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Kelvin-Voigt equations with memory: existence, uniqueness and regularity of solutions

In general, the study of inverse problems is realizable only in the case when the corresponding direct problems have the unique solution with some necessary properties such as continuity and regularity. In this paper, we study initial-boundary value problems for the system of 2D-3D nonlinear Kelvin-Voigt equations with memory, which describes a motion of an incompressible homogeneous non-Newtonian fluids with viscoelastic and relaxation properties. The investigation of these direct problems is related to the study of inverse problems for this system, which requires the continuity and regularity of solutions to these direct problems and their derivatives. The system, in addition to the initial condition, is supplemented with one of the boundary conditions: stick and slip boundary conditions. In both cases of these boundary conditions, the global in time existence and uniqueness of strong solutions to these initial-boundary value problems were proved. Moreover, under suitable assumptions on the data, the regularity of solutions and their derivatives were established.

Keywords: Kelvin-Voigt system, slip and stick boundary conditions, strong solutions, global existence and uniqueness, smoothness.

Introduction

Let $\Omega \in \mathbb{R}^d,\ d = 2, 3$, be a bounded domain with a smooth boundary $\partial\Omega$, and $Q_T = \Omega \times (0, T)$ be a cylinder with a lateral $\Gamma_T = \partial\Omega \times [0, T]$. Let us consider the following initial-boundary value problem for the system of nonlinear Kelvin-Voigt (Navier-Stokes-Voigt) equations with memory

$$\mathbf{v}_t + (\mathbf{v} \cdot \nabla)\,\mathbf{v} - \varkappa\Delta\mathbf{v}_t - \nu\Delta\mathbf{v} - \int\limits_0^t K(t-\tau)\Delta\mathbf{v}(\mathbf{x},\tau)d\tau + \nabla p = f, \quad (x,t) \in Q_T, \tag{1}$$

$$\operatorname{div}\mathbf{v}(\mathbf{x},t) = 0, \quad (x,t) \in Q_T, \tag{2}$$

supplemented with the initial condition

$$\mathbf{v}(\mathbf{x},0) = \mathbf{v}_0\,(\mathbf{x})\,, \quad \mathbf{x} \in \Omega \tag{3}$$

and one of the following boundary conditions: stick boundary condition

$$\mathbf{v}(\mathbf{x},t) = 0, \quad (\mathbf{x},t) \in \Gamma_T \tag{4}$$

or slip boundary condition

$$\mathbf{v_n}(\mathbf{x},t) = \mathbf{v}\cdot\mathbf{n} = 0, \quad \operatorname{rot}\mathbf{v}\times\mathbf{n} = 0,\ (\mathbf{x},t) \in \Gamma_T. \tag{5}$$

System (1)-(2) is called a Kelvin-Voigt (also called Navier-Stokes-Voigt) system with memory or an integro-differential Kelvin-Voigt system, and models a motion of viscoelastic incompressible

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non-Newtonian fluids [1–5]. Most of hydrodynamics problems were considered with stick-boundary condition (4), however, in recent years works have appeared on initial-boundary value problems with a slip-boundary condition like (5), see for instance [6–8] et al. Because this is related to the fact that these boundary conditions have an important meaning for non-Newtonian fluids [9, 10]. In the case of slip boundary condition (5), we assume that Ω is a simply connected bounded domain [11]. System (1)-(2), in some particular cases, can be considered as a nonlinear pseudoparabolic equation due to the term $\Delta \mathbf{v}_t$, therefore all established below results will be hold true also for initial-boundary value problems for such type PDEs.

The issue of study of problems (1)–(4) and (1)–(3), (5) is aroused due to the investigation of inverse problems for system (1)-(2) that is supplemented with some additional conditions on solutions of the corresponding direct problem. In generally, the study of inverse problems are realizable only there is information such as unique solvability of the corresponding direct problems and smoothness of their solutions [12–14]. The direct problems for (1)-(2) with various statements have been studied before in some works as [5, 7, 15, 16], where the existence and uniqueness of weak solutions were established. The existence, uniqueness, and the regularities of smooth solutions of the initial-boundary value problems for system (1)-(2) without the memory term have been investigated in [17] for homogeneous fluids, and in [18], in the case for non-homogeneous fluids. However, by our knowledge, there is not work for smooth solutions for problems (1)–(4) and (1)–(3), (5). By this purpose, in this paper, we investigate the existence and uniqueness of strong solutions of problems (1)–(4) and (1)–(3), (5), and their regularities. First we work on problem (1)–(4) and the study problem (1)–(3), (5) is similar to the first one, therefore, we omit some details of proofs.

1 Preliminaries

In this section, we introduce the main functional spaces and some useful inequalities related to boundary conditions (4) and (5) from [8]. We distinguish vectors from scalars by using boldface letters. The symbol C will denote a generic constant – generally a positive one, value of which will not be specified; it can change from one inequality to another. We denote by $\mathbf{L}^2(\Omega)$ the usual Lebesgue space of square integrable vector-valued functions on Ω, and by $\mathbf{W}^{m,2}(\Omega)$ the Sobolev space of functions in $\mathbf{L}^2(\Omega)$ whose weak derivatives of an order not greater than m are in $\mathbf{L}^2(\Omega)$. The norm and inner product in $\mathbf{L}^2(\Omega)$ denoted by $\|\cdot\|_{2,\Omega}$ and $(\cdot,\cdot)_{2,\Omega}$, respectively.

Let us introduce the function spaces regarding to the slip and stick boundary conditions (5) and (4), respectively (see [3, 6]):

$$\mathbf{H_n}(\Omega) \equiv \{\mathbf{v} \in \mathbf{L}^2(\Omega) : \operatorname{div} \mathbf{v} = 0,\ \mathbf{v_n}|_{\partial\Omega} = 0\};\quad \mathbf{H}(\Omega) \equiv \{\mathbf{v} \in \mathbf{L}^2(\Omega) : \operatorname{div} \mathbf{v} = 0, \mathbf{v}|_{\partial\Omega} = 0\};$$
$$\mathbf{H}^1_{\mathbf{n}}(\Omega) \equiv \{\mathbf{v} \in \mathbf{W}^1_2(\Omega) : \operatorname{div} \mathbf{v} = 0, \mathbf{v_n}|_{\partial\Omega} = 0\};\quad \mathbf{H}^1(\Omega) \equiv \{\mathbf{v} \in \mathbf{W}^1_2(\Omega) : \operatorname{div} \mathbf{v} = 0, \mathbf{v}|_{\partial\Omega} = 0\};$$
$$\mathbf{H}^2_{\mathbf{n}}(\Omega) \equiv \{\mathbf{v} \in \mathbf{H}^1_{\mathbf{n}}(\Omega) \cap \mathbf{W}^{2,2}(\Omega) : (\operatorname{rot}\ \mathbf{v} \times \mathbf{n})|_{\partial\Omega} = 0\};\quad \mathbf{H}^2(\Omega) \equiv \{\mathbf{v} \in \mathbf{H}^1(\Omega) \cap \mathbf{W}^{2,2}(\Omega)\}$$

and for the simplicity, we use the following common notation for both cases

$$\mathbf{V} := \begin{cases} \mathbf{H}(\Omega), \text{ in the case (4);} \\ \mathbf{H_n}(\Omega), \text{ in the case (5),} \end{cases} \quad \mathbf{V}^i := \begin{cases} \mathbf{H}^i(\Omega), \text{ in the case (4);} \\ \mathbf{H}^i_{\mathbf{n}}(\Omega), \text{ in the case (5), } i = 1, 2. \end{cases}$$

The scalar product and the norm in $\mathbf{V}^1_{\mathbf{n}}(\Omega)$ we define by $(\operatorname{rot}\mathbf{v}, \operatorname{rot}\mathbf{u})_{2,\Omega}$ and $\|\mathbf{v}\|_{\mathbf{V}^1_{\mathbf{n}}(\Omega)} := \|\operatorname{rot}\ \mathbf{v}\|_{2,\Omega}$, respectively. According to [3, 6, 8, 11] and the references cited in them (see for example [9, 19]), the following inequalities are hold:
Poincare's inequality

$$\|\mathbf{v}\|_{2,\Omega} \le C_1(\Omega)\, \|\nabla \mathbf{v}\|_{2,\Omega}\,,\ \mathbf{v} \in \mathbf{V}^1(\Omega); \tag{6}$$

$$N_1(\Omega)\, \|\mathbf{v}\|_{\mathbf{W}^{1,2}(\Omega)} \le \|\operatorname{rot}\mathbf{v}\|_{2,\Omega} \le N_2(\Omega)\, \|\mathbf{v}\|_{\mathbf{W}^{1,2}(\Omega)}\,,\quad \forall \mathbf{v} \in \mathbf{V}^1(\Omega);$$

$$N_3(\Omega)\,\|\mathbf{v}\|_{\mathbf{W}^{2,2}(\Omega)} \le \|\Delta\mathbf{v}\|_{2,\Omega} = \|\mathrm{rot}\,\mathrm{rot}\,\mathbf{v}\|_{2,\Omega} \le N_4(\Omega)\,\|\mathbf{v}\|_{\mathbf{W}^{2,2}(\Omega)}\,, \quad \forall\mathbf{v}\in\mathbf{V}^2(\Omega); \tag{7}$$

and Ladyzhenskaya inequalities [6].

Let us introduce a bilinear and continuous form $\mathbf{a}$ on $\mathbf{V}^1$, associated with the operator Δ:

$$\mathbf{a}(\mathbf{v},\mathbf{u}) = (\nabla\mathbf{v},\nabla\mathbf{u})_{2,\Omega}\,, \quad \forall\mathbf{v},\mathbf{u}\in\mathbf{V}^1(\Omega) \tag{8}$$

in case (4), and

$$\mathbf{a}(\mathbf{v},\mathbf{u}) = (\mathrm{rot}\,\mathbf{v},\mathrm{rot}\,\mathbf{u})_{2,\Omega}\,, \quad \forall\mathbf{v},\mathbf{u}\in\mathbf{V}^1(\Omega) \tag{9}$$

in case (5). It is clear that $\mathbf{a}(\mathbf{v},\mathbf{v})$ is a norm on $\mathbf{V}^1(\Omega)$, which is equivalent to $\mathbf{W}^{1,2}(\Omega)$-norm. In particular, due to (6), in $\mathbf{V}^1$ the norm $\|\,\mathrm{rot}\,\mathbf{v}\|_{2,\Omega}$ is equivalent to the norm $\|\mathbf{v}\|_{\mathbf{W}^{1,2}(\Omega)}$, and therefore equivalent to the norm $\|\nabla\mathbf{v}\|_{2,\Omega}$.

Thus, $\mathbf{a}$ defines an isomorphism A from $\mathbf{V}^1(\Omega)$ to $\mathbf{V}^{-1}(\Omega)$,

$$\langle A\mathbf{v},\mathbf{u}\rangle \equiv \mathbf{a}(\mathbf{v},\mathbf{u}), \quad \forall\mathbf{v},\mathbf{u}\in\mathbf{V}^1(\Omega),$$

where $\langle\cdot,\cdot\rangle$ denotes the pairing of $\mathbf{V}^1$ and $\mathbf{V}^{-1}$. There hold the following continuous inclusions

$$\mathbf{V}^1(\Omega)\hookrightarrow\mathbf{L}^2(\Omega)\hookrightarrow\mathbf{V}^{-1}(\Omega),$$

where each of the first two spaces is dense in the next one.

It follows from (7) also that in $\mathbf{V}^2$ the norm $\|\Delta\mathbf{v}\|_{2,\Omega} = \|\mathrm{rotrot}\,\mathbf{v}\|_{2,\Omega}$ is equivalent to the norm $\|\mathbf{v}\|_{\mathbf{W}^{2,2}(\Omega)}$.

Regarding to sliding condition (5), we have the Green formulas (see [6] and [8,9]):

$$\begin{aligned}(-\Delta\mathbf{v},\mathbf{u})_{2,\Omega} &= -\left(\nabla\,\mathrm{div}\,\mathbf{v},\mathbf{u}\right)_{2,\Omega} + \left(\mathrm{rot}^2\,\mathbf{v},\mathbf{u}\right)_{2,\Omega} = -\int_{\partial\Omega}\mathrm{div}\,\mathbf{v}\cdot\mathbf{u}_n\,dS+\\ &+(\mathrm{div}\,\mathbf{v},\mathrm{div}\,\mathbf{u})_{2,\Omega} + \int_{\partial\Omega}\mathbf{u}\cdot(\mathrm{rot}\,\mathbf{v}\times\mathbf{n})\,dS + (\mathrm{rot}\,\mathbf{v},\mathrm{rot}\,\mathbf{u})_{2,\Omega} = (\mathrm{rot}\,\mathbf{v},\mathrm{rot}\,\mathbf{u})_{2,\Omega}\end{aligned} \tag{10}$$

in case $d=3$, and

$$\begin{aligned}(-\Delta\mathbf{v},\mathbf{u})_{2,\Omega} &= (\mathrm{div}\,\mathbf{v},\mathrm{div}\,\mathbf{u})_{2,\Omega} + \left(\overline{\mathrm{rot}}(\mathrm{rot}\,\mathbf{v}),\mathbf{u}\right)_{2,\Omega} =\\ &= \int_{\partial\Omega}(\mathrm{rot}\,\mathbf{v}\times\mathbf{n})\,\mathbf{u}\,dS + (\mathrm{rot}\,\mathbf{v},\mathrm{rot}\,\mathbf{u})_{2,\Omega} = (\mathrm{rot}\,\mathbf{v},\mathrm{rot}\,\mathbf{u})_{2,\Omega}\,,\end{aligned} \tag{11}$$

in case $d=2$, where $\overline{\mathrm{rot}}\varphi$ is the vector $(\varphi_{x_2},-\varphi_{x_1})_{2,\Omega}$ for the scalar function φ.

The regularity properties of solutions will be proved under the following lemma, which the proof is given in [20].

Lemma 1. If $f\in L^p(0,T;X)$ and $\frac{\partial f}{\partial t}\in L^p(0,T;X)$ $(1\le p\le\infty)$, then f, after, which can be changing on a set of measure zero (from segment $(0,T)$) be a continuous mapping $[0,T]\to X$.

Definition 1. A vector function $\mathbf{v}(\mathbf{x},t)$ is a strong solution to problem (1)–(4) ((1)–(3), (5)) if:

1 $\mathbf{v}(\mathbf{x},t)\in\mathbf{C}(0,T;\mathbf{V}^1(\Omega)\cap\mathbf{V}^2(\Omega))\cap\mathbf{W}^1_2(0,T;\mathbf{V}^1(\Omega)\cap\mathbf{V}^2(\Omega))$;

2 Each equation in (1)–(4) ((1)–(3), (5)) holds in the distribution sense in the their corresponding domain.

2 Main results

Throughout the work, we assume that

$$K(t) \in L^2([0,T]) \text{ and } \|K\|_{L^2([0,T])} \equiv K_0 < \infty. \tag{12}$$

For the problems (1)–(4) and (1)–(3), (5) the following results are hold.

Theorem 1. Suppose that

$$\mathbf{f} \in \mathbf{L}^2(0,T;\mathbf{L}^2(\Omega)), \quad \mathbf{v}_0 \in \mathbf{V}^1(\Omega) \cap \mathbf{V}^2(\Omega),$$

and (12) are hold. Then problems (1)–(4) and (1)–(3), (5) have a unique strong solution and the following estimate is valid

$$\|\mathbf{v}\|^2_{\mathbf{L}^\infty(0,T;\mathbf{V}^1(\Omega)\cap\mathbf{V}^2(\Omega))} + \|\mathbf{v}_t\|^2_{\mathbf{L}^2(0,T;\mathbf{V}^1(\Omega)\cap\mathbf{V}^2(\Omega))} + \|\mathbf{v}\|^2_{\mathbf{L}^2(0,T;\mathbf{V}^1(\Omega))} \leq C < \infty,$$

where C is a positive constant depending on data of the problem.

Proof. The proof consists of following steps: by Gelerkin's method constructing a sequence of approximated solutions; obtaining a priori estimates and passage to limit.

2.1 Galerkin's approximations

To prove the existence of a strong solution to problem (1)–(5), we use the Faedo–Galerkin method with a special basis of eigenfunctions of the spectral problem

$$-\Delta\varphi_j + \nabla q = \lambda_j \varphi_j, \ \varphi_j \in \mathbf{V}^2(\Omega)$$

is closely connected with problems (1)–(4) and (1)–(3), (5). In case (5), it is equivalent to the problem [6, 8]

$$\mathbb{A}\varphi_j \equiv -\Delta\varphi_j = \lambda_j \varphi_j, \ \varphi_j \in \mathbf{V}^2(\Omega)$$

since $\nabla q \equiv 0$ due to the fact

$$(\Delta\varphi, \nabla p) = 0, \text{ for any } \varphi \in \mathbf{V}^2(\Omega) \text{ and any } p \in W_2^1(\Omega).$$

For the problem (1)–(3), (4), $\mathbb{A}\varphi_j \equiv \tilde{\Delta}\varphi_j$ [21]. Given $m \in N$, let us consider the m-dimensional spaces $\mathbf{X}^m$ spanned by the first m eigenfunctions $\varphi_1, ..., \varphi_m$. For each $m \in N$, we search for approximate solutions in the form

$$\mathbf{v}^m(x,t) = \sum_{j=1}^{m} c_j^m(t)\varphi_j(x), \quad \varphi_j \in \mathbf{X}^m,$$

where unknown coefficients $c_j^m(t)$, $j = 1, ..., m$ are defined as solutions of the following system of ordinary differential equations derived from

$$\begin{aligned} &\frac{d}{dt}\left((\mathbf{v}^m, \varphi_k)_{2,\Omega} - \varkappa\,(\Delta\mathbf{v}^m, \varphi_k)_{2,\Omega}\right) + ((\mathbf{v}^m \cdot \nabla)\,\mathbf{v}^m, \varphi_k)_{2,\Omega} - \nu\,(\Delta\mathbf{v}^m, \varphi_k)_{2,\Omega} - \\ &- \int_0^t K(t-\tau)\,(\Delta\mathbf{v}^m, \varphi_k)_{2,\Omega}\,d\tau = (\mathbf{f}, \varphi_k)_{2,\Omega}\,, \end{aligned} \tag{13}$$

for $k = 1,\ 2,\ \ldots,\ m$. System (13) is supplemented with the Cauchy data

$$\mathbf{v}^m(0) = \mathbf{v}_0^m, \tag{14}$$

where

$$\mathbf{v}_0^m = \sum_{j=1}^{m} (\mathbf{v}_0, \varphi_j)_{2,\Omega}\, \varphi_j$$

is a sequence in $\mathbf{L}^2(\Omega) \cap \mathbf{V}^1(\Omega)$ such that

$$\mathbf{v}_0^m \to \mathbf{v}_0(x) \text{ strong as } m \to \infty \ \text{ in } \ \mathbf{V}^1(\Omega) \cap \mathbf{V}^2(\Omega). \tag{15}$$

According to a general theory of ordinary differential equations, Cauchy problem (13)–(14) has a solution $c_j^m(t)$ in $[0, T_*]$. By a priori estimates which we shall establish below, $[0, T_*]$ can be extended to $[0, T]$.

2.2 A priori estimates

Lemma 2. Assume that

$$\mathbf{f} \in \mathbf{L}^2(0, T; \mathbf{L}^2(\Omega)),\ \mathbf{v}_0(x) \in \mathbf{V}^1(\Omega),$$

and the conditions (12) and (15) are fulfilled. Then, for all $t \in [0, T]$, the following a priori estimate is valid

$$\|\mathbf{v}^m\|^2_{\mathbf{L}^\infty(0,T;\mathbf{V}(\Omega)\cap\mathbf{V}^1(\Omega))} + \|\mathbf{v}^m\|^2_{\mathbf{L}^2(0,T;\mathbf{V}^1(\Omega))} \le M_0 < \infty, \tag{16}$$

where M_0 is a positive constant depending only on data of the problem.

Proof. Multiply k-th equation of (13) by $c_k^m(t)$ and summing up from 1 to m, then using Green's formulas (10)-(11), we obtain

$$\frac{d}{dt}\left(\|\mathbf{v}^m\|^2_{2,\Omega} + \varkappa \|\mathbf{v}^m\|^2_{\mathbf{V}^1(\Omega)}\right) + \nu \|\mathbf{v}^m\|^2_{\mathbf{V}^1(\Omega)} =$$

$$= \int_0^t K(t-\tau)\mathbf{a}(\mathbf{v}^m(t), \mathbf{v}^m(\tau))\, d\tau + (\mathbf{f}, \mathbf{v}^m)_{2,\Omega} \equiv I_1, \tag{17}$$

where $\mathbf{a}$ is defined by (8) and (9), regarding to the boundary conditions. Next, we estimate the terms on the right-hand side of (17) by Hölder's and Young's inequalities

$$I_1 \le \int_0^t |K(t-\tau)|\, \|\mathbf{v}^m(\tau)\|_{\mathbf{V}^1(\Omega)}\, \|\mathbf{v}^m(t)\|_{\mathbf{V}^1(\Omega)}\, d\tau + \|\mathbf{v}^m\|_{2,\Omega}\, \|\mathbf{f}\|_{2,\Omega} \le$$

$$\le \frac{\nu}{2}\|\mathbf{v}^m\|^2_{\mathbf{V}^1(\Omega)} + \frac{K_0^2}{2\nu}\int_0^t \|\mathbf{v}^m(\tau)\|^2_{\mathbf{V}^1(\Omega)}\, d\tau + \frac{1}{2}\|\mathbf{v}^m\|^2_{2,\Omega} + \frac{1}{2}\|\mathbf{f}\|^2_{2,\Omega}.$$

Substituting last inequality into (17), and integrating by s from 0 to t, we obtain

$$\begin{aligned} &\|\mathbf{v}^m\|^2_{2,\Omega} + \varkappa\|\mathbf{v}^m\|^2_{\mathbf{V}^1(\Omega)} + \nu\int_0^t \|\mathbf{v}^m\|^2_{\mathbf{V}^1(\Omega)}\, ds \le \|\mathbf{v}_0(x)\|^2_{2,\Omega} + \varkappa\|\mathbf{v}_0(x)\|^2_{\mathbf{V}^1(\Omega)} + \int_0^t \|\mathbf{v}^m\|^2_{2,\Omega}\, ds + \\ &+ \frac{K_0^2}{\nu}\int_0^t\int_0^s \|\mathbf{v}^m(s)\|^2_{\mathbf{V}^1(\Omega)}\, d\tau ds + \|\mathbf{f}\|^2_{2,Q_T} \le C_1\int_0^t \left(\|\mathbf{v}^m\|^2_{2,\Omega} + \varkappa\|\mathbf{v}^m\|^2_{\mathbf{V}^1(\Omega)}\right) ds + C_2, \end{aligned} \tag{18}$$

where

$$C_1 = \max\{1, \frac{K_0^2 T}{\nu \varkappa}\}, \quad C_2 = \|\mathbf{v}_0(x)\|_{2,\Omega}^2 + \varkappa \|\mathbf{v}_0(x)\|_{\mathbf{V}^1(\Omega)}^2 + \|\mathbf{f}\|_{2,Q_T}^2 .$$

Omitting the third term on the left hand side of (18) and applying Grönwall's lemma and taking supremum, we get

$$\sup_{t\in(0,T]} \left(\|\mathbf{v}^m\|_{2,\Omega}^2 + \|\mathbf{v}^m\|_{\mathbf{V}^1(\Omega)}^2 \right) \le C_3 < \infty, \tag{19}$$

where $C_3 = C_3(\nu, \varkappa, T, C_1, C_2)$. Plugging (19) with (18), we obtain the first energy estimate (16).

Lemma 3. Assume that all conditions of Lemma 2 are fulfilled. Then the following estimate is valid

$$\|\mathbf{v}^m\|_{\mathbf{L}^\infty(0,T;\mathbf{V}(\Omega)\cap\mathbf{V}^1(\Omega))}^2 + \|\mathbf{v}_t^m\|_{\mathbf{L}^2(0,T;\mathbf{V}(\Omega)\cap\mathbf{V}^1(\Omega))}^2 \le M_1 < \infty, \quad \forall t \in [0,T], \tag{20}$$

where M_1 is a positive constant depending on data of the problem.

Proof. Multiplying both sides of k-th equation of (13) by $\frac{dc_k^m}{dt}$ and summing up from $k = 1$ to $k = m$, we obtain

$$\begin{aligned} \|\mathbf{v}_t^m\|_{2,\Omega}^2 + \varkappa \|\mathbf{v}_t^m\|_{\mathbf{V}^1(\Omega)}^2 + \frac{\nu}{2}\frac{d}{dt}\|\mathbf{v}^m\|_{\mathbf{V}^1(\Omega)}^2 &= ((\mathbf{v}^m \cdot \nabla)\,\mathbf{v}_t^m, \mathbf{v}^m)_{2,\Omega} - \\ &- \int_0^t K(t-\tau)\mathbf{a}\left(\mathbf{v}^m(\tau), \mathbf{v}_t^m(t)\right) d\tau + (\mathbf{f}, \mathbf{v}_t^m)_{2,\Omega} \equiv I_{21} + I_{22}. \end{aligned} \tag{21}$$

Using Hölder and Ladyzhenskaya together with Young inequalities, we have

$$I_{21} \le |((\mathbf{v}^m \cdot \nabla)\,\mathbf{v}_t^m, \mathbf{v}^m)| \le \|\mathbf{v}_t^m\|_{4,\Omega} \|\mathbf{v}^m\|_{4,\Omega}^2 \le \frac{\varepsilon_1}{2}\|\mathbf{v}_t^m\|_{\mathbf{V}^1(\Omega)}^2 + \frac{C^4(\Omega)}{2\varepsilon_1}\|\mathbf{v}^m\|_{\mathbf{V}^1(\Omega)}^4, \tag{22}$$

$$I_{22} \le \frac{\varepsilon_2}{2}\|\mathbf{v}_t^m(t)\|_{\mathbf{V}^1(\Omega)}^2 + \frac{K_0^2}{2\varepsilon_2}\int_0^t \|\mathbf{v}^m(\tau)\|_{\mathbf{V}^1(\Omega)}^2 d\tau + \frac{\varepsilon_3}{2}\|\mathbf{v}_t^m(t)\|_{2,\Omega}^2 + \frac{1}{2\varepsilon_3}\|\,\mathbf{f}\|_{2,\Omega}^2 . \tag{23}$$

Plugging (22)-(23) with $\varepsilon_i = \frac{\varkappa}{3}, i = 1, 2, 3$ into (21), and integrating the result by s in $[0, t]$, $t \le T$, we have

$$\begin{aligned} &\nu \|\mathbf{v}^m\|_{\mathbf{V}^1(\Omega)}^2 + 2\|\mathbf{v}_t^m(t)\|_{2,Q_T}^2 + \varkappa \|\mathbf{v}_t^m(t)\|_{\mathbf{L}^2(0,T;\mathbf{V}^1(\Omega))}^2 \le \\ &\le \nu \|\mathbf{v}_0\|_{\mathbf{V}^1(\Omega)}^2 + \frac{3C^4(\Omega)}{\varkappa} M_0^2 + \frac{3K_0^2}{\varkappa} M_0 T + \frac{3}{\varkappa}\|\mathbf{f}\|_{2,Q_T}^2 := C_4 < \infty \end{aligned}$$

which follows that (20).

Lemma 4. Assume that in addition to the conditions of Lemma 2 holds

$$\mathbf{v}_0 \in \mathbf{V}(\Omega) \cap \mathbf{V}^2(\Omega).$$

Then for all $t \in [0, T]$, the estimate is valid

$$\sup_{t\in[0,T]} \|\mathbb{A}\mathbf{v}^m\|_{2,\Omega}^2 + \|\mathbb{A}\mathbf{v}_t^m\|_{2,Q_T}^2 \le M_2 < \infty, \tag{24}$$

where $\mathbb{A} \equiv \tilde{\Delta}$ for the problem with (4), and $\mathbb{A} \equiv \Delta$ for the problem with (5).

Proof. Let us multiply the kth equation of (13) by $-\mu_k \frac{dc_k^m(t)}{dt}$ and sum with respect to k, from 1 to m to obtain

$$\|\mathbf{v}_t^m\|^2_{\mathbf{V}^1(\Omega)} + \varkappa \|\mathbb{A}\mathbf{v}_t^m\|^2_{2,\Omega} + \frac{\nu}{2}\frac{d}{dt}\|\mathbb{A}\mathbf{v}^m\|^2_{2,\Omega} = ((\mathbf{v}^m \cdot \nabla)\mathbf{v}^m, \mathbb{A}\mathbf{v}_t^m)_{2,\Omega} -$$
$$- \int_0^t K(t-\tau)(\mathbf{\Delta}\mathbf{v}^m(\tau), \mathbb{A}\mathbf{v}_t^m(t))_{2,\Omega}\, d\tau + (\mathbf{f}(t), \mathbb{A}\mathbf{v}_t^m)_{2,\Omega} \equiv I_{31} + I_{32}. \tag{25}$$

Estimating the terms on right hand side (25) by using Hölder, Ladyzhenskaya, Sobolev and Young inequalities, we get the following inequalities

$$I_{31} \le \|\mathbb{A}\mathbf{v}_t^m\|_{2,\Omega}\|\mathbf{v}^m\|_{4,\Omega}\|\nabla\mathbf{v}^{\mathbf{m}}\|_{4,\Omega} \le \frac{\varepsilon_1}{2}\|\mathbb{A}\mathbf{v}_t^m\|^2_{2,\Omega} + \frac{C^2(\Omega)}{2\varepsilon_1}\|\mathbf{v}^m\|^2_{\mathbf{V}^1(\Omega)}\|\mathbb{A}\mathbf{v}^m\|^2_{2,\Omega},$$

$$I_{32} \le \frac{\varepsilon_2}{2}\|\mathbb{A}\mathbf{v}_t^m(t)\|^2_{2,\Omega} + \frac{K_0^2}{2\varepsilon_2}\int_0^t \|\mathbb{A}\mathbf{v}^m(\tau)\|^2_{2,\Omega}\, d\tau + \frac{\varepsilon_3}{2}\|\mathbb{A}\mathbf{v}_t^m(t)\|^2_{2,\Omega} + \frac{1}{2\varepsilon_3}\|\mathbf{f}\|^2_{2,\Omega}, \tag{26}$$

where $C(\Omega)$ is a constant from embedding inequalities.

Substituting (26) with $\varepsilon_i = \frac{\varkappa}{3}, i = 1,2,3$ into (25) and integrating the result by $\tau \in (0,t)$ and using estimates (16), (20), we have

$$\nu\|\mathbb{A}\mathbf{v}^m\|^2_{2,\Omega} + \int_0^t \|\mathbf{v}_t^m(\tau)\|^2_{\mathbf{V}^1(\Omega)}\, d\tau + \varkappa\int_0^t \|\mathbb{A}\mathbf{v}_t^m(\tau)\|^2_{2,\Omega}\, d\tau \le C_5\int_0^t \|\mathbb{A}\mathbf{v}^m(\tau)\|^2_{2,\Omega}\, d\tau + C_6, \tag{27}$$

where

$$C_5 = \frac{3K_0}{\varkappa}\left(C^2(\Omega) + K_0 T\right), \quad C_6 = \nu\|\mathbf{v}_0\|^2_{\mathbf{V}^2(\Omega)} + \frac{3}{\varkappa}\|\mathbf{f}\|^2_{2,Q_T}.$$

By applying Granwall's lemma and the standard techniques, we get from (27) estimate (24).

Along with the above estimates, one can establish the following more regular estimate assuming an additional smoothness for data.

Lemma 5. Assume that in addition to the conditions of Lemma 5 holds

$$\mathbf{f} \in \mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega)).$$

Then for all $t \in [0,T]$ the following estimate is valid

$$\sup_{t\in[0,T]}\|\mathbf{v}_t^n\|^2_{\mathbf{V}^1(\Omega)} + \sup_{t\in[0,T]}\|\mathbb{A}\mathbf{v}_t^n\|^2_{2,\Omega} \le M_3 < \infty. \tag{28}$$

Proof. Let us multiply the kth equation of (13) by $-\mu_k \frac{dc_k^m(t)}{dt}$, and sum with respect to k, from 1 to m. Then we have

$$\|\mathbf{v}_t^m\|^2_{\mathbf{V}^1(\Omega)} + \varkappa\|\mathbb{A}\mathbf{v}_t^m\|^2_{2,\Omega} = ((\mathbf{v}^m \cdot \nabla)\mathbf{v}^m, \mathbb{A}\mathbf{v}_t^m)_{2,\Omega} -$$
$$- \int_0^t K(t-\tau)(\mathbf{\Delta}\mathbf{v}^m(\tau), \mathbb{A}\mathbf{v}_t^m(t))_{2,\Omega}\, d\tau + (\mathbf{f}(t), \mathbb{A}\mathbf{v}_t^m)_{2,\Omega} - \nu(\mathbf{\Delta}\mathbf{v}^m, \mathbb{A}\mathbf{v}_t^m)_{2,\Omega} \equiv I_{41} + I_{42}, \tag{29}$$

where

$$I_{41} = ((\mathbf{v}^m \cdot \nabla)\mathbf{v}^m, \mathbb{A}\mathbf{v}_t^m)_{2,\Omega},$$

$$I_{42} = -\int_0^t K(t-\tau)\left(\mathbf{\Delta v}^m(\tau), \mathbb{A}\mathbf{v}_t^m(t)\right)_{2,\Omega} d\tau + \left(\mathbf{f}(t), \mathbb{A}\mathbf{v}_t^m\right)_{2,\Omega} - \nu\left(\mathbf{\Delta v}^m, \mathbb{A}\mathbf{v}_t^m\right)_{2,\Omega}.$$

Estimating I_{41} and I_{42} by using Hölder and Cauchy inequalities as above, we obtain the following inequality

$$I_{41} \le \|\mathbb{A}\mathbf{v}_t^m\|_{2,\Omega} \|\mathbf{v}^m\|_{4,\Omega} \|\nabla \mathbf{v}^m\|_{4,\Omega} \le \frac{\varepsilon_1}{2} \|\mathbb{A}\mathbf{v}_t^m\|_{2,\Omega}^2 + \frac{C^2(\Omega)}{2\varepsilon_1} \|\mathbf{v}^m\|_{\mathbf{V}^1(\Omega)}^2 \|\mathbb{A}\mathbf{v}^m\|_{2,\Omega}^2, \tag{30}$$

$$\begin{aligned} I_{42} \le{}& \frac{\varepsilon_2}{2} \|\mathbb{A}\mathbf{v}_t^m(t)\|_{2,\Omega}^2 + \frac{K_0^2}{2\varepsilon_2} \int_0^t \|\Delta \mathbf{v}^m(\tau)\|_{2,\Omega}^2 d\tau + \frac{\varepsilon_3}{2} \|\mathbb{A}\mathbf{v}_t^m(t)\|_{2,\Omega}^2 + \frac{1}{2\varepsilon_3} \|\mathbf{f}\|_{2,\Omega}^2 + \\ &+ \frac{\nu}{2\varepsilon_4} \|\Delta \mathbf{v}^m\|_{2,\Omega}^2 + \frac{\varepsilon_4}{2} \|\mathbb{A}\mathbf{v}_t^m(t)\|_{2,\Omega}^2. \end{aligned} \tag{31}$$

Choosing $\varepsilon_i = \frac{\varkappa}{4}, i = 1,2,3,4$ in (30), (31), and substituting into (29), we get

$$\begin{aligned} \|\mathbf{v}_t^m\|_{\mathbf{V}^1(\Omega)}^2 + \frac{\varkappa}{2} \|\mathbb{A}\mathbf{v}_t^m\|_{2,\Omega}^2 \le{}& \frac{2C^2(\Omega)}{\varkappa} \|\mathbf{v}^m\|_{\mathbf{V}^1(\Omega)}^2 \|\mathbb{A}\mathbf{v}^m\|_{2,\Omega}^2 + \\ &+ \frac{2K_0^2}{\varkappa} \int_0^t \|\Delta \mathbf{v}^m(\tau)\|_{2,\Omega}^2 d\tau + \frac{2}{\varkappa} \|\mathbf{f}\|_{2,\Omega}^2 + \frac{2\nu}{\varkappa} \|\Delta \mathbf{v}^m\|_{2,\Omega}^2. \end{aligned} \tag{32}$$

Now, taking the supremum by $t \in [0,T]$ on both sides of (32), and using (20) and (24), we obtain

$$\sup_{t\in[0,T]} \left(\|\mathbf{v}_t^m\|_{\mathbf{V}^1(\Omega)}^2 + \frac{\varkappa}{2} \|\mathbb{A}\mathbf{v}_t^m\|_{2,\Omega}^2\right) \le \frac{2C^2(\Omega)}{\varkappa} M_1 M_2 + \frac{2K_0^2}{\varkappa} M_2 T + \frac{2}{\varkappa} \|\mathbf{f}\|_{L^\infty(0,T;L(\Omega))}^2 + \frac{2\nu}{\varkappa} M_2 \le K < \infty.$$

2.3 Passage to the limit as $m \to \infty$

By means of reflexivity and up to some subsequences, estimates (16), (20), (28) imply that

$$\mathbf{v}^m \rightharpoonup \mathbf{v} \quad \text{weakly-}* \text{ in } L^\infty(0,T;\mathbf{V}(\Omega)\cap\mathbf{V}^1(\Omega)), \quad \text{as } m\to\infty, \tag{33}$$

$$\mathbf{v}^m \rightharpoonup \mathbf{v} \quad \text{weakly in } L^2(0,T;\mathbf{V}(\Omega)\cap\mathbf{V}^1(\Omega)), \quad \text{as } m\to\infty, \tag{34}$$

$$\mathbf{v}_t^m \rightharpoonup \mathbf{v}_t \quad \text{weakly in } L^2(0,T;\mathbf{V}(\Omega)\cap\mathbf{V}^1(\Omega)), \quad \text{as } m\to\infty, \tag{35}$$

$$\mathbf{v}^m \rightharpoonup \mathbf{v} \quad \text{weakly-}* \text{ in } L^\infty(0,T;\mathbf{V}^2(\Omega)), \quad \text{as } m\to\infty, \tag{36}$$

$$\mathbf{v}^m \rightharpoonup \mathbf{v} \quad \text{weakly in } L^2(0,T;\mathbf{V}^2(\Omega)), \quad \text{as } m\to\infty, \tag{37}$$

$$\mathbf{v}_t^m \rightharpoonup \mathbf{v}_t \quad \text{weakly in } L^2(0,T;\mathbf{V}^2(\Omega)), \quad \text{as } m\to\infty. \tag{38}$$

On the other hand, due to the compact embedding $\mathbf{W}_0^{1,2}(\Omega) \hookrightarrow\hookrightarrow \mathbf{L}^2(\Omega)$ and the Aubin-Lions compactness lemma, it follows that

$$\mathbf{v}^m \longrightarrow \mathbf{v} \quad \text{strongly in } \mathbf{L}^2(Q_T) \quad \text{as } m\to\infty. \tag{39}$$

Let $\zeta(t) \in C_0^\infty([0,T])$ be an arbitrary function. Multiplying (13) by $\zeta(t)$ and integrating the result by t from 0 to T, we obtain

$$\begin{aligned} &\int_{Q_T} \mathbf{v}_t^m \cdot \varphi_k \zeta d\mathbf{x}dt + \int_{Q_T} (\mathbf{v}^m \cdot \nabla)\mathbf{v}^m \cdot \varphi_k \zeta d\mathbf{x}dt + \nu \int_{Q_T} \Delta \mathbf{v}^m \cdot \varphi_k \zeta d\mathbf{x}dt + \\ &+ \varkappa \int_{Q_T} \Delta \mathbf{v}_t^m \cdot \varphi_k \zeta d\mathbf{x}dt = \int_{Q_T} \int_0^\tau K(\tau - s) \Delta \mathbf{v}^m \cdot \varphi_k \zeta ds d\tau + \int_{Q_T} \mathbf{f} \cdot \varphi_k \zeta d\mathbf{x}dt \end{aligned} \tag{40}$$

for $k \in \{1, \ldots, m\}$. Then, fixing k, we can pass in equation (40) to the limit $m \to \infty$, by using the convergence results (33)–(39). Then, we obtain

$$\int_{Q_T} \mathbf{v}_t \cdot \varphi_k \zeta d\mathbf{x}dt + \int_{Q_T} (\mathbf{v} \cdot \nabla) \mathbf{v} \cdot \varphi_k \zeta d\mathbf{x}dt + \nu \int_{Q_T} \Delta \mathbf{v} \cdot \varphi_k \zeta d\mathbf{x}dt + \\ + \varkappa \int_{Q_T} \Delta \mathbf{v}_t \cdot \varphi_k \zeta d\mathbf{x}dt = \int_{Q_T} \int_0^\tau K(\tau - s) \Delta \mathbf{v} \cdot \varphi_k \zeta ds d\tau + \int_{Q_T} \mathbf{f} \cdot \varphi_k \zeta d\mathbf{x}dt. \tag{41}$$

for $k \in \{1, \ldots, m\}$.

By linearity, equation (41) holds for any finite linear combination of $\{\mathbf{z}_k = \varphi_k \cdot \zeta(t)\}_{k=1}^m$ with $\zeta(t) \in C_0^\infty([0,T])$, and, by a continuity argument, it is still true for any $\mathbf{z} \in L^2(0,T;\mathbf{V}(\Omega))$. Hence, we can see that $\mathbf{v}$ satisfies to

$$\int_{Q_T} \mathbf{v}_t \cdot \mathbf{z} d\mathbf{x}dt + \int_{Q_T} (\mathbf{v} \cdot \nabla) \mathbf{v} \cdot \mathbf{z} d\mathbf{x}dt + \nu \int_{Q_T} \Delta \mathbf{v} \cdot \mathbf{z} \zeta d\mathbf{x}dt + \\ + \varkappa \int_{Q_T} \Delta \mathbf{v}_t \cdot \mathbf{z} d\mathbf{x}dt = \int_{Q_T} \int_0^\tau K(\tau - s) \Delta \mathbf{v} \cdot \mathbf{z} ds d\tau + \int_{Q_T} \mathbf{f} \cdot \mathbf{z} d\mathbf{x}dt,$$

i.e. $\mathbf{v}$ is a strong solution to problem (1)–(4).

3 Regularity of solutions

Theorem 2. Let all conditions of Theorem 1 be fulfilled. Then

$$\mathbf{v} \in C(0,T;\mathbf{V}(\Omega) \cap \mathbf{V}^2(\Omega)),\ p \in C(0,T;G(\Omega)).$$

If, in addition,

$$\mathbf{f} \in C(0,T;L^2(\Omega))$$

holds, then for all $t \in (0,T)$

$$\mathbf{v} \in C^1(0,T;\mathbf{V}(\Omega) \cap \mathbf{V}^2(\Omega)),\ p \in C(0,T;G(\Omega)) \tag{42}$$

holds.

Proof. Embedding (42) follows from Lemma 1, under estimates (16), (20), (24). The second assertion follows from the embedding theorems under the estimates from [20, 22].

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Жады мүшесі бар Кельвин-Фойгт теңдеулері: шешімдердің бар болуы, жалғыздығы және регулярлығы

Жалпы алғанда кері есептерді зерттеу оларға сәйкес келетіп тура есептердің бірмәнді шешімділігі және шешімдерішің ұзіліссіздігі мен жоғары регулярлығы сияқты кейбір қажетті қасиеттерге ие болған жағдайда ғана жүзеге асырылады. Мақалада тұтқыр серпімді және релаксациялық қасиеттері ескерілген сығылмайтын біртекті Ньютондық емес сұйықтардың қозғалысын сипаттайтын жады мүшесі бар 2D-3D өлшемді сызықты емес Кельвин-Фойгт теңдеулер жүйесі үшін қойылған бастапқы-шектік есептер зерттелген. Бұл тура есептерді зерттеу осы жүйе үшін қойылған кері есептерді зерттеумен байланысты. Себебі, онда осы тура есептердіщ шешімдерінің және олардың туындыларының үзіліссіздігі мен реґулярлығы сияқты қасиеттері қажет етіледі. Қарастырылып отырған есептерде теңдеулер жүйесі бастапқы шартпен қатар жүгу және сырғанау сияқты шекаралық шарттарының бірімен толықтырылады. Осы екі шекаралық шарттар жағдайында бастапқы-шекаралық есептердің әлді шешімдерінің уақыт бойынша ґлобальды бар болуы және жалғыздығы дәлелденген. Сонымен қатар, есептің берілгендері үшін қолайлы үйғарымдар жасай отырып, шешімдермен олардың туындыларының реґулярлығы көрсетілді.

Кілт сөздер: Кельвин-Фойгт жүйесі, жүгу және сырғанаудың шекаралық шарттары, әлді шешімдер, ґлобальды бар болуы және жалғыздығы, тегістік.

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Уравнения Кельвина–Фойгта с памятью: существование, единственность и регулярность решений

В общем случае изучение обратных задач осуществимо только в том случае, когда соответствующие прямые задачи имеют единственное решение, обладающее некоторыми необходимыми свойствами, такими как непрерывность и реґулярность. В статье исследованы начально-краевые задачи для системы 2D–3D нелинейных уравнений Кельвина–Фойґта с памятью, описывающей движение несжимаемой однородной неньютоновской жидкости с вязкоупруґими и релаксационными свойствами. Исследование таких прямых задач связано с изучением соответствующих обратных задач для данной системы, которое требует свойств как непрерывности и реґулярности решения и их производных этих прямых задач решений. Система уравнений, помимо начального условия, дополняется одним из ґраничных

условий: условием прилипания или скольжения. В обоих случаях доказаны глобальное во времени существование и единственность сильных решений этих начально-краевых задач. Более того, при соответствующих предположениях на данные была установлена регулярность решений и их производных.

Ключевые слова: система Кельвина–Фойгта, граничные условия скольжения и прилипания, сильные решения, глобальное существование и единственность, гладкость.

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