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On Graded J_{gr}-Prime Submodules

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In this paper, several results concerning graded $\mathfrak{J}_{gr}$-prime submodules over a commutative graded ring were obtained. For example, we give characterization of graded $\mathfrak{J}_{gr}$-prime submodules and results related to residual of graded $\mathfrak{J}_{gr}$-prime submodules. Also, the relations between graded $\mathfrak{J}_{gr}$-prime submodules and graded prime submodules of $\mathfrak{D}$ were studied. In addition, we present the necessary and sufficient condition for graded submodules to be graded $\mathfrak{J}_{gr}$-prime submodules.

Keywords: graded $\mathfrak{J}_{gr}$-prime submodule, graded prime submodule, graded submodule.

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Introduction

The study of graded rings and modules has attracted the attentions of many researchers for a long time due to their important applications in many fields in such as geometry and physics. For example, graded Lie algebra plays a significant role in differential geometry such as Frolicher-Nijenhuis as well as Nijenhuis-Richardson bracket [1]. In addition, they solve many physical problems related to supermanifolds, supersymmetries and quantizations of systems with symmetry [2, 3].

In recent years, graded prime submodules have attracted the attention of many mathematicians, for example [4–8]. In addition, many other generalizations of graded prime have been investigated. For example, in [9], the authors introduce the concept of graded weakly prime submodules of graded modules as a generalization of graded prime submodule. In [10] Al-Zoubi and Alghueiri mentioned the concept of graded $\mathfrak{J}_{gr}$-prime submodules. Here, we discuss the concept of graded $\mathfrak{J}_{gr}$-prime submodule and we study several results concerning it. For example, we characterize graded $\mathfrak{J}_{gr}$-prime submodules. Also, the relations between graded $\mathfrak{J}_{gr}$-prime submodules and graded prime submodules were studied. In addition, the necessary and sufficient condition for graded submodules to be graded $\mathfrak{J}_{gr}$-prime submodules were investigated.

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1 *Preliminaries*

Throughout this article, we assume that $\mathfrak{A}$ is a commutative $\mathfrak{G}$-graded ring with identity and $\mathfrak{D}$ is a unitary graded $\mathcal{A}$-module. A left $\mathfrak{A}$-module $\mathfrak{D}$ is called a graded $\mathfrak{A}$-module if there exists a family of additive subgroups $\{\mathfrak{D}_\alpha\}_{\alpha\in\mathfrak{G}}$ of $\mathfrak{D}$ such that $\mathfrak{D} = \bigoplus_{\alpha\in\mathfrak{G}} \mathfrak{D}_\alpha$ and $\mathfrak{A}_\alpha\, \mathfrak{D}_\beta \subseteq \mathfrak{D}_{\alpha\beta}$ for all $\alpha, \beta \in \mathfrak{G}$. Also if an element of $\mathfrak{D}$ belongs to $\cup_{\alpha\in\mathfrak{G}} \mathfrak{D}_\alpha = h(\mathfrak{D})$, then it is called a homogeneous. Let $\mathfrak{A} = \bigoplus_{\alpha\in\mathfrak{G}} \mathfrak{A}_\alpha$ be a $\mathfrak{G}$-graded ring. A submodule $\mathcal{V}$ of $\mathfrak{D}$ is said to be a graded submodule of $\mathfrak{D}$ if $\mathcal{V} = \bigoplus_{\alpha\in\mathfrak{G}} (\mathcal{V} \cap \mathfrak{D}_\alpha) := \bigoplus_{\alpha\in\mathfrak{G}} \mathcal{V}_\alpha$. In this case, $\mathcal{V}_\alpha$ is called the α-component of $\mathcal{V}$ [11,12]. Let $\mathfrak{A}$ be a $\mathfrak{G}$-graded ring and $\mathfrak{D}$ — a graded $\mathfrak{A}$-module. A graded submodule $\mathcal{V}$ of $\mathfrak{D}$ is said to be a graded maximal (briefly, Gr-maximal) submodule if $\mathcal{V} \neq \mathfrak{D}$ and if there is a graded submodule L of $\mathfrak{D}$ such that $\mathcal{V} \subseteq L \subseteq \mathfrak{D}$, then $\mathcal{V} = L$ or $L = \mathfrak{D}$ [13]. The graded Jacobson radical of a graded module $\mathfrak{D}$, denoted by $\mathfrak{J}_{gr}(\mathfrak{D})$, is defined to be the intersection of all Gr-maximal submodules of $\mathfrak{D}$, if $\mathfrak{D}$ has no Gr-maximal submodule then we shall take, by definition, $\mathfrak{J}_{gr}(\mathfrak{D}) = \mathfrak{D}$ [12]. A proper graded submodule $\mathcal{V}$ of $\mathfrak{D}$ is called a graded prime submodule if whenever $rm \in \mathcal{V}$ where $r \in h(\mathfrak{A})$ and $m \in h(\mathfrak{D})$, then $r \in (\mathcal{V} :_{\mathfrak{A}} \mathfrak{D})$ or $m \in \mathcal{V}$ [6]. A proper graded submodule $\mathcal{V}$ of $\mathfrak{D}$ is called a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$ if whenever $r_g \in h(\mathfrak{A})$ and $m_\lambda \in h(\mathfrak{D})$ with $r_g\, m_\lambda \in \mathcal{V}$, then either $m_\lambda \in \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$ or $r_g \in (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$, where $\mathfrak{J}_{gr}(\mathfrak{D})$ is the graded Jacobson radical of $\mathfrak{D}$ [10].

2 *Results*

Theorem 1. If $\mathcal{V}$ is a graded prime submodule of $\mathfrak{D}$, then $\mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$.

Proof. Let $rm \in \mathcal{V}$, where $r \in h(\mathfrak{A})$ and $m \in h(\mathfrak{D})$, since $\mathcal{V}$ is a graded prime submodule of $\mathfrak{D}$, then $r \in (\mathcal{V} :_{\mathfrak{A}} \mathfrak{D})$ or $m \in \mathcal{V}$. If $r \in (\mathcal{V} :_{\mathfrak{A}} \mathfrak{D})$, then $rM \subseteq \mathcal{V}$, but $\mathcal{V} \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$, thus $rM \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$, it follows that $r \in (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$. If $m \in \mathcal{V}$, since $\mathcal{V} \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$, then $m \in \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$. Hence $\mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$.

In the following example, it is shown that the converse of Theorem 1 is not necessarily true.

Example 1. Let $\mathfrak{G} = \mathbb{Z}_2$, $\mathfrak{A} = \mathbb{Z}$ be a $\mathfrak{G}$-graded ring with $\mathfrak{A}_0 = \mathbb{Z}$, $\mathfrak{A}_1 = \{0\}$. Let $\mathfrak{D} = \mathbb{Z}_{12}$ be a graded $\mathfrak{A}$-module with $\mathfrak{D}_0 = \mathbb{Z}_{12}$ and $\mathfrak{D}_1 = \{\overline{0}\}$. Now, consider $\mathcal{V} = \{\overline{0}, \overline{4}, \overline{8}\} = \langle \overline{4} \rangle$ be a graded submodule of $\mathbb{Z}_{12}$. Then $\mathcal{V}$ is not graded prime submodule of $\mathfrak{D}$, since there exist $2 \in h(\mathfrak{A})$ and $\overline{2} \in h(\mathfrak{D})$ such that $2 \cdot \overline{2} = \overline{4} \in \mathcal{V}$, but $\overline{2} \notin \mathcal{V}$ and $2 \notin (\mathcal{V} :_{\mathfrak{A}} \mathfrak{D}) = 4\mathbb{Z}$. However, an easy computation shows that $\mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$.

Example 2. Let $\mathfrak{G} = \mathbb{Z}_2$, $\mathfrak{A} = \mathbb{Z}$ be a $\mathfrak{G}$-graded ring with $\mathfrak{A}_0 = \mathbb{Z}$, $\mathfrak{A}_1 = \{0\}$, and $\mathfrak{D} = \mathbb{Z} \times \mathbb{Z}$ be a graded $\mathfrak{A}$-module with $\mathfrak{D}_0 = \mathbb{Z} \times \mathbb{Z}$, $\mathfrak{D}_1 = \{(0,0)\}$. The graded submodule $\mathcal{V} = 2\mathbb{Z} \times \langle 0 \rangle$ is not graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$. Since $(6,0) = 2(3,0) \in \mathcal{V}$, but $(3,0) \notin \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) = \mathcal{V} + \{(0,0)\} = \mathcal{V}$ and $2 \notin (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D}) = (\mathcal{V} :_{\mathfrak{A}} \mathfrak{D}) = (2\mathbb{Z} \times \langle 0 \rangle :_{\mathfrak{A}} \mathbb{Z} \times \mathbb{Z}) = \langle 0 \rangle$, hence $\mathcal{V} = 2\mathbb{Z} \times \langle 0 \rangle$ is not graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$.

Remark 1. Let $\mathfrak{A}$ be a $\mathfrak{G}$-graded ring and $\mathfrak{D}$ a graded $\mathfrak{A}$-module.

1) If $\mathfrak{J}_{gr}(\mathfrak{D}) = 0$, then every graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$ is a graded prime submodule of $\mathfrak{D}$.

2) If $\mathfrak{J}_{gr}(\mathfrak{D})$ is contained in every graded submodule of $\mathfrak{D}$, then every graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$ is a graded prime submodule of $\mathfrak{D}$.

A graded $\mathfrak{A}$-module $\mathfrak{D}$ is called a Gr-torsion free if whenever $r \in h(\mathfrak{A})$ and $m \in h(\mathfrak{D})$ with $rm = 0$, then either $r = 0$ or $m = 0$ [5].

The following theorem characterizes graded $\mathfrak{J}_{gr}$-prime submodules.

Theorem 2. Let $\mathcal{V}$ be a proper graded submodule of $\mathfrak{D}$ and $P = (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$. Then the following statements are equivalent:

1) $\mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-prime submodule.

2) For every graded submodule $\mathcal{K}$ of $\mathfrak{D}$ and for every graded ideal $\mathcal{U}$ of $\mathfrak{A}$ such that $\mathcal{U}\mathcal{K} \subseteq \mathcal{V}$ implies that either $\mathcal{K} \subseteq \mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D})$ or $\mathcal{U} \subseteq P = (\mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$.

3) $\mathfrak{D}/(\mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D}))$ is a Gr-torsion free $\mathfrak{A}/P$-module.

4) The graded submodule $(\mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D}) :_{\mathfrak{D}} \langle r\rangle) = \mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D})$, for each $r \in h(\mathfrak{A}) - P$.

5) The graded ideal $(\mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \langle x\rangle) = P$, for each $x \in h(\mathfrak{D}) - (\mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D}))$.

Proof. (1)⇒(2) Let $\mathcal{K}$ be a graded submodule of $\mathfrak{D}$ and $\mathcal{U}$ be a graded ideal of $\mathfrak{A}$ such that $\mathcal{U}\mathcal{K} \subseteq \mathcal{V}$. Suppose $\mathcal{K} \nsubseteq \mathcal{V}+\mathfrak{I}_{gr}(\mathfrak{D})$, then there exists $k \in \mathcal{K}\cap h(\mathfrak{D}) - (\mathcal{V}+\mathfrak{I}_{gr}(\mathfrak{D}))$. Let $i \in \mathcal{U}\cap h(\mathfrak{A})$. Since $k \in \mathcal{K}$, then $ik \in \mathcal{U}\mathcal{K} \subseteq \mathcal{V}$, so $ik \in \mathcal{V}$. But $\mathcal{V}$ is a graded $\mathfrak{I}_{gr}$-prime submodule, then either $i \in (\mathcal{V}+\mathfrak{I}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$ or $k \in \mathcal{V}+\mathfrak{I}_{gr}(\mathfrak{D})$. But $k \notin \mathcal{V}+\mathfrak{I}_{gr}(\mathfrak{D})$, thus $i \in (\mathcal{V}+\mathfrak{I}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$. Hence $\mathcal{U} \subseteq (\mathcal{V}+\mathfrak{I}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D}) = P$.

(2)⇒(3) Assume that $(r+P)(m+\mathcal{V}+\mathfrak{I}_{gr}(\mathfrak{D})) = \mathcal{V}+\mathfrak{I}_{gr}(\mathfrak{D})$ and $r+P \neq P$, where $r+P \in h(\mathfrak{A}/P)$ and $m+\mathcal{V}+\mathfrak{I}_{gr}(\mathfrak{D}) \in h(\mathfrak{D}/(\mathcal{V}+\mathfrak{I}_{gr}(\mathfrak{D})))$. Then $rm+\mathcal{V}+\mathfrak{I}_{gr}(\mathfrak{D}) = \mathcal{V}+\mathfrak{I}_{gr}(\mathfrak{D})$, thus $rm \in \mathcal{V}+\mathfrak{I}_{gr}(\mathfrak{D})$ it follows that $\langle r\rangle\langle m\rangle \subseteq \mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D})$, by hypothesis, we get either $\langle m\rangle \subseteq \mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D}) + \mathfrak{I}_{gr}(\mathfrak{D})$ or $\langle r\rangle \subseteq (\mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D}) + \mathfrak{I}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$. That is either $\langle m\rangle \subseteq \mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D})$ or $\langle r\rangle \subseteq (\mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$. If $\langle r\rangle \subseteq (\mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D}) = P$, then $r \in P$, thus $r + P = P$ as a contradiction. So we have $\langle m\rangle \subseteq \mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D})$ implies $m \in \mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D})$. Hence $m + \mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D}) = \mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D})$. Therefore, $\mathfrak{D}/(\mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D}))$ is a Gr-torsion free $\mathfrak{A}/P$-module.

(3)⇒(4) Let $r \in h(\mathfrak{A})-P$ and let $m \in (\mathcal{V}+\mathfrak{I}_{gr}(\mathfrak{D}) :_{\mathfrak{D}} \langle r\rangle)\cap h(\mathfrak{D})$. Then $\langle r\rangle m \subseteq \mathcal{V}+\mathfrak{I}_{gr}(\mathfrak{D})$ it follows that $rm \in \mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D})$. Thus $(r+P)(m+\mathcal{V}+\mathfrak{I}_{gr}(\mathfrak{D})) = \mathcal{V}+\mathfrak{I}_{gr}(\mathfrak{D})$, since $r \notin P$ and $\mathfrak{D}/(\mathcal{V}+\mathfrak{I}_{gr}(\mathfrak{D}))$ is a Gr-torsion free $\mathfrak{A}/P$-module we get $m + \mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D}) = \mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D})$, thus $m \in \mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D})$. Hence $(\mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D}) :_{\mathfrak{D}} \langle r\rangle) \subseteq \mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D})$ for each $r \in h(\mathfrak{A}) - P$. Now, let $m \in (\mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D})) \cap h(\mathfrak{D})$ and $r \in h(\mathfrak{A}) - P$, then $rm \in \mathcal{V}+\mathfrak{I}_{gr}(\mathfrak{D})$ it follows that $\langle r\rangle m \subseteq \mathcal{V}+\mathfrak{I}_{gr}(\mathfrak{D})$, thus $m \in (\mathcal{V}+\mathfrak{I}_{gr}(\mathfrak{D}) :_{\mathfrak{D}} \langle r\rangle)$. Hence $\mathcal{V}+\mathfrak{I}_{gr}(\mathfrak{D}) \subseteq (\mathcal{V}+\mathfrak{I}_{gr}(\mathfrak{D}) :_{\mathfrak{D}} \langle r\rangle)$ for each $r \in h(\mathfrak{A}) - P$.

(4)⇒(5) Let $x \in h(\mathfrak{D}) - (\mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D}))$. Let $r \in (\mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \langle x\rangle) \cap h(\mathfrak{A})$. Suppose the contrary, $r \notin P$. Since $r \in (\mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \langle x\rangle) \cap h(\mathfrak{A})$, then $r\langle x\rangle \subseteq \mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D})$ it follows that $rx \in \mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D})$, thus $\langle r\rangle x \subseteq \mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D})$. That is $x \in (\mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D}) :_{\mathfrak{D}} \langle r\rangle)$ but by hypothesis $(\mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D}) :_{\mathfrak{D}} \langle r\rangle) = \mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D})$, for each $r \in h(\mathfrak{A}) - P$, so we get $x \in \mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D})$ a contradiction. Hence, $r \in P$. Therefore, $(\mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \langle x\rangle) \subseteq P$. Now, let $r \in P \cap h(\mathfrak{A}) = (\mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D}) \cap h(\mathfrak{A})$. Then $rM \subseteq \mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D})$. In particular, $rx \in \mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D})$, thus $r\langle x\rangle \subseteq \mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D})$ implies $r \in (\mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \langle x\rangle)$. Hence $P \subseteq (\mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \langle x\rangle)$. Therefore $(\mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \langle x\rangle) = P$.

(5)⇒(1) Let $rm \in \mathcal{V}$, where $r \in h(\mathfrak{A})$ and $m \in h(\mathfrak{D})$. Suppose $m \notin \mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D})$, we need to prove that $r \in P$. Since $rm \in \mathcal{V} \subseteq \mathcal{V}+\mathfrak{I}_{gr}(\mathfrak{D})$, then $r\langle m\rangle \subseteq \mathcal{V}+\mathfrak{I}_{gr}(\mathfrak{D})$ it follows that $r \in (\mathcal{V}+\mathfrak{I}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \langle m\rangle)$, apply hypothesis, we have $(\mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \langle m\rangle) = (\mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D}) = P$, hence $r \in P$. Therefore, $\mathcal{V}$ is a graded $\mathfrak{I}_{gr}$-prime submodule of $\mathfrak{D}$.

Theorem 3. If $\mathcal{V}$ is a graded $\mathfrak{I}_{gr}$-prime submodule of $\mathfrak{D}$, then $(\mathcal{V}+\mathfrak{I}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$ is a graded $\mathfrak{I}_{gr}$-prime ideal of $\mathfrak{A}$.

Proof. We show that P is a graded prime ideal of $\mathfrak{A}$, where $P = (\mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$. Let $ab \in P$, where $a, b \in h(\mathfrak{A})$. Suppose $a \notin P$, then there exists $x \in h(\mathfrak{D})$ such that $ax \notin \mathcal{V}+\mathfrak{I}_{gr}(\mathfrak{D})$. Since $ab \in P$, then $abM \subseteq \mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D})$. In particular, $b(ax) \in \mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D})$. Thus $b(ax) + \mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D}) = \mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D})$, it follows that $(b + P)(ax + \mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D})) = \mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D})$. Since $\mathcal{V}$ is a graded $\mathfrak{I}_{gr}$-prime submodule, by Theorem 2, we get $\mathfrak{D}/(\mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D}))$ is a Gr-torsion free $\mathfrak{A}/P$-module. But $ax \notin \mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D})$, then $b + P = P$, so we have $b \in P$. Therefore, $P = (\mathcal{V} + \mathfrak{I}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$ is a graded prime ideal of $\mathfrak{A}$, then P is a graded $\mathfrak{I}_{gr}$-prime ideal of $\mathfrak{A}$, by Theorem 1.

A graded ring $\mathfrak{A}$ is called a graded integral domain if whenever $ab = 0$, where a, $b \in h(\mathfrak{A})$, then either $a = 0$ or $b = 0$ [10].

In the following example, it is shown that the converse of Theorem 3 is not necessarily true.

Example 3. Let $\mathfrak{G} = \mathbb{Z}_2$, $\mathfrak{A} = \mathbb{Z}$ be a $\mathfrak{G}$-graded ring with $\mathfrak{A}_0 = \mathbb{Z}$, $\mathfrak{A}_1 = \{0\}$, and $\mathfrak{D} = \mathbb{Z} \times \mathbb{Z}$ be a graded $\mathfrak{A}$-module with $\mathfrak{D}_0 = \mathbb{Z} \times \mathbb{Z}$, $\mathfrak{D}_1 = \{(0,0)\}$. The graded submodule $\mathcal{V} = 2\mathbb{Z} \times \langle 0 \rangle$ is not graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$, by Example 2. However, $P = (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D}) = (2\mathbb{Z} \times \langle 0 \rangle :_{\mathfrak{A}} \mathbb{Z} \times \mathbb{Z}) = \langle 0 \rangle$ is a graded prime ideal of $\mathbb{Z}$. Since if $ab \in P = \langle 0 \rangle$, where $a, b \in h(\mathbb{Z})$, then $ab = 0$ implies either $a = 0$ or $b = 0$ as $\mathbb{Z}$ is a graded integral domain. Thus $a \in P$ or $b \in P$, by Theorem 1, we have P is a graded $\mathfrak{J}_{gr}$-prime ideal of $\mathbb{Z}$.

The following example shows that the residual of graded $\mathfrak{J}_{gr}$-prime submodule is not necessarily a graded $\mathfrak{J}_{gr}$-prime ideal of $\mathfrak{A}$.

Example 4. Let $\mathfrak{G} = \mathbb{Z}_2$, $\mathfrak{A} = \mathbb{Z}$ be a $\mathfrak{G}$-graded ring with $\mathfrak{A}_0 = \mathbb{Z}$ and $\mathfrak{A}_1 = \{0\}$. Let $\mathfrak{D} = \mathbb{Z}_{12}$ be a graded $\mathfrak{A}$-module with $\mathfrak{D}_0 = \mathbb{Z}_{12}$ and $\mathfrak{D}_1 = \{\overline{0}\}$. Consider $\mathcal{V} = \{\overline{0}, \overline{4}, \overline{8}\} = \langle \overline{4} \rangle$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathbb{Z}$-module $\mathbb{Z}_{12}$, but $(\mathcal{V} :_{\mathbb{Z}} \mathbb{Z}_{12})$ is not graded $\mathfrak{J}_{gr}$-prime ideal of $\mathbb{Z}$, since there exists $2 \in h(\mathbb{Z})$ such that $2 \cdot 2 = 4 \in (\mathcal{V} :_{\mathbb{Z}} \mathbb{Z}_{12})$, but $2 \notin (\mathcal{V} :_{\mathbb{Z}} \mathbb{Z}_{12}) + \mathfrak{J}_{gr}(\mathbb{Z}) = (\mathcal{V} :_{\mathbb{Z}} \mathbb{Z}_{12}) + \{0\} = (\mathcal{V} :_{\mathbb{Z}} \mathbb{Z}_{12}) = 4\mathbb{Z}$ and $2 \notin ((\mathcal{V} :_{\mathbb{Z}} \mathbb{Z}_{12}) + \mathfrak{J}_{gr}(\mathbb{Z}) :_{\mathbb{Z}} \mathbb{Z}) = ((\mathcal{V} :_{\mathbb{Z}} \mathbb{Z}_{12}) :_{\mathbb{Z}} \mathbb{Z})$.

Theorem 4. If $\mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$ with $\mathfrak{J}_{gr}(\mathfrak{D}) \subseteq \mathcal{V}$, then $(\mathcal{V} :_{\mathfrak{A}} \mathfrak{D})$ is a graded $\mathfrak{J}_{gr}$-prime ideal of $\mathfrak{A}$.

Proof. Since $\mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$, then $(\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$ is a graded $\mathfrak{J}_{gr}$-prime ideal of $\mathfrak{A}$ by Theorem 3. But $\mathfrak{J}_{gr}(\mathfrak{D}) \subseteq \mathcal{V}$, thus $(\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D}) = (\mathcal{V} :_{\mathfrak{A}} \mathfrak{D})$. Therefore, $(\mathcal{V} :_{\mathfrak{A}} \mathfrak{D})$ is a graded $\mathfrak{J}_{gr}$-prime ideal of $\mathfrak{A}$.

Theorem 5. Let $\mathfrak{A}$ be a $\mathfrak{G}$-graded ring, $\mathfrak{D}$ — a graded $\mathfrak{A}$-module and $\mathcal{V}$ — a proper graded submodule of $\mathfrak{D}$. Then the following statements are equivalent:

1) $\mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$.

2) $(\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D}) = (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \langle c \rangle)$ for each $c \in h(\mathfrak{D}) - (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}))$.

3) $(\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D}) = (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathcal{K})$ for each graded submodule $\mathcal{K}$ of $\mathfrak{D}$ such that $\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) \subsetneq \mathcal{K}$.

Proof. (1)$\Rightarrow$(2) By Theorem 2.

(2) $\Rightarrow$(3) Let $\mathcal{K}$ be a graded submodule of $\mathfrak{D}$ such that $\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) \subsetneq \mathcal{K}$. It is clear that $(\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D}) \subseteq (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathcal{K})$ since if $r \in (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D}) \cap h(\mathfrak{A})$, then $rM \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$, but $\mathcal{K} \subseteq \mathfrak{D}$ implies $rK \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$, thus $r \in (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathcal{K})$, hence $(\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D}) \subseteq (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathcal{K})$. Now, let $s \in (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathcal{K}) \cap h(\mathfrak{A})$, then $sK \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$, but $\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) \subsetneq \mathcal{K}$ so there exists $x \in \mathcal{K} \cap h(\mathfrak{D})$ and $x \notin \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$. In particular $sx \in \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$, it follows that $s\langle x \rangle \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$ implies $s \in (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \langle x \rangle)$ but by hypothesis we have $(\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \langle x \rangle) = (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$, so $s \in (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$, hence $(\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathcal{K}) \subseteq (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$. Therefore, $(\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D}) = (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathcal{K})$ for each $\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) \subsetneq \mathcal{K}$.

(3)$\Rightarrow$(1) Let $rm \in \mathcal{V}$ and $m \notin \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$, where $r \in h(\mathfrak{A})$ and $m \in h(\mathfrak{D})$. Take $\mathcal{K} = \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) + \langle m \rangle$, where $\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) \subsetneq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) + \langle m \rangle$ (since $m \in \mathcal{K} - (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}))$), it follows that $rK = r(\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})) + r\langle m \rangle \subseteq (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})) + \mathcal{V} = \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$, so $r \in (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathcal{K})$. But by hypothesis, we have $(\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathcal{K}) = (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$, thus $r \in (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$. Therefore, $\mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$.

A proper graded submodule $\mathcal{V}$ is called a graded small (Gr-small) of $\mathfrak{D}$ if $\mathfrak{D} = \mathcal{V} + L$ for some graded submodule L of $\mathfrak{D}$ implies that $L = \mathfrak{D}$. A graded $\mathfrak{A}$-module $\mathfrak{D}$ is said to be a graded hollow (Gr-hollow) module if every proper graded submodule $\mathcal{V}$ of $\mathfrak{D}$ is a Gr-small [13].

Theorem 6. Let $\mathfrak{A}$ be a $\mathfrak{G}$-graded ring, $\mathfrak{D}$ a Gr-hollow $\mathfrak{A}$-module and $\mathfrak{J}_{gr}(\mathfrak{D})$ a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$, then every proper graded submodule of $\mathfrak{D}$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$.

Proof. Let $\mathcal{V}$ be a proper graded submodule of $\mathfrak{D}$ and let $rm \in \mathcal{V}$ where $r \in h(\mathfrak{A})$ and $m \in h(\mathfrak{D})$. Since $\mathfrak{D}$ is a Gr-hollow then $\mathcal{V}$ is a Gr-small, so $rm \in \mathcal{V} \subseteq \sum\{A : A \text{ is a } Gr\text{-small}\} = \mathfrak{J}_{gr}(\mathfrak{D})$ by [14; Theorem 2.10]. But $\mathfrak{J}_{gr}(\mathfrak{D})$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$. Thus either $m \in \mathfrak{J}_{gr}(\mathfrak{D}) + \mathfrak{J}_{gr}(\mathfrak{D}) = \mathfrak{J}_{gr}(\mathfrak{D}) \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$ or $rM \subseteq \mathfrak{J}_{gr}(\mathfrak{D}) + \mathfrak{J}_{gr}(\mathfrak{D}) = \mathfrak{J}_{gr}(\mathfrak{D}) \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$. So either $m \in \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$ or $r \in (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$. Therefore, $\mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$.

A nonempty subset $S \subseteq h(\mathfrak{A})$ of a $\mathfrak{G}$-graded ring $\mathfrak{A}$ is called multiplicatively closed subset (briefly, $m.c.s.$) of $\mathfrak{A}$ if $0 \notin S$, $1 \in S$ and $x \cdot y \in S$ for all $x, y \in S$. Let $S \subseteq h(\mathfrak{A})$ be a multiplicatively closed subset of $\mathfrak{A}$ and $\mathcal{V}$ be a graded submodule of $\mathfrak{D}$ then $\mathcal{V}(S) = \{x \in \mathfrak{D} : \text{there exists } t \in S \text{ such that } tx \in \mathcal{V}\}$ be a graded submodule of $\mathfrak{D}$ is said to be the component of $\mathcal{V}$ determined by S, or simply the S-component of $\mathcal{V}$. We conclude from definition $\mathcal{V} \subseteq \mathcal{V}(S)$.

Lemma 1. Let P be a proper graded ideal of $\mathfrak{A}$. Then P is a graded prime ideal of a graded ring $\mathfrak{A}$ if and only if $h(\mathfrak{A}) - P$ is a $m.c.s.$ of $\mathfrak{A}$.

Proof. Let P is a proper graded submodule of $\mathfrak{D}$, then $0 \in P$, $1 \notin P$ (if $1 \in P$, then $P = \mathfrak{D}$, thus P is not proper a contradiction) and since P is a graded prime ideal of $\mathfrak{A}$, we have $0 \notin h(\mathfrak{A}) - P$, $1 \in h(\mathfrak{A}) - P$ and $ab \in h(\mathfrak{A}) - P$ for each $a, b \in h(\mathfrak{A}) - P$. Therefore, $h(\mathfrak{A}) - P$ is a $m.c.s.$ of $\mathfrak{A}$. Conversely, suppose the contrary, P is not graded prime ideal of $\mathfrak{A}$, then there exist $x, y \in h(\mathfrak{A}) - P$ with $xy \in P$. Since $h(\mathfrak{A}) - P$ is $m.c.s.$ of $\mathfrak{A}$, then $xy \in h(\mathfrak{A}) - P$ which is a contradiction.

Theorem 7. Let $\mathfrak{A}$ be a $\mathfrak{G}$-graded ring, $\mathfrak{D}$ — a graded $\mathfrak{A}$-module and $\mathcal{V}$ — a graded submodule of $\mathfrak{D}$. Then $\mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$ if and only if the graded ideal $P = (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$ is a graded prime of $\mathfrak{A}$ and $\mathcal{V}(S) \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$ for each $S \subseteq h(\mathfrak{A})$ a $m.c.s.$ of $\mathfrak{A}$ such that $S \cap P = \phi$.

Proof. Let $\mathcal{V}$ be a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$ and $S \subseteq h(\mathfrak{A})$ be $m.c.s.$ of $\mathfrak{A}$ with $S \cap P = \phi$. Let $ab \in P$, where $a, b \in h(\mathfrak{A})$. Suppose $a \notin P$, then there exists $x \in h(\mathfrak{D})$ such that $ax \notin \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$. Since $ab \in P$, then $abM \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$. In particular, $b(ax) \in \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$, thus $b(ax) + \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) = \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$, it follows that $(b+P)(ax + \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})) = \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$. Since $\mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-prime submodule, by Theorem 2, we get $\mathfrak{D}/(\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}))$ is a Gr-torsion free $\mathfrak{A}/P$-module. But $ax \notin \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$, then $b + P = P$, so we have $b \in P$. Therefore, $P = (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$ is a graded prime ideal of $\mathfrak{A}$. Now, let $a \in \mathcal{V}(S) \cap h(\mathfrak{D})$, then there exists $s \in S$ such that $sa \in \mathcal{V}$. Since $\mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$, $S \cap (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D}) = \phi$ and $s \notin (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$, we have $a \in \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$. Hence $\mathcal{V}(S) \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$. Conversely, suppose not, let $rm \in \mathcal{V}$, where $r \in h(\mathfrak{A})$ and $m \in h(\mathfrak{D})$, but $r \notin (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$ and $m \notin \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$. Assume that P is a graded prime ideal of $\mathfrak{A}$, by Lemma 1, we have $h(\mathfrak{A}) - P$ is a $m.c.s.$ of $\mathfrak{A}$. Since $(h(\mathfrak{A}) - P) \cap P = \phi$, by assumption we have $\mathcal{V}(h(\mathfrak{A}) - P) \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$. But $m \notin \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$ and $\mathcal{V}(h(\mathfrak{A}) - P) \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$ it follows that $m \notin \mathcal{V}(h(\mathfrak{A}) - P)$. This yields that for each $s \in h(\mathfrak{A}) - P$ we have $sm \notin \mathcal{V}$, but $r \in h(\mathfrak{A}) - P$, then $rm \notin \mathcal{V}$ which is a contradiction. Therefore, $\mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$.

The following example shows that the intersection of two graded $\mathfrak{J}_{gr}$-prime submodules needs, not to be a graded $\mathfrak{J}_{gr}$-prime submodule.

Example 5. Let $\mathfrak{G} = \mathbb{Z}_2$ and $\mathfrak{A} = \mathbb{Z}$ be a $\mathfrak{G}$-graded ring with $\mathfrak{A}_0 = \mathbb{Z}$ and $\mathfrak{A}_1 = \{0\}$. Let $\mathfrak{D} = \mathbb{Z}_6$ be a graded $\mathfrak{A}$-module with $\mathfrak{D}_0 = \mathbb{Z}_6$ and $\mathfrak{D}_1 = \{\overline{0}\}$. Consider $\mathcal{V} = \langle\overline{2}\rangle = \{\overline{0}, \overline{2}, \overline{4}\}$ and $L = \langle\overline{3}\rangle = \{\overline{0}, \overline{3}\}$ are graded submodules of $\mathbb{Z}_6$. Then $\mathcal{V} \cap L = \langle\overline{0}\rangle$ is not a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$, since there exist $3 \in h(\mathbb{Z})$ and $\overline{2} \in h(\mathbb{Z}_6)$ such that $3 \cdot \overline{2} = \overline{0} \in \mathcal{V} \cap L$, but $\overline{2} \notin (\mathcal{V} \cap L) + \mathfrak{J}_{gr}(\mathbb{Z}_6) = \langle\overline{0}\rangle + \langle\overline{0}\rangle = \langle\overline{0}\rangle$ and $3 \notin ((\mathcal{V} \cap L) + \mathfrak{J}_{gr}(\mathbb{Z}_6) :_{\mathbb{Z}} \mathbb{Z}_6) = 6\mathbb{Z}$. However, an easy computation and using the definition of graded $\mathfrak{J}_{gr}$-prime submodule to show that $\mathcal{V}$ and L are graded $\mathfrak{J}_{gr}$-prime submodules of $\mathfrak{D}$.

The next theorem shows that the intersection of two graded $\mathfrak{J}_{gr}$-prime submodules is a graded $\mathfrak{J}_{gr}$-prime submodule under conditions.

Theorem 8. Let $\mathfrak{A}$ be a $\mathfrak{G}$-graded ring, $\mathfrak{D}$ a graded $\mathfrak{A}$-module and $\mathcal{V}$, L be two graded submodules of $\mathfrak{D}$ such that $\mathcal{V} \subseteq \mathfrak{J}_{gr}(\mathfrak{D})$ or $L \subseteq \mathfrak{J}_{gr}(\mathfrak{D})$. If $\mathcal{V}$ and L are graded $\mathfrak{J}_{gr}$-prime submodules of $\mathfrak{D}$, then $\mathcal{V} \cap L$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$.

Proof. Assume that $\mathcal{V}$ and L are graded $\mathfrak{J}_{gr}$-prime submodules of $\mathfrak{D}$. Let $rm \in \mathcal{V} \cap L$, where $r \in h(\mathfrak{A})$ and $m \in h(\mathfrak{D})$, then $rm \in \mathcal{V}$ and $rm \in L$. If $\mathcal{V} \subseteq \mathfrak{J}_{gr}(\mathfrak{D})$, since $\mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$, then either $rM \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) = \mathfrak{J}_{gr}(\mathfrak{D}) \subseteq (\mathcal{V} \cap L) + \mathfrak{J}_{gr}(\mathfrak{D})$ or $m \in \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) = = \mathfrak{J}_{gr}(\mathfrak{D}) \subseteq (\mathcal{V} \cap L) + \mathfrak{J}_{gr}(\mathfrak{D})$. Thus either $r \in ((\mathcal{V} \cap L) + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$ or $m \in (\mathcal{V} \cap L) + \mathfrak{J}_{gr}(\mathfrak{D})$. Hence $\mathcal{V} \cap L$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$. Similarly, If $L \subseteq \mathfrak{J}_{gr}(\mathfrak{D})$, we get $\mathcal{V} \cap L$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$.

Theorem 9. Let $\mathfrak{A}$ be a $\mathfrak{G}$-graded ring, $\mathfrak{D}$ and $\mathfrak{D}'$ be two graded $\mathfrak{A}$-modules and $\mathcal{V}$, $\mathcal{V}'$ be two proper graded submodules of $\mathfrak{D}$, $\mathfrak{D}'$,respectively. If $\mathcal{V} \times \mathcal{V}'$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D} \times \mathfrak{D}'$, then $\mathcal{V}$ and $\mathcal{V}'$ are graded $\mathfrak{J}_{gr}$-prime submodules of $\mathfrak{D}$ and $\mathfrak{D}'$, respectively.

Proof. To prove $\mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$, let $rm \in \mathcal{V}$, where $r \in h(\mathfrak{A})$ and $m \in h(\mathfrak{D})$, then $r(m, 0) \in \mathcal{V} \times \mathcal{V}'$ as $r(m, 0) = (rm, 0) \in \mathcal{V} \times \mathcal{V}'$. Since $\mathcal{V} \times \mathcal{V}'$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D} \times \mathfrak{D}'$, so either $r \in ((\mathcal{V} \times \mathcal{V}') + \mathfrak{J}_{gr}(\mathfrak{D} \times \mathfrak{D}') :_{\mathfrak{A}} \mathfrak{D} \times \mathfrak{D}')$ or $(m, 0) \in (\mathcal{V} \times \mathcal{V}') + \mathfrak{J}_{gr}(\mathfrak{D} \times \mathfrak{D}')$. If $r \in ((\mathcal{V} \times \mathcal{V}') + \mathfrak{J}_{gr}(\mathfrak{D} \times \mathfrak{D}') :_{\mathfrak{A}} \mathfrak{D} \times \mathfrak{D}')$, then $r(\mathfrak{D} \times \mathfrak{D}') \subseteq (\mathcal{V} \times \mathcal{V}') + \mathfrak{J}_{gr}(\mathfrak{D} \times \mathfrak{D}') = (\mathcal{V} \times \mathcal{V}') + + (\mathfrak{J}_{gr}(\mathfrak{D}) \times \mathfrak{J}_{gr}(\mathfrak{D}'))$, it follows that $(rM \times rM') \subseteq (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})) \times (\mathcal{V}' + \mathfrak{J}_{gr}(\mathfrak{D}'))$, so $rM \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$ and $rM' \subseteq \mathcal{V}' + \mathfrak{J}_{gr}(\mathfrak{D}')$. This implies that $r \in (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$ and $r \in (\mathcal{V}' + \mathfrak{J}_{gr}(\mathfrak{D}') :_{\mathfrak{A}} \mathfrak{D}')$. If $(m, 0) \in (\mathcal{V} \times \mathcal{V}') + \mathfrak{J}_{gr}(\mathfrak{D} \times \mathfrak{D}')$, then $(m, 0) \in (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})) \times (\mathcal{V}' + \mathfrak{J}_{gr}(\mathfrak{D}')$. Thus $m \in \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$ and $0 \in \mathcal{V}' + \mathfrak{J}_{gr}(\mathfrak{D}')$. Hence $\mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$. In a similar manner, we can prove that $\mathcal{V}'$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}'$.

Theorem 10. Let $\mathfrak{A}$ be a $\mathfrak{G}$-graded ring, $\mathfrak{D}$ and $\mathfrak{D}'$ be two graded $\mathfrak{A}$-modules and $f : \mathfrak{D} \longrightarrow \mathfrak{D}'$ be a graded epimorphism. If $\mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$ containing $kerf$, then $f(\mathcal{V})$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}'$.

Proof. Since $\mathcal{V}$ is a proper graded submodule of $\mathfrak{D}$, by [15; Lemma 4.8], we have $f(\mathcal{V})$ is a proper graded submodule of $\mathfrak{D}'$. Let $rm' \in f(\mathcal{V})$, where $r \in h(\mathfrak{A})$ and $m' \in h(\mathfrak{D}')$, since f is onto and $m' \in h(\mathfrak{D}')$, then there exists $m \in h(\mathfrak{D})$ such that $f(m) = m'$. Thus $rm' = rf(m) = f(rm) \in f(\mathcal{V})$, so there exists $n \in \mathcal{V} \cap h(\mathfrak{D})$ such that $f(rm) = f(n)$, thus $f(rm - n) = 0$, it follows that $rm - n \in kerf \subseteq \mathcal{V}$ so $rm + \mathcal{V} = n + \mathcal{V} = \mathcal{V}$. That is $rm \in \mathcal{V}$, but $\mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$, then either $r \in (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$ or $m \in \mathcal{V} + \mathfrak{J}_{gr}(\mathcal{V})$. If $r \in (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$, then $rM \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$, thus $f(rM) \subseteq f(\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})) = f(\mathcal{V}) + f(\mathfrak{J}_{gr}(\mathfrak{D}))$, implies that $rf(\mathfrak{D}) = rM' \subseteq f(\mathcal{V}) + f(\mathfrak{J}_{gr}(\mathfrak{D}))$, by [14; Theorem 2.12], we get $f(\mathfrak{J}_{gr}(\mathfrak{D})) \subseteq \mathfrak{J}_{gr}(\mathfrak{D}')$. So $rM' \subseteq f(\mathcal{V}) + \mathfrak{J}_{gr}(\mathfrak{D}')$, then $r \in (f(\mathcal{V}) + \mathfrak{J}_{gr}(\mathfrak{D}') :_{\mathfrak{A}} \mathfrak{D}')$. If $m \in \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$, then $f(m) \in f(\mathcal{V}) + f(\mathfrak{J}_{gr}(\mathfrak{D}))$, but $f(m) = m'$, by [14; Theorem 2.12], we have $m' \in f(\mathcal{V}) + f(\mathfrak{J}_{gr}(\mathfrak{D})) \subseteq f(\mathcal{V}) + \mathfrak{J}_{gr}(\mathfrak{D}')$. Hence $f(\mathcal{V})$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}'$.

Theorem 11. Let $\mathfrak{A}$ be a $\mathfrak{G}$-graded ring, $\mathfrak{D}$ and $\mathfrak{D}'$ be a graded $\mathfrak{A}$-modules. Let $f : \mathfrak{D} \longrightarrow \mathfrak{D}'$ be a graded epimorphism with $kerf$ is a Gr-small submodule of $\mathfrak{D}$. If $\mathcal{V}'$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}'$, then $f^{-1}(\mathcal{V}')$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$.

Proof. Since $\mathcal{V}'$ is a proper graded submodule of $\mathfrak{D}'$, by [15; Lemma 5.2], we have $f^{-1}(\mathcal{V}')$ is a proper graded submodule of $\mathfrak{D}$. Let $rm \in f^{-1}(\mathcal{V}')$, where $r \in h(\mathfrak{A})$ and $m \in h(\mathfrak{D})$, then $f(rm) \in \mathcal{V}'$, thus $rf(m) \in \mathcal{V}'$ since $\mathcal{V}'$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}'$, then either $r \in (\mathcal{V}' + \mathfrak{J}_{gr}(\mathfrak{D}') :_{\mathfrak{A}} \mathfrak{D}')$ or $f(m) \in \mathcal{V}' + \mathfrak{J}_{gr}(\mathfrak{D}')$. If $r \in (\mathcal{V}' + \mathfrak{J}_{gr}(\mathfrak{D}') :_{\mathfrak{A}} \mathfrak{D}')$, then $rM' \subseteq \mathcal{V}' + \mathfrak{J}_{gr}(\mathfrak{D}')$ since f is a graded epimorphism, then f is onto, so $\mathfrak{D}' = f(\mathfrak{D})$ implies that $rf(\mathfrak{D}) \subseteq \mathcal{V}' + \mathfrak{J}_{gr}(\mathfrak{D}')$ then $f(rM) \subseteq \mathcal{V}' + + \mathfrak{J}_{gr}(\mathfrak{D}')$, it follows that $rM \subseteq f^{-1}(\mathcal{V}' + \mathfrak{J}_{gr}(\mathfrak{D}')) = f^{-1}(\mathcal{V}') + f^{-1}(\mathfrak{J}_{gr}(\mathfrak{D}'))$, since f is a graded epimorphism and $kerf$ is a Gr-small of $\mathfrak{D}$ [14; Theorem 2.12], we get $f(\mathfrak{J}_{gr}(\mathfrak{D})) = \mathfrak{J}_{gr}(\mathfrak{D}')$. Thus $\mathfrak{J}_{gr}(\mathfrak{D}) = f^{-1}(\mathfrak{J}_{gr}(\mathfrak{D}'))$, so $rM \subseteq f^{-1}(\mathcal{V}') + \mathfrak{J}_{gr}(\mathfrak{D})$, it follows that $r \in (f^{-1}(\mathcal{V}') + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$. If

$f(m) \in \mathcal{V}' + \mathfrak{J}_{gr}(\mathfrak{D}')$, then $m \in f^{-1}(\mathcal{V}') + f^{-1}(\mathfrak{J}_{gr}(\mathfrak{D}')) = f^{-1}(\mathcal{V}') + \mathfrak{J}_{gr}(\mathfrak{D})$. Hence $f^{-1}(\mathcal{V}')$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$.

Corollary 1. Let $\mathfrak{A}$ be a $\mathfrak{G}$-graded ring, $\mathfrak{D}$ a graded $\mathfrak{A}$-module and $\mathcal{V}$, $\mathcal{K}$ proper graded submodules of $\mathfrak{D}$ such that $\mathcal{K} \subseteq \mathcal{V}$ and $kerf$ is Gr-small of $\mathfrak{D}$. If $\mathcal{V}/\mathcal{K}$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}/\mathcal{K}$, then $\mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$.

Proof. Define $f : \mathfrak{D} \longrightarrow \mathfrak{D}/\mathcal{K}$ by $f(x) = x + \mathcal{K}$. Then f is a graded epimorphism, so by Theorem 11, we get $f^{-1}(\mathcal{V}/\mathcal{K}) = \mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$.

Recall that a proper graded submodule $\mathcal{V}$ of $\mathfrak{D}$ is called a graded $\mathfrak{J}_{gr}$-pure submodule of $\mathfrak{D}$, if $\mathcal{V} \cap \mathcal{U}\mathfrak{D} = \mathcal{U}\mathcal{V} + (\mathfrak{J}_{gr}(\mathfrak{D}) \cap \mathcal{V} \cap \mathcal{U}\mathfrak{D})$ for each proper graded ideal $\mathcal{U}$ of $\mathfrak{A}$, see [14; Definition 2.19].

The following example shows that a graded $\mathfrak{J}_{gr}$-pure submodule of $\mathfrak{D}$ not necessarily a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$.

Example 6. Let $\mathfrak{A} = \mathbb{Z}$ be a $\mathfrak{G}$-graded ring with $\mathfrak{A}_0 = \{0\}$ and $\mathfrak{A}_1 = \mathbb{Z}$, where $\mathfrak{G} = \mathbb{Z}_2$. Let $\mathfrak{D} = \mathbb{Z}_6$ be a graded $\mathfrak{A}$-module with $\mathfrak{D}_0 = \{\overline{0}\}$ and $\mathfrak{D}_1 = \mathbb{Z}_6$. $\mathcal{V} = \{\overline{0}\}$ is a graded $\mathfrak{J}_{gr}$-pure submodule of $\mathfrak{D}$. However $\mathcal{V}$ is not graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$ since there exist $3 \in h(\mathfrak{A})$ and $\overline{2} \in h(\mathfrak{D})$ such that $3 \cdot \overline{2} = \overline{0} \in \mathcal{V}$ but $3 \notin (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$ and $\overline{2} \notin \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) = \{\overline{0}\}$.

The next theorem shows that a graded $\mathfrak{J}_{gr}$-pure submodule of $\mathfrak{D}$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$ with under some conditions.

Theorem 12. Let $\mathfrak{A}$ be a $\mathfrak{G}$-graded ring, $\mathfrak{D}$ a Gr-torsion free $\mathfrak{A}$-module and $\mathcal{V}$ a proper graded submodule of $\mathfrak{D}$ with $\mathfrak{J}_{gr}(\mathfrak{D}) = \{0\}$. If $\mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-pure submodule of $\mathfrak{D}$, then $\mathcal{V}$ is a $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$.

Proof. Let $rm \in \mathcal{V}$, where $r \in h(\mathfrak{A})$ and $m \in h(\mathfrak{D})$, assume that $r \notin (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$. Thus $rm \in \mathcal{V} \cap \langle r\rangle\mathfrak{D} = \langle r\rangle\mathcal{V} + (\mathfrak{J}_{gr}(\mathfrak{D}) \cap \mathcal{V} \cap \langle r\rangle\mathfrak{D})$ as $\mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-pure submodule of $\mathfrak{D}$. But $\mathfrak{J}_{gr}(\mathfrak{D}) = \{0\}$. Thus $rm \in \langle r\rangle\mathcal{V}$, it follows that there exists $n \in \mathcal{V} \cap h(\mathfrak{D})$ and $r' \in h(\mathfrak{A})$ such that $rm = rr'n$. Thus $rm - rr'n = 0$ implies $r(m - r'n) = 0$. Since $\mathfrak{D}$ is a Gr-torsion free and $r \notin (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$, then $m = r'n \in \mathcal{V} \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$. Hence $m \in \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$. Therefore, $\mathcal{V}$ is a $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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