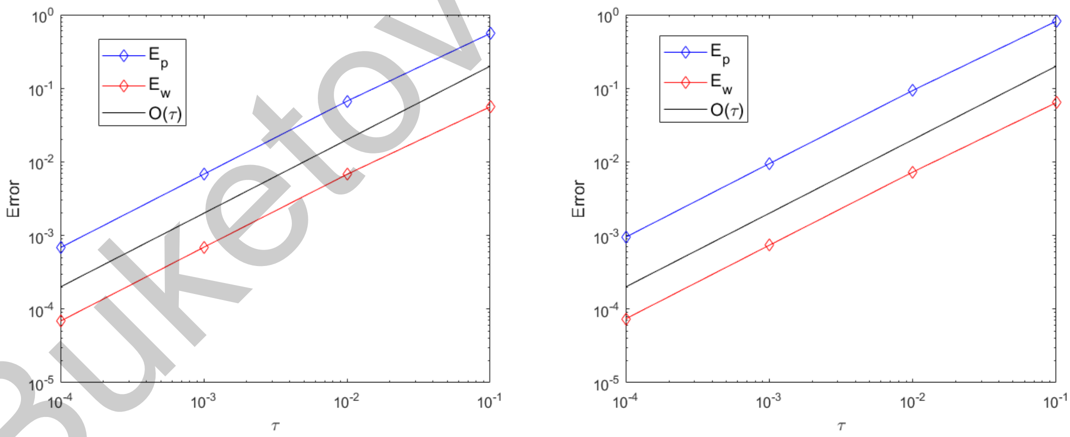

Figure 1. The errors between the analytical solutions of problems (11) (on the left) and (12) (on the right) and their numerical solutions, computed using the first-order DS for various values of the time step τ

Conclusion

In this work, we developed an algorithm for the numerical solution of one-dimensional telegraph-parabolic equations with an unknown source term dependent on a spatial variable. The local inverse problems considered are transformed into corresponding nonlocal direct problems, which are then solved using an iterative technique similar to the shooting method. Numerical experiments are provided to illustrate the procedure in practice.

Our results demonstrate first-order convergence of the proposed numerical method. It is of practical importance to develop higher order accuracy stable DSs so that more accurate results can be obtained in less computational time. Future work will also focus on the investigation of SIPs for telegraph-parabolic equations with time-dependent sources.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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