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A stable difference scheme for the solution of a source identification problem for telegraph-parabolic equations

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In the present paper, we construct a first order of accuracy difference scheme for the approximate solution of the inverse problem for telegraph-parabolic equations with an unknown spacewise dependent source term. The unique solvability of constructed difference scheme and the stability estimates for its solution were obtained. The proofs are based on the spectral representation of the self-adjoint positive definite operator in a Hilbert space.

Keywords: Difference scheme, source identification problem, telegraph-parabolic equation, stability estimates.

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Introduction

Differential equations with unknown source terms are widely used in the mathematical modelling of real-life phenomena in many different fields of science and have been broadly investigated over the years (see, e.g., [1]–[9] and the references therein).

The problems for differential equations containing a time- and/or space-dependent parameter (source term) are called source identification problems. These types of problems are inverse and their solutions cannot be determined uniquely from imposed initial and/or boundary conditions. To achieve a well-posedness of a source identification problem, one needs to provide some additional condition(s). Source identification problems for mixed type differential equations have been receiving a great deal of attention recently (see, e.g., [10]–[19] for hyperbolic-parabolic, [20]–[22] for elliptic-hyperbolic, and [23] for parabolic-elliptic source identification problems).

Numerous local and nonlocal boundary value problems for telegraph-parabolic equations with unknown source terms can be reduced to the following abstract problem for the differential equation with a spacewise dependent parameter p

$$\begin{cases} u''(t) + \alpha u'(t) + Au(t) = p + f(t), \ 0 < t < 1, \\ u'(t) + Au(t) = p + g(t), \ -1 < t < 0, \\ u(0+) = u(0-), \ u'(0+) = u'(0-), \\ u(-1) = \varphi, \ u(\lambda) = \psi, \ -1 < \lambda \le 1 \end{cases} \tag{1}$$

in a Hilbert space H with a self-adjoint positive definite operator A satisfying $A \ge \delta I$, where $\delta > \frac{\alpha^2}{4}$ and $\alpha \ge 0$. The last condition in (1) is considered in order to compensate the uncertainty in the problem due to unknown term p.

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The unique solvability of problem (1) in the space $C(H)$ of the continuous H-valued functions $u(t)$ defined on $[-1,1]$, equipped with the norm

$$\|u\|_{C(H)} = \max_{-1\le t\le 1} \|u(t)\|_H$$

was established in [24], and the following theorem on the continuous dependence of the solution on the given data was proven.

Theorem 1 ([24]). Assume that $\varphi,\ \psi \in D(A)$. Let $f(t)$ and $g(t)$ be continuously differentiable functions on $[0,1]$ and $[-1,0]$, respectively. Then, for the solution $\{u(t),p\}$ of problem (1) in $C(H)\times H$ the following stability inequalities

$$\|u\|_{C(H)} + \|A^{-1}p\|_H \le M(\delta,\lambda)\Big[\|\varphi\|_H + \|\psi\|_H + \max_{0\le t\le 1}\|f(t)\|_H + \max_{-1\le t\le 0}\|g(t)\|_H\Big],$$

$$\max_{0\le t\le 1}\|u''(t)\|_H + \max_{0\le t\le 1}\|\alpha u'(t)\|_H + \max_{-1\le t\le 0}\|u'(t)\|_H + \|Au\|_{C(H)} + \|p\|_H$$
$$\le M(\delta,\lambda)\Big[\|A\varphi\|_H + \|A\psi\|_H + \max_{0\le t\le 1}\|f'(t)\|_H + \|f(0)\|_H + \max_{-1\le t\le 0}\|g'(t)\|_H + \|g(0)\|_H\Big]$$

hold, where $M(\delta,\lambda)$ does not depend on φ, ψ, $f(t)$ and $g(t)$.

In general, the differential equations with unknown parameters are not solvable analytically and therefore one needs to use numerical methods to approximate their solutions. The main goal of this study is to construct and investigate a first order of accuracy stable difference scheme for the approximate solution of abstract problem (1). We prove the unique solvability of the constructed difference scheme and obtain the stability estimates for its solution. The analysis is based on the operator approach and the proofs of the stability estimates are based on the spectral representation of the self-adjoint positive definite operator in a Hilbert space.

1 First order of accuracy stable difference scheme

Let $\tau = 1/N$ be sufficiently small positive number satisfying $\lambda \ge -1+\tau$. Let us define the grid points $t_k = k\tau,\ -N \le k \le N$. For the approximate solution of problem (1), we construct the first order of accuracy stable difference scheme

$$\begin{cases} \frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2} + \alpha\frac{u_{k+1}-u_k}{\tau} + Au_{k+1} = p + f_k, \quad 1\le k\le N-1,\\ \frac{u_k-u_{k-1}}{\tau} + Au_k = p + g_k, \quad -N+1\le k\le 0,\\ \frac{u_1-u_0}{\tau} = p - Au_0 + g_0, \quad u_{-N}=\varphi, \quad u_\ell = \psi, \end{cases} \tag{2}$$

where $\ell = \lfloor \lambda/\tau \rfloor$, $f_k = f(t_k)$, $1\le k\le N-1$ and $g_k = g(t_k)$, $-N+1\le k\le 0$.

We first present some lemmas, which we will need in the remaining part of this paper. Here and everywhere else, we denote

$$R = \left(\left(1+\frac{\alpha\tau}{2}\right)I + i\tau\left(A-\frac{\alpha^2}{4}I\right)^{1/2}\right)^{-1}, \quad \tilde{R} = \left(\left(1+\frac{\alpha\tau}{2}\right)I - i\tau\left(A-\frac{\alpha^2}{4}I\right)^{1/2}\right)^{-1}$$

and

$$Q = (I+\tau A)^{-1}.$$

Lemma 1 ([25]). The following estimates hold

$$\|R\|_{H\to H}\le 1, \qquad \|\tilde{R}\|_{H\to H}\le 1, \qquad \|\tilde{R}^{-1}R\|_{H\to H}\le 1, \qquad \left\|R^{-1}\tilde{R}\right\|_{H\to H}\le 1. \tag{3}$$

Lemma 2 ([25]). The following estimates hold

$$\|Q^m\|_{H\to H} \le \frac{1}{1+m\tau\delta} < 1, \quad m \ge 1, \tag{4}$$

$$\|A^{1/2}Q^m\|_{H\to H} \le \frac{1}{2\sqrt{m\tau}}, \quad m \ge 1. \tag{5}$$

Lemma 3. If $-1+\tau \le \lambda < \tau$, then $-N+1 \le \ell \le 0$, and the following estimate holds

$$\left\| \left(I - Q^{N+\ell}\right)^{-1} \right\|_{H\to H} \le M_1(\delta,\lambda). \tag{6}$$

Proof. The proof of estimate (6) is based on estimate (4).

Lemma 4. The following estimates hold for $m \ge 1$

$$\left\| \left[\frac{R^{m-1}}{2} \left(I - \frac{\alpha}{2i}\left(A - \frac{\alpha^2}{4}I\right)^{-1/2} \right) + \frac{\tilde{R}^{m-1}}{2} \left(I + \frac{\alpha}{2i}\left(A - \frac{\alpha^2}{4}I\right)^{-1/2} \right) \right.\right.$$
$$\left.\left. + \frac{A}{2i}\left(A - \frac{\alpha^2}{4}I\right)^{-1/2} \left(\tilde{R}^{-1}R^{m-1} - R^{-1}\tilde{R}^{m-1}\right) \right] Q^N \right\|_{H\to H} < 1. \tag{7}$$

Proof. Since

$$\frac{R^{m-1}}{2} \left(I - \frac{\alpha}{2i}\left(A - \frac{\alpha^2}{4}I\right)^{-1/2} \right) + \frac{\tilde{R}^{m-1}}{2} \left(I + \frac{\alpha}{2i}\left(A - \frac{\alpha^2}{4}I\right)^{-1/2} \right)$$
$$+ \frac{A}{2i}\left(A - \frac{\alpha^2}{4}I\right)^{-1/2} \left(\tilde{R}^{-1}R^{m-1} - R^{-1}\tilde{R}^{m-1}\right)$$
$$= \left[I - \tau A + i\left\{ \frac{\alpha}{2}I - \left(1 + \frac{\alpha\tau}{2}\right)A \right\} \left(A - \frac{\alpha^2}{4}I\right)^{-1/2} \right] \frac{R^{m-1}}{2}$$
$$+ \left[I - \tau A - i\left\{ \frac{\alpha}{2}I - \left(1 + \frac{\alpha\tau}{2}\right)A \right\} \left(A - \frac{\alpha^2}{4}I\right)^{-1/2} \right] \frac{\tilde{R}^{m-1}}{2},$$

using (3) and the following estimates

$$\left\| \left[I - \tau A \pm i\left\{ \frac{\alpha}{2}I - \left(1 + \frac{\alpha\tau}{2}\right)A \right\} \left(A - \frac{\alpha^2}{4}I\right)^{-1/2} \right] Q^N \right\|_{H\to H} < 1, \tag{8}$$

we obtain (7). The proof of estimates (8) is based on the spectral representation of the self-adjoint positive definite operator A in a Hilbert space H [25].

Lemma 5. If $\tau \le \lambda \le 1$, then $1 \le \ell \le N$ and the following estimate holds

$$\left\| \left(I - \left[\frac{R^{\ell-1}}{2} \left(I - \frac{\alpha}{2i}\left(A - \frac{\alpha^2}{4}I\right)^{-1/2} \right) + \frac{\tilde{R}^{\ell-1}}{2} \left(I + \frac{\alpha}{2i}\left(A - \frac{\alpha^2}{4}I\right)^{-1/2} \right) \right.\right.\right.$$
$$\left.\left.\left. + \frac{A}{2i}\left(A - \frac{\alpha^2}{4}I\right)^{-1/2} \left(\tilde{R}^{-1}R^{\ell-1} - R^{-1}\tilde{R}^{\ell-1}\right) \right] Q^N \right)^{-1} \right\|_{H\to H} \le M_2(\delta,\lambda,\alpha). \tag{9}$$

Proof. The proof of estimate (9) is based on estimate (7).

We now present the main theorem for the solution of the first order of accuracy difference scheme (2).

Theorem 2. The difference scheme (2) has a unique solution and the following stability estimate holds

$$\max_{-N\le k\le N}\|u_k\|_H+\left\|A^{-1}p\right\|_H$$
$$\le M^*(\delta,\lambda,\alpha)\Big[\|\varphi\|_H+\|\psi\|_H+\max_{1\le k\le N-1}\|f_k\|_H+\max_{-N+1\le k\le 0}\|g_k\|_H\Big], \quad (10)$$

where $M^*(\delta,\lambda,\alpha)$ is independent of φ, ψ, τ, f_k and g_k.

Proof. Let us denote

$$u_k=v_k+A^{-1}p,\quad -N\le k\le N. \quad (11)$$

Then, the difference scheme (2) results in the following auxiliary difference scheme

$$\begin{cases}\frac{v_{k+1}-2v_k+v_{k-1}}{\tau^2}+\alpha\frac{v_{k+1}-v_k}{\tau}+Av_{k+1}=f_k,\quad 1\le k\le N-1,\\ \frac{v_k-v_{k-1}}{\tau}+Av_k=g_k,\quad -N+1\le k\le 0,\\ \frac{v_1-v_0}{\tau}=-Av_0+g_0,\quad v_\ell=v_{-N}+\psi-\varphi.\end{cases} \quad (12)$$

First, we obtain the formulas for solution of scheme (12). For the given v_0 the following difference scheme

$$\begin{cases}\frac{v_{k+1}-2v_k+v_{k-1}}{\tau^2}+\alpha\frac{v_{k+1}-v_k}{\tau}+Av_{k+1}=f_k,\quad 1\le k\le N-1,\\ \frac{v_1-v_0}{\tau}=-Av_0+g_0\end{cases}$$

has a solution

$$v_k=\left[\frac{R^{k-1}}{2}\left(I-\frac{\alpha}{2i}\left(A-\frac{\alpha^2}{4}I\right)^{-1/2}\right)+\frac{\tilde{R}^{k-1}}{2}\left(I+\frac{\alpha}{2i}\left(A-\frac{\alpha^2}{4}I\right)^{-1/2}\right)\right]v_0$$
$$+\left(R-\tilde{R}\right)^{-1}\tau\left(R^k-\tilde{R}^k\right)(-Av_0+g_0)$$
$$-\frac{1}{2i}\sum_{j=1}^{k}\left(A-\frac{\alpha^2}{4}I\right)^{-1/2}\left(R^{k-j}-\tilde{R}^{k-j}\right)f_j\tau,\quad 1\le k\le N. \quad (13)$$

Furthermore, for the given v_{-N}, the following difference scheme

$$\frac{v_k-v_{k-1}}{\tau}+Av_k=g_k,\ -N+1\le k\le 0$$

has a solution

$$v_k=Q^{N+k}v_{-N}+\sum_{j=-N+1}^{k}Q^{k-j+1}g_j\tau,\quad -N+1\le k\le 0. \quad (14)$$

In particular, putting $k=0$ in (14), we get

$$v_0=Q^Nv_{-N}+\sum_{j=-N+1}^{0}Q^{-j+1}g_j\tau.$$

Then, by putting this expression for v_0 in (13), we obtain

$$\begin{aligned} v_k =& \left[\frac{R^{k-1}}{2} \left(I - \frac{\alpha}{2i} \left(A - \frac{\alpha^2}{4} I \right)^{-1/2} \right) + \frac{\tilde{R}^{k-1}}{2} \left(I + \frac{\alpha}{2i} \left(A - \frac{\alpha^2}{4} I \right)^{-1/2} \right) \right. \\ & \left. - \tau A \left(R - \tilde{R} \right)^{-1} \left(R^k - \tilde{R}^k \right) \right] \left(Q^N v_{-N} + \sum_{j=-N+1}^{0} Q^{-j+1} g_j \tau \right) \\ & + \left(R - \tilde{R} \right)^{-1} \left(R^k - \tilde{R}^k \right) \tau g_0 - \frac{1}{2i} \sum_{j=1}^{k} \left(A - \frac{\alpha^2}{4} I \right)^{-1/2} \left(R^{k-j} - \tilde{R}^{k-j} \right) f_j \tau, \quad 1 \le k \le N. \end{aligned}$$

Using $R - \tilde{R} = -2i\tau \left(A - \frac{\alpha^2}{4} I \right)^{1/2} R\tilde{R}$, we have

$$\begin{aligned} v_k =& \left[\frac{R^{k-1}}{2} \left(I - \frac{\alpha}{2i} \left(A - \frac{\alpha^2}{4} I \right)^{-1/2} \right) + \frac{\tilde{R}^{k-1}}{2} \left(I + \frac{\alpha}{2i} \left(A - \frac{\alpha^2}{4} I \right)^{-1/2} \right) \right. \\ & \left. + \frac{A}{2i} \left(A - \frac{\alpha^2}{4} I \right)^{-1/2} \left(\tilde{R}^{-1} R^{k-1} - R^{-1} \tilde{R}^{k-1} \right) \right] \left(Q^N v_{-N} + \sum_{j=-N+1}^{0} Q^{-j+1} g_j \tau \right) \\ & - \frac{1}{2i} \left(A - \frac{\alpha^2}{4} I \right)^{-1/2} \left(\tilde{R}^{-1} R^{k-1} - R^{-1} \tilde{R}^{k-1} \right) g_0 \\ & - \frac{1}{2i} \sum_{j=1}^{k} \left(A - \frac{\alpha^2}{4} I \right)^{-1/2} \left(R^{k-j} - \tilde{R}^{k-j} \right) f_j \tau, \quad 1 \le k \le N. \end{aligned} \tag{15}$$

If $-1 + \tau \le \lambda < \tau$, then $-N + 1 \le \ell \le 0$, and therefore from (12) and (14) it follows

$$v_\ell = v_{-N} + \psi - \varphi = Q^{N+\ell} v_{-N} + \sum_{j=-N+1}^{\ell} Q^{\ell-j+1} g_j \tau,$$

so that

$$v_{-N} = \left(I - Q^{N+\ell} \right)^{-1} \left(\sum_{j=-N+1}^{\ell} Q^{\ell-j+1} g_j \tau + \varphi - \psi \right). \tag{16}$$

If $\tau \le \lambda \le 1$, then $1 \le \ell \le N$, and therefore from (12) and (15) it follows

$$\begin{aligned} v_\ell =& v_{-N} + \psi - \varphi = \left[\frac{R^{\ell-1}}{2} \left(I - \frac{\alpha}{2i} \left(A - \frac{\alpha^2}{4} I \right)^{-1/2} \right) + \frac{\tilde{R}^{\ell-1}}{2} \left(I + \frac{\alpha}{2i} \left(A - \frac{\alpha^2}{4} I \right)^{-1/2} \right) \right. \\ & \left. + \frac{A}{2i} \left(A - \frac{\alpha^2}{4} I \right)^{-1/2} \left(\tilde{R}^{-1} R^{\ell-1} - R^{-1} \tilde{R}^{\ell-1} \right) \right] \left(Q^N v_{-N} + \sum_{j=-N+1}^{0} Q^{-j+1} g_j \tau \right) \\ & - \frac{1}{2i} \left(A - \frac{\alpha^2}{4} I \right)^{-1/2} \left(\tilde{R}^{-1} R^{\ell-1} - R^{-1} \tilde{R}^{\ell-1} \right) g_0 - \frac{1}{2i} \sum_{j=1}^{\ell} \left(A - \frac{\alpha^2}{4} I \right)^{-1/2} \left(R^{\ell-j} - \tilde{R}^{\ell-j} \right) f_j \tau, \end{aligned}$$

so that

$$v_{-N}=\left(I-\left[\frac{R^{\ell-1}}{2}\left(I-\frac{\alpha}{2i}\left(A-\frac{\alpha^2}{4}I\right)^{-1/2}\right)+\frac{\tilde{R}^{\ell-1}}{2}\left(I+\frac{\alpha}{2i}\left(A-\frac{\alpha^2}{4}I\right)^{-1/2}\right)\right.\right.$$
$$\left.\left.+\frac{A}{2i}\left(A-\frac{\alpha^2}{4}I\right)^{-1/2}\left(\tilde{R}^{-1}R^{\ell-1}-R^{-1}\tilde{R}^{\ell-1}\right)\right]Q^N\right)^{-1}$$
$$\times\left[\left\{\frac{R^{\ell-1}}{2}\left(I-\frac{\alpha}{2i}\left(A-\frac{\alpha^2}{4}I\right)^{-1/2}\right)+\frac{\tilde{R}^{\ell-1}}{2}\left(I+\frac{\alpha}{2i}\left(A-\frac{\alpha^2}{4}I\right)^{-1/2}\right)\right.\right.$$
$$\left.+\frac{A}{2i}\left(A-\frac{\alpha^2}{4}I\right)^{-1/2}\left(\tilde{R}^{-1}R^{\ell-1}-R^{-1}\tilde{R}^{\ell-1}\right)\right\}\sum_{j=-N+1}^{0}Q^{-j+1}g_j\tau$$
$$-\frac{1}{2i}\left(A-\frac{\alpha^2}{4}I\right)^{-1/2}\left(\tilde{R}^{-1}R^{\ell-1}-R^{-1}\tilde{R}^{\ell-1}\right)g_0$$
$$\left.-\frac{1}{2i}\sum_{j=1}^{\ell}\left(A-\frac{\alpha^2}{4}I\right)^{-1/2}\left(R^{\ell-j}-\tilde{R}^{\ell-j}\right)f_j\tau+\varphi-\psi\right]. \quad (17)$$

Thus, for the solution of auxiliary difference scheme (12), we have formulas (14) and (15), with v_{-N} being found by formula (16) if $-1+\tau\leq\lambda<\tau$ and formula (17), if $\tau\leq\lambda\leq 1$. Now, taking into account that $u_{-N}=\varphi$, we have $A^{-1}p=\varphi-v_{-N}$. Then, using (11), we obtain the solution of difference scheme (2).

Now, let us obtain the estimate (10). Using (16) and estimates (4) and (6), we obtain

$$\|v_{-N}\|_H\leq M_1(\delta,\lambda)\Big[\|\varphi\|_H+\|\psi\|_H+\max_{-N+1\leq k\leq 0}\|g_k\|_H\Big]. \quad (18)$$

Next, using (17) and the estimates (3), (4), (5), and (9), we obtain

$$\|v_{-N}\|_H\leq M_2(\delta,\lambda,\alpha)\Big[\|\varphi\|_H+\|\psi\|_H+\max_{1\leq k\leq N-1}\|f_k\|_H+\max_{-N+1\leq k\leq 0}\|g_k\|_H\Big]. \quad (19)$$

Then, using (14) and the estimates (4), (18), and (19), we get

$$\begin{aligned}\|v_k\|_H&\leq\|v_{-N}\|_H+\max_{-N+1\leq k\leq 0}\|g_k\|_H\\&\leq M_3(\delta,\lambda,\alpha)\Big[\|\varphi\|_H+\|\psi\|_H+\max_{1\leq k\leq N-1}\|f_k\|_H+\max_{-N+1\leq k\leq 0}\|g_k\|_H\Big]\end{aligned} \quad (20)$$

for $k=-N+1,\dots,0$. Using (15) and the estimates (3), (4), (5), (7), (18), and (19), we obtain

$$\begin{aligned}\|v_k\|_H&\leq\|v_{-N}\|_H+M_4(\delta,\alpha)\Big(\max_{1\leq k\leq N-1}\|f_k\|_H+\max_{-N+1\leq k\leq 0}\|g_k\|_H\Big)\\&\leq M_5(\delta,\lambda,\alpha)\Big[\|\varphi\|_H+\|\psi\|_H+\max_{1\leq k\leq N-1}\|f_k\|_H+\max_{-N+1\leq k\leq 0}\|g_k\|_H\Big]\end{aligned} \quad (21)$$

for $k=1,\dots,N$. Since $A^{-1}p=\varphi-v_{-N}$, using (18), (19), and the triangle inequality, we have

$$\begin{aligned}\left\|A^{-1}p\right\|_H&\leq\|\varphi\|_H+\|v_{-N}\|_H\\&\leq M_6(\delta,\lambda,\alpha)\Big[\|\varphi\|_H+\|\psi\|_H+\max_{1\leq k\leq N-1}\|f_k\|_H+\max_{-N+1\leq k\leq 0}\|g_k\|_H\Big].\end{aligned} \quad (22)$$

Finally, using (11), (20), (21), and (22), we prove the estimate

$$\begin{aligned}\|u_k\|_H \leq & \left\|A^{-1}p\right\|_H + \|v_k\|_H \\ \leq & M_7(\delta,\lambda,\alpha)\Big[\|\varphi\|_H + \|\psi\|_H + \max_{1\leq k\leq N-1}\|f_k\|_H + \max_{-N+1\leq k\leq 0}\|g_k\|_H\Big]\end{aligned} \tag{23}$$

for $k=-N,\dots,N$. Estimate (10) follows from (22) and (23).

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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