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Boundary value problem for the heat equation with a load as the Riemann-Liouville fractional derivative

A boundary value problem for a fractionally loaded heat equation is considered in the first quadrant. The loaded term has the form of the Riemann-Liouville's fractional derivative with respect to the time variable, and the order of the derivative in the loaded term is less than the order of the differential part. The study is based on reducing the boundary value problem to a Volterra integral equation. The kernel of the obtained integral equation contains a special function, namely, the Wright function. The kernel is estimated, and the conditions for the unique solvability of the integral equation are obtained.

Keywords: loaded equation, fractional derivative, Volterra integral equation, Wright function, unique solvability.

Introduction

The study of fractional differential equations has been the subject of intense research attention [1–7]. This is due both to the development of the fractional integration and differentiation theory, and to the use of the apparatus of fractional integration and differentiation in various fields of science. Considerable interest in the study of fractional differential equations, among other things, is also fueled by various applications in physics, mechanics, and simulation [8–14]. Of particular note are some recent applications of the fractional diffusion equation to economics and financial modeling (see e.g., [15]). Monographs [16–18] contain vast bibliographies concerning the issue. Also, an important section in the theory of differential equations is the class of loaded equations. The study of loaded partial differential equations has a long history and occupies an important place in the modern theory of differential equations. In [19], on numerous examples A.M. Nakhushev showed the practical and theoretical importance of studies on loaded equations. In [20–23], the theory of loaded equations was further developed. In [22, 23] loaded differential equations are considered as weak or strong perturbations of differential equations depending on the derivative order of the loaded summand.

In the works [24–27], BVPs with a loaded heat equation are investigated, when the loaded term is represented in the form of a fractional derivative. In [24, 25], the load moves with a constant velocity. The loaded term is the trace of the fractional order derivative on the line $x = t$. It is represented as a Riemann-Liouville fractional derivative. The obtained Volterra singular integral equation has a nonempty spectrum for certain values of the fractional derivative order. Volterra integral equations of the second kind with singularities in the kernel arising from the study were considered in [26, 27] In the papers [28, 29], the loaded term is represented in the form of the Caputo fractional derivative with respect to the time variable and the spatial variable, and the derivative order of the loaded term is less than the order of the differential part.

In this paper, a BVP is considered in the open right upper quadrant. The problem is reduced to an integral equation that, in some cases, belongs to the pseudo-Volterra type, and its solvability depends on the order of differentiation in the loaded term and the behavior of the load line in a neighborhood of the origin. The BVP is reduced to a Volterra integral equation of the second kind with a kernel containing a special function. The solvability of the integral equation in the class of continuous functions is established depending on the nature of the load for small values of time.

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The article has 4 sections. Section 1 contains notations some previously known concepts and several auxiliary assertions. In Section 2, we formulate the problem we are going to solve. In Section 3, the problem is reduced to an integral equation. In Section 4, we study the resulting integral equation by evaluating its kernel and formulate the corresponding results on the solvability of the problem.

1 Preliminaries

Let us first recall some previously known concepts and results. The first one is the definition of the Riemann-Liouville fractional derivative.

Definition 1 ([1]). Let $f(t) \in L_1[a,b]$. Then, the Riemann-Liouville derivative of the order β is defined by the following formula

$$ {}_rD_{a,t}^{\beta} f(t) = \frac{1}{\Gamma(n-\beta)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\beta-n+1}} d\tau, \ \beta, a \in R, \ n-1 < \beta < n. \tag{1} $$

From formula (1) it follows that

$$ {}_rD_{a,t}^{0} f(t) = f(t), \quad {}_rD_{a,t}^{n} f(t) = f^{(n)}(t), \quad n \in N. \tag{2} $$

In [30], when considering the limiting cases of the order of the fractional derivative in the loaded term of the equation, formula (2) is used to investigate the continuity in the order of the fractional derivative.

We study boundary value problems for the loaded heat equation when the loaded term is represented in the form of a fractional derivative. The considered problem is reduced to an integral equation by inverting the differential part.

In the domain $Q = \{(x,t) \mid x > 0, \quad t > 0\}$ the solution to the boundary value problem ([31]; 57) of heat conduction

$$ u_t = a^2 u_{xx} + F(x,t), $$

$$ u\left|_{t=0}\right. = f(x), \quad u\left|_{x=0}\right. = g(x), $$

is described by the formula

$$ u(x,t) = \int_0^\infty G(x,\xi,t)\, f(\xi)\, d\xi + \int_0^t H(x,t-\tau)\, g(\tau)\, d\tau + $$

$$ + \int_0^t \int_0^\infty G(x,\xi,t-\tau)\, F(\xi,\tau)\, d\xi d\tau, \tag{3} $$

where

$$ G(x,\,\xi,\,t) = \frac{1}{2\sqrt{\pi\, a\, t}} \left\{ \exp\left(-\frac{(x-\xi)^2}{4\,a\,t} \right) - \exp\left(-\frac{(x+\xi)^2}{4\,a\,t} \right) \right\}, $$

$$ H(x,\,t) = \frac{1}{2\sqrt{\pi\, a}\, t^{3/2}} \exp\left(-\frac{x^2}{4\,a\,t} \right). $$

The Green's function $G(x,\xi,t-\tau)$ satisfies the relation

$$ \int_0^\infty G(x,\xi,t)\, d\xi = \operatorname{erf}\left(\frac{x}{2\sqrt{t}} \right), \tag{4} $$

where

$$ \operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-\xi^2} d\xi. \tag{5} $$

Fractional calculus can be considered as a "laboratory" for special functions.

We get a reduced integral equation with a kernel containing the Wright function. Accordingly, we determine the conditions for the solvability of this equation using the kernel estimate from the works [32, 33].

ϕ is the Wright function:

$$ \phi(a,b;\,z) = \sum_{k=0}^{\infty} \frac{z^k}{k!\,\Gamma(ak+b)} \qquad (a > -1). \tag{6} $$

The differentiation formula is valid:

$$\left(\frac{d}{dz}\right)^n \phi\left(\alpha,\beta;z\right) = \phi\left(\alpha,\alpha+n\beta;z\right), n\in N. \tag{7}$$

For all $\alpha \in \left]0,1\right[, \beta\in R, x>0, y>0$ the following inequality holds

$$\left|y^{\beta-1}\phi\left(-\alpha,\beta;-xy^{-\alpha}\right)\right| \le Cx^{-\theta}y^{\beta+\alpha\theta-1}, \tag{8}$$

where

$$\theta \ge \begin{cases} 0, (-\beta)\notin N\bigcup\{0\}, \\ -1, (-\beta)\in N\bigcup\{0\}. \end{cases}$$

2 Statement of the fractionally loaded BVP of heat conduction

In the domain $Q=\{(x,t): x>0, t>0\}$ we consider a BVP

$$u_t - u_{xx} + \lambda\left\{ {}_rD^{\beta}_{0,t}u\left(x,t\right)\right\}|_{x=\gamma(t)} = f\left(x,t\right), \tag{9}$$

$$u\left(x,0\right)=0, \quad u\left(0,t\right)=0, \tag{10}$$

where λ is a complex parameter, ${}_rD^{\beta}_{0,\ t}\,u(x,\ t)$ is the Riemann-Liouville derivative (1) of an order β, $0<\beta<1$, $\gamma(t)$ is a continuous increasing function, $\gamma(0)=0$.

The problem is studied in the class of continuous functions.

Let us introduce the notation

$$D^{\nu}_{at}g\left(t\right) = \frac{1}{\Gamma\left(-\nu\right)}\int_a^t \frac{g\left(\xi\right)d\xi}{\left(t-\xi\right)^{\nu+1}}, \quad \nu<0.$$

When $\nu=0$ $D^0_{at}g\left(t\right)=g\left(t\right)$ then

$$D^{\nu}_{at}g\left(t\right) = \frac{d^n}{dt^n}D^{\nu-n}_{at}g\left(t\right), \quad n-1<\nu\le n, \quad n\in N.$$

$a=0$, $n=1$, $\nu=\beta \Rightarrow$

$${}_rD^{\beta}_{0t}u\left(x,t\right) = \frac{d}{dt}D^{\beta-1}_{0t}u\left(x,t\right) \tag{11}$$

or

$${}_rD^{\beta}_{0t}u\left(x,t\right) = \frac{d}{dt}\left(\frac{1}{\Gamma\left(1-\beta\right)}\int_0^t \frac{u\left(x,\tau\right)d\tau}{\left(t-\tau\right)^{\beta}}\right). \tag{12}$$

The derivative in the loaded term of equation (9) is determined by the formula (12).

3 Reducing the boundary value problem to an integral equation

According to the formula (3) a solution to BVP (9)–(10) can be represented as

$$u\left(x,t\right) = -\lambda\int_0^t\int_0^{\infty} G\left(x,\xi,t-\tau\right)\mu\left(\tau\right)d\xi d\tau + f_1\left(x,t\right), \tag{13}$$

where

$$\mu\left(t\right) = \left\{ {}_rD^{\beta}_{0t}u\left(x,t\right)\right\}|_{x=\gamma(t)} \tag{14}$$

$$f_1\left(x,t\right) = \int_0^t\int_0^{+\infty} G\left(x,\xi,t-\tau\right)f\left(\xi,\tau\right)d\xi d\tau. \tag{15}$$

According to the formula (4) and

$$e^{-\xi^2} = \sqrt{\pi}\phi\left(-\frac{1}{2},\frac{1}{2},-2\xi\right), \tag{16}$$

where

$$\phi\left(a,b,z\right)=\sum_{\kappa=0}^{\infty}\frac{z^{\kappa}}{\kappa!\Gamma\left(a\kappa+b\right)},\quad a>-1,\quad b\in C \tag{17}$$

is the Wright function (6), we have [34]

$$\operatorname{erf}\left(z\right)=2\int_0^z\phi\left(-\frac{1}{2},\frac{1}{2},-2\xi\right)d\xi=1-\phi\left(-\frac{1}{2},1,-2z\right). \tag{18}$$

Indeed, since

$$\Gamma\left(1-z\right)\cdot\Gamma\left(z\right)=\frac{\pi}{\sin\pi z}\Rightarrow$$

$$\Gamma\left(-\frac{\kappa}{2}+\frac{1}{2}\right)=\frac{\pi}{\Gamma\left(\frac{\kappa}{2}+\frac{1}{2}\right)\sin\left(\frac{\pi\kappa}{2}+\frac{\pi}{2}\right)}=\frac{\pi}{\Gamma\left(\frac{\kappa}{2}+\frac{1}{2}\right)\cos\frac{\pi\kappa}{2}};$$

$$\cos\frac{\pi\kappa}{2}=\begin{cases}0, & \text{if}\quad \kappa=2n+1,\\ \left(-1\right)^{n}, & \text{if}\quad \kappa=2n.\end{cases}$$

$$\Rightarrow\frac{1}{\Gamma\left(-\frac{\kappa}{2}+\frac{1}{2}\right)}=\begin{cases}0, & \text{if}\quad \kappa=2n+1,\\ \dfrac{(-1)^{n}\Gamma\left(\frac{\kappa}{2}+\frac{1}{2}\right)}{\pi}, & \text{if}\quad \kappa=2n,\end{cases}$$

where $n=0;1;2;...$, then from (17) we have:

$$\phi\left(-\frac{1}{2},\frac{1}{2},-2\xi\right)=\sum_{\kappa=0}^{\infty}\frac{(-2\xi)^{\kappa}}{\kappa!\Gamma\left(-\frac{\kappa}{2}+\frac{1}{2}\right)}=\sum_{n=0}^{\infty}\frac{\left(-1\right)^{n}\Gamma\left(n+\frac{1}{2}\right)}{(2n)!\cdot\pi}\cdot\left(-2\xi\right)^{2n}=$$

$$\left\|\Gamma\left(n+\frac{1}{2}\right)=\frac{(2n)!}{4^{n}\cdot n!}\sqrt{\pi}\right\|$$

$$=\frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\left(-1\right)^{n}}{4^{n}\cdot n!}\cdot 4^{n}\left(\xi^{2}\right)^{n}=\frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\left(-\xi^{2}\right)^{n}}{n!}=\frac{1}{\sqrt{\pi}}e^{-\xi^{2}}.$$

We obtain formula (16).

From (5) we have:

$$\operatorname{erf}\left(z\right)=\frac{2}{\sqrt{\pi}}\int_0^z e^{-\xi^{2}}d\xi=2\int_0^z\phi\left(-\frac{1}{2},\frac{1}{2};-2\xi\right)d\xi=-\int_0^{-2z}\phi\left(-\frac{1}{2},\frac{1}{2};\zeta\right)d\zeta=$$

$$=-\sum_{\kappa=0}^{\infty}\int_0^{2z}\frac{\zeta^{\kappa}}{\kappa!\cdot\Gamma\left(-\frac{\kappa}{2}+\frac{1}{2}\right)}d\zeta=-\sum_{\kappa=0}^{\infty}\frac{\zeta^{\kappa+1}}{(\kappa+1)!\cdot\Gamma\left(-\frac{\kappa+1}{2}+1\right)}\Big|_{\zeta=0}^{\zeta=-2z}=$$

$$=-\sum_{\kappa=0}^{\infty}\frac{\left(-2z\right)^{\kappa+1}}{(\kappa+1)!\cdot\Gamma\left(-\frac{\kappa+1}{2}+1\right)}=-\sum_{n=0}^{\infty}\frac{\left(-2z\right)^{n}}{n!\cdot\Gamma\left(-\frac{n}{2}+1\right)}+1=1-\phi\left(-\frac{1}{2},1,-2z\right).$$

We get formula (18).

Then, taking into account formulas (16) and (18), representation (13) can be rewritten as:

$$u\left(x,t\right)=-\lambda\int_0^t K\left(\frac{x}{2\sqrt{t-\tau}}\right)\mu\left(\tau\right)d\tau+f_1\left(x,t\right), \tag{19}$$

where

$$K\left(\frac{x}{2\sqrt{t-\tau}}\right)=1-\phi\left(-\frac{1}{2},1,-\frac{x}{\sqrt{t-\tau}}\right) \tag{20}$$

and $\mu(t)$ and $f_1(t)$ are defined by formulas (14) and (15), respectively.

For formula (19) we implement the fractional differentiation formula of order β $(0<\beta<1)$ in the sense of Riemann-Liouville.

Denote $K\left(\frac{x}{2\sqrt{t}}\right)=g\left(x,t\right)$.

As:

$$\int_0^t K\left(\frac{x}{2\sqrt{t-\tau}}\right)\mu\left(\tau\right)d\tau=\int_0^t g\left(x,t-\tau\right)\mu\left(\tau\right)d\tau=\left(g\left(x,t\right)*\mu\left(t\right)\right)\left(t\right),$$

and

$$\frac{d}{dt}\left(g*\mu\right)\left(t\right)=\left(\frac{dg}{dt}*\mu\right)\left(t\right)+g\left|_{t=0}\right.\cdot\mu\left(t\right),$$

then by formulas (11), (7) and (20) we have

$$D_{0t}^{\beta}\left(\int_0^t K\left(\frac{x}{2\sqrt{t-\tau}}\right)\mu\left(\tau\right)d\tau\right)=D_{0t}^{\beta}\left(K\left(\frac{x}{2\sqrt{t}}\right)*\mu\left(t\right)\right)=$$

$$=D_{0t}^{\beta}\left(1-\phi\left(-\frac{1}{2},1,-\frac{x}{\sqrt{t}}\right)\right)*\mu\left(t\right)+\mathrm{K}\left(\frac{x}{2\sqrt{t}}\right)\Bigg|_{t=0}\cdot\mu(t). \tag{21}$$

As

$$D_{0t}^{\beta}\left(1\right)=\frac{1}{\Gamma\left(1-\beta\right)}\cdot t^{-\beta},$$

then when $0<\beta<1$

$$D_{0t}^{\beta}t^{\mu}=\frac{\Gamma\left(\mu+1\right)}{\Gamma\left(\mu-\beta+1\right)}t^{\mu-\beta}.$$

From here

$$D_{0t}^{\beta}\phi\left(-\frac{1}{2},1;-x\cdot t^{-\frac{1}{2}}\right)=\sum_{n=0}^{\infty}\frac{\left(-x\right)^{n}}{n!\cdot\Gamma\left(-\frac{n}{2}+1\right)}D_{0t}^{\beta}\left(t^{-\frac{n}{2}}\right)=$$

$$=\sum_{n=0}^{\infty}\frac{\left(-x\right)^{n}}{n!\cdot\Gamma\left(-\frac{n}{2}+1\right)}\frac{\Gamma\left(-\frac{n}{2}+1\right)}{\Gamma\left(-\frac{n}{2}+1-\beta\right)}\cdot t^{-\frac{n}{2}-\beta}=t^{-\beta}\cdot\sum_{n=0}^{\infty}\frac{\left(-\frac{x}{\sqrt{t}}\right)^{n}}{n!\cdot\Gamma\left(-\frac{n}{2}+1-\beta\right)}=$$

$$=t^{-\beta}\cdot\phi\left(-\frac{1}{2},1-\beta;-\frac{x}{\sqrt{t}}\right). \tag{22}$$

$$g\left(x,t\right)\left|_{t=0}\right.=\mathrm{K}\left(\frac{x}{2\sqrt{t}}\right)\Bigg|_{t=0}=\left(1-\phi\left(-\frac{1}{2},1;-\frac{x}{\sqrt{t}}\right)\right)\Bigg|_{t=0}. \tag{23}$$

Since in the given problem (9), (10) the line along which the load is moving has the form $x=\gamma\left(t\right)$, and $\gamma\left(t\right)$ increases and $\gamma\left(0\right)=0$ then there are different cases of behavior for $\frac{x}{\sqrt{t}}\left|_{x=\gamma(t)}\right.$ when $t\to 0$.

Let $0<x=\gamma(t)\sim t^{\omega}$ when $t\to 0$. Then $\frac{x}{\sqrt{t}}\to+\infty$ when $t\to 0$, if $\omega<\frac{1}{2}$.

Cases $\omega>\frac{1}{2}$ and $\omega=\frac{1}{2}$ we consider later.

From [12, p. 6] we have an asymptotic expansion for $z\to+\infty$:

$$\phi\left(-\frac{1}{2},1;-z\right)=e^{-\frac{z^2}{4}}\left[\sum_{j=0}^{m}A_j\cdot 2^{2j+1}\cdot z^{-2j-1}+O\left(2^{2m}\cdot z^{-2m-1}\right)\right].$$

Then if $\omega<\frac{1}{2}$ for formula (23) we get when $t\to 0$

$$g\left(x,t\right)=\left(1-\phi\left(-\frac{1}{2},1;-\frac{x}{\sqrt{t}}\right)\right)\to 1. \tag{24}$$

So, applying to (19) the fractional differentiation of the order β by formula (11) taking into account the formula (21)–(24), when $x=\gamma\left(t\right)$, where $\gamma\left(t\right))\sim t^{\omega}$ when $t\to 0$, $\omega<\frac{1}{2}$, we get when $\lambda\neq-1$

$$\mu\left(t\right)+\frac{\lambda}{\lambda+1}\int_0^t K\left(t,\tau\right)\mu\left(\tau\right)d\tau=f_3\left(t\right), \tag{25}$$

where

$$f_3(t) = \frac{\lambda}{\lambda+1} D_{0t}^{\beta}\left(f_1(x,t)\right)\Bigg|_{x=\gamma(t)}, \tag{26}$$

$$K(t,\tau) = \frac{1}{\Gamma(1-\beta)(t-\tau)^{\beta}} - \frac{1}{(t-\tau)^{\beta}} \cdot \phi\left(-\frac{1}{2}, 1-\beta; -\frac{\gamma(t)}{\sqrt{t-\tau}}\right). \tag{27}$$

4 Integral equation research. Main result

Let us estimate kernel (27) of integral equation (25). The Wright function for $\forall\alpha \in (0;1)$, $b \in R$, $x > 0$, $y > 0$ satisfies the inequality (8) [16]

$$\left|y^{b-1} \cdot \phi\left(-\alpha, b; -xy^{-\alpha}\right)\right| \leq C \cdot x^{-\theta} \cdot y^{b+\alpha\theta-1},$$

where $\theta \geq 0$, when $-b \notin N \bigcup \{0\}$.

Then

$$\left|\frac{1}{(t-\tau)^{\beta}} \cdot \phi\left(-\frac{1}{2}, 1-\beta; -\frac{\gamma(t)}{\sqrt{t-\tau}}\right)\right| \leq C\left(\gamma(t)\right)^{-\theta}(t-\tau)^{-\beta+\frac{\theta}{2}}, \theta \geq 0.$$

At $\theta = 0$ we obtain:

$$|K(t,\tau)| \leq \left(\frac{1}{\Gamma(1-\beta)} + 1\right) \cdot (t-\tau)^{-\beta},$$

when $0 < \beta < 1$.

From here we get that the kernel of the integral equation has an integrable singularity if $\gamma(t)) \sim t^{\omega}$ when $t \to 0$, $\omega < \frac{1}{2}$.

Thus, the following theorem has been proved.

Theorem. Integral equation (25) with kernel (27) for $0 < \beta < 1$ and with $\gamma(t) \sim t^{\omega}$ in the neighborhood of $t = 0$ is uniquely solvable in the class of continuous functions for any continuous right-hand side $f_3(t)$ defined by formula (26), if $0 \leq \omega < \frac{1}{2}$.

This result coincided with the result obtained in [30].

Conclusions

According to the theorem, the integral equation (25) has a kernel with a weak singularity. Therefore, to find a unique solution to the equation (25) in the class of continuous functions, we can apply the method of successive approximations. After finding the solution $\mu(\tau)$ to equation (25), the solution to the original boundary value problem is found uniquely by formula (13). For the boundary value problem, the loaded term is a weak perturbation.

In other cases of values of the parameters β and ω, the method of successive approximations is not applicable for solving the integral equation (25). It is possible that the corresponding homogeneous equation will have nontrivial solutions for some values of the parameter λ, i.e. the spectrum of the problem will appear. Then the load can be interpreted as a strong perturbation. The existence and uniqueness of solutions to the integral equation depends on the fractional derivative order of the loaded summand. For $\lambda = -1$, BVP (9), (10) is reduced to The Volterra integral equation of the first kind.

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Риман-Лиувилль бөлшек туындысы түріндегі жүктемемен берілген жылуөткізгіштік теңдеуі үшін шекаралық есеп

Бірінші квадрантта үздіксіз функциялар класында бөлшекті-жүктелген жылуөткізгіштік теңдеуі үшін шекаралық есеп қарастырылған. Жүктелген қосылғыш уақытша айнымалы бойынша Риман-Лиувиллдің бөлшек туындысы түрінде болады, ал жүктелген қосылғыштағы туынды реті дифференциалдық бөліктің ретінен аз болады. Зерттеу шеттік есепті Вольтерр интегралдық теңдеуіне келтіруге негізделген. Алынған интегралдық теңдеудің ядросында арнайы функция бар, атап айтқанда Райт функциясы. Ядро бағаланып, интегралдық теңдеудің біркелкі шешілу шарттары алынды.

Кілт сөздер: жүктелген теңдеу, бөлшек туынды, Вольтеррдің интегралдық теңдеуі, Райт функциясы, бірмәнді шешімділік.

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Граничная задача для уравнения теплопроводности с нагрузкой в виде дробной производной Римана–Лиувилля

В первом квадранте рассмотрена краевая задача для дробно-нагруженного уравнения теплопроводности в классе непрерывных функций. Нагруженное слагаемое имеет форму дробной производной Римана–Лиувилля по временной переменной, и порядок производной в нагруженном слагаемом меньше порядка дифференциальной части. Исследование основано на сведении краевой задачи к интегральному уравнению Вольтерра. Ядро полученного интегрального уравнения содержит специальную функцию, а именно, функцию Райта. Произведена оценка ядра, и получены условия однозначной разрешимости интегрального уравнения.

Ключевые слова: нагруженное уравнение, дробная производная, интегральное уравнение Вольтерра, функция Райта, однозначная разрешимость.