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On a New Class of Function Systems of Faber–Schauder Type

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Abstract—In this paper, we introduce a new class of function systems generalizing the classical Faber–Schauder system. Under the condition that the generating sequence is bounded, we show that systems of such a class constitute bases in the space of continuous functions and prove some properties of series expansions of functions in these systems.

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In 1910, Faber [1] constructed a function system, which, in 1927, was rediscovered by Schauder [2] and is now called the “Faber–Schauder system” (see also [3]). This system, consisting of continuous piecewise linear functions, is one of the simplest bases in the space of continuous functions on $[0,1]$. More recently, various properties of the series expansions of functions in this system were studied by a number of authors [4]–[9].

In the present paper, we introduce a new class Φ of function systems $\Phi\{p_n\}$ containing the Faber–Schauder system.

Suppose that we are given a sequence $\{p_n\}$ of natural numbers such that $p_0 = 1$, and $p_n \geq 2$, $n = 1, 2, \dots$.

Set $m_n = p_0 p_1 \cdots p_n$, $n = 0, 1, 2, \dots$. Then, for any point $x \in [0,1] \setminus Q$, where

$$Q = \left\{ \frac{l}{m_n} \right\}, \qquad 0 \leq l \leq m_n, \quad n \geq 0, \quad l \in \mathbb{Z},$$

there exists a unique expansion

$$x = \sum_{k=1}^{\infty} \frac{\alpha_k(x)}{m_k},$$

where $0 \leq \alpha_k(x) \leq p_k - 1$, and $\alpha_k(x)$ are integers.

Any integer $k \geq 2$ can be uniquely expressed as

$$\begin{gathered} k = m_n + r(p_n + 1 - 1) + s, \\ n = 0, 1, 2, \dots, \qquad r = 0, 1, \dots, m_n - 1, \qquad s = 1, 2, \dots, p_{n+1} - 1. \end{gathered} \tag{1}$$

Let us define the function system $\Phi\{p_n\} = \{\varphi_k(x)\}_{k=0}^{\infty}$, $x \in [0,1]$ in which $\varphi_0(x) = 1$, $\varphi_1(x) = x$, $x \in [0,1]$,

$$\varphi_k(x) = \varphi_{n,r}^{(s)}(x) = \begin{cases} (m_{n+1}x - p_{n+1}r - \alpha_{n+1}(x)) \exp \dfrac{2\pi i s \alpha_{n+1}(x)}{p_{n+1}} & \\ \qquad + \dfrac{1 - \exp(2\pi i s \alpha_{n+1}(x)/p_{n+1})}{1 - \exp(2\pi i s/p_{n+1})}, & x \in \left(\dfrac{r}{m_n}, \dfrac{r+1}{m_n} \right) \setminus Q, \\ 0, & x \bar{\in} \left[\dfrac{r}{m_n}, \dfrac{r+1}{m_n} \right], \end{cases} \tag{2}$$

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where $k \geq 2$ and n, r, s are defined in (1).

Using the fact that the set $[0,1] \setminus Q$ is everywhere dense on $[0,1]$, we extend the function $\varphi_{n,r}^{(s)}(x)$ by continuity to the interval $[r/m_n, (r+1)/m_n]$.

Thus, the system $\Phi\{p_n\}$ is completely defined and consists of continuous piecewise linear functions. For $p_n = 2$, $n = 1, 2, \dots$, the function system $\Phi\{p_n\}$ coincides with the Faber–Schauder system.

Consider the following series in the system $\Phi\{p_n\}$,

$$\sum_{k=0}^{\infty} a_k \varphi_k(x) = a_0\varphi_0(x) + a_1\varphi_1(x) + \sum_{n=1}^{\infty} \sum_{r=0}^{m_n-1} \sum_{s=1}^{p_{n+1}-1} a_{n,r}^{(s)} \varphi_{n,r}^{(s)}(x) \tag{3}$$

and suppose that it is convergent at each point of the interval $[0,1]$ to some function $f(x)$. Then the coefficients a_k are uniquely defined by the function $f(x)$.

Indeed, it follows from (2) that

$$a_0 = a_0\varphi_0(0) = \sum_{k=0}^{\infty} a_k\varphi_k(0) = f(0),$$

$$a_0 + a_1 = a_0\varphi_0(1) + a_1\varphi_1(1) = \sum_{k=0}^{\infty} a_k\varphi_k(1) = f(1), \tag{4}$$

because $\varphi_k(0) = 0$ for all $k \geq 1$ and $\varphi_k(1) = 0$ for all $k \geq 2$. Hence

$$a_1 = f(1) - f(0).$$

But if $k \geq 2$, then $a_k \equiv a_{n,r}^{(s)}$ (see (1)) and, for k such that

$$m_n + r(p_{n+1} - 1) < k \leq m_n + (r+1)(p_{n+1} - 1),$$

the following system of linear equations is valid:

$$\begin{aligned}
&\sum_{s=1}^{p_{n+1}-1} a_{n,r}^{(s)} \sum_{l=0}^{R} \exp\frac{2\pi i s l}{p_{n+1}} \\
&\quad= \sum_{k=m_{n+1}}^{\infty} a_k\varphi_k\biggl(\frac{r}{m_n} + \frac{R+1}{m_{n+1}}\biggr) = f\biggl(\frac{r}{m_n} + \frac{R+1}{m_{n+1}}\biggr) - S_{m_n}\biggl(\frac{r}{m_n} + \frac{R+1}{m_{n+1}}\biggr) \\
&\quad= f\biggl(\frac{r}{m_n} + \frac{R+1}{m_{n+1}}\biggr) - \frac{1}{p_{n+1}}\biggl[(p_{n+1} - R - 1) f\biggl(\frac{r}{m_n}\biggr) + (R+1) f\biggl(\frac{r+1}{m_n}\biggr)\biggr],
\end{aligned} \tag{5}$$

where $R = 0, 1, \dots, p_{n+1} - 2$.

By elementary transformations, the system of linear equations (5) can be reduced to the following system of equations:

$$\begin{aligned}
\sum_{s=1}^{p_{n+1}-1} a_{n,r}^{(s)} \exp\frac{2\pi i s R}{p_{n+1}} &= f\biggl(\frac{r}{m_n} + \frac{R+1}{m_{n+1}}\biggr) - f\biggl(\frac{r}{m_n} + \frac{R}{m_{n+1}}\biggr) \\
&\qquad + \frac{1}{p_{n+1}}\biggl[f\biggl(\frac{r}{m_n}\biggr) - f\biggl(\frac{r+1}{m_n}\biggr)\biggr],
\end{aligned} \tag{6}$$

where $R = 0, 1, \dots, p_{n+1} - 2$.

The determinant of system (6) is the Vandermonde determinant and is equal to the following expression:

$$\prod_{1 \leq k < l \leq p_{n+1}-1} \biggl(\exp\frac{2\pi i l}{p_{n+1}} - \exp\frac{2\pi i k}{p_{n+1}}\biggr),$$

which is not zero.

Thus, for a fixed n and r, the coefficients $a_{n,r}^{(s)}$, $s = 1, 2, \dots, p_{n+1} - 1$, are uniquely defined by the system of equations (6).

Theorem 1. *Suppose that* $f(x) \in C[0,1]$ *and* $f(x)$ *can be expanded in a series in the system* $\Phi\{p_n\}$, *i.e.,*

$$f(x) = \sum_{k=0}^{\infty} a_k(f)\varphi_k(x).$$

Then, the coefficients $a_k(f) \equiv a_{n,r}^{(s)}(f)$ *(see* (1)*) satisfy the estimate*

$$|a_{n,r}^{(s)}(f)| \le \frac{2\omega(1/m_{n+1}, f)}{(\sin \pi/p_{n+1})^{p_{n+1}-2}}. \tag{7}$$

Here $\omega(\delta, f)$ *is the modulus of continuity of a function* $f \in C[0,1]$.

Proof. Suppose that n and r are fixed, $n = 1, 2, \dots,$ $r = 0, 1, \dots, m_{n-1}$, and $p_{n+1} = k+1$, $k \ge 1$. Denote

$$\lambda = \exp\frac{2\pi i}{k+1}, \qquad x_s = a_{n,r}^{(s)}(f), \quad s = 1, 2, \dots, k,$$

$$b_R = f\biggl(\frac{r}{m_n} + \frac{R}{m_{n+1}}\biggr) - f\biggl(\frac{r}{m_n} + \frac{R-1}{m_{n+1}}\biggr) + \frac{1}{k+1}\biggl[f\biggl(\frac{r}{m_n}\biggr) - f\biggl(\frac{r+1}{m_n}\biggr)\biggr],$$

where $R = 1, 2, \dots, k$. Then, in the new notation, the system of equations (6) takes the following form:

$$\begin{cases} x_1 + x_2 + \dots + x_k = b_1, \\ \lambda x_1 + \lambda^2 x_2 + \dots + \lambda^k x_k = b_2, \\ \lambda^2 x_1 + \lambda^4 x_2 + \dots + \lambda^{2k} x_k = b_3, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \lambda^{k-1} x_1 + \lambda^{2(k-1)} x_2 + \dots + \lambda^{k(k-1)} x_k = b_k. \end{cases}$$

The determinant of this system is the Vandermonde determinant (see [10])

$$\Delta = \begin{vmatrix} 1 & 1 & \dots & 1 \\ \lambda & \lambda^2 & \dots & \lambda^k \\ \lambda^2 & \lambda^4 & \dots & \lambda^{2k} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda^{k-1} & \lambda^{2(k-1)} & \dots & \lambda^{k(k-1)} \end{vmatrix} = \prod_{1 \le i < l \le k} (\lambda^l - \lambda^i)$$

and, obviously, is not zero in our case. Therefore, this system of linear equations has a unique solution, which can be found by using Cramer's rule.

Suppose that the determinant Δ_j is obtained from the determinant Δ by replacing the jth column by a column composed of constant terms. Expanding the determinant Δ_j according to the elements of the jth column, we obtain

$$\Delta_j = \sum_{i=1}^{k} b_i A_{ij},$$

where the A_{ij} are the cofactors of the elements of the determinant Δ.

Consider these cofactors

$$A_{ij} = (-1)^{i+j} \begin{vmatrix} 1 & 1 & \dots & 1 & 1 & \dots & 1 \\ \lambda & \lambda^2 & \dots & \lambda^{j-1} & \lambda^{j+1} & \dots & \lambda^k \\ \lambda^2 & \lambda^4 & \dots & \lambda^{2(j-1)} & \lambda^{2(j+1)} & \dots & \lambda^{2k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda^{i-2} & \lambda^{2(i-2)} & \dots & \lambda^{(j-1)(i-2)} & \lambda^{(j+1)(i-2)} & \dots & \lambda^{k(i-2)} \\ \lambda^i & \lambda^{2i} & \dots & \lambda^{(j-1)i} & \lambda^{(j+1)i} & \dots & \lambda^{ki} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda^{k-1} & \lambda^{2(k-1)} & \dots & \lambda^{(j-1)(k-1)} & \lambda^{(j+1)(k-1)} & \dots & \lambda^{k(k-1)} \end{vmatrix}$$
$$= (-1)^{i+j}\sigma_{k-i} \prod_{1\le s<l\le k,\, s\ne j,\, l\ne j} (\lambda^l - \lambda^s),$$

where σ_{k-i} is the sum of all possible products of the numbers $\lambda, \lambda^2, \dots, \lambda^{j-1}, \lambda^{j+1}, \dots, \lambda^k$ taken $k-i$ at a time (see [11, p. 40]). Since $|\lambda| = 1$, the modulus of each term in the sum σ_{k-i} is 1. The number of terms in the sum σ_{k-i} is equal to C_{k-1}^{k-i}, $i = 1, 2, \dots, k$. [1] Therefore,

$$|\sigma_{k-i}| \le C_{k-1}^{k-i}. \tag{8}$$

Now consider the ratios

$$\frac{\prod_{1\le s<l\le k,\, s\ne j,\, l\ne j} (\lambda^l - \lambda^s)}{\prod_{1\le s<l\le k} (\lambda^l - \lambda^s)}, \qquad j = 1, 2, \dots, k.$$

After cancelations, the denominator contains products of factors of the form $\lambda^l - \lambda^j$, $l = 1, 2, \dots, j-1, j+1, \dots, k$. Let us estimate $|\lambda^l - \lambda^j|$ from below:

$$|\lambda^l - \lambda^j| = \left|\exp\frac{2\pi il}{k+1} - \exp\frac{2\pi ij}{k+1}\right| = \left|\exp\frac{2\pi i(l-j)}{k+1} - 1\right|$$
$$= \left|\cos\frac{2\pi(l-j)}{k+1} - 1 + i\sin\frac{2\pi(l-j)}{k+1}\right| = 2\left|\sin\frac{\pi(l-j)}{k+1}\right| \ge 2\sin\frac{\pi}{k+1}.$$

Hence, for any $j = 1, 2, \dots, k$, we have

$$\frac{\prod_{1\le s<l\le k,\, s\ne j,\, l\ne j} (\lambda^l - \lambda^s)}{\prod_{1\le s<l\le k} (\lambda^l - \lambda^s)} \le \frac{1}{(2\sin\pi/(k+1))^{k-1}}. \tag{9}$$

The quantities b_i can be estimated by definition as follows:

$$|b_i| \le 2\omega\biggl(\frac{1}{m_{n+1}}, f\biggr), \qquad i = 1, 2, \dots, k. \tag{10}$$

It follows from (8) that

$$\sum_{i=1}^{k} |\sigma_{k-i}| \le \sum_{i=1}^{k} C_{k-1}^{k-i} = 2^{k-1}. \tag{11}$$

Further, from (9)–(11), we find that

$$|x_j| = \left|\frac{\Delta_j}{\Delta}\right| \le 2^k \frac{\omega(1/m_{n+1}, f)}{(2\sin\pi/(k+1))^{k-1}} = \frac{2\omega(1/m_{n+1}, f)}{(\sin\pi/(k+1))^{k-1}}.$$

Returning to the original notation, we obtain (7). Theorem 1 is proved. ☐

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[1] Here and elsewhere, C_n^m denotes the binomial coefficient $\binom{n}{m}$.

Denote by $\omega_2(\delta, f)$ the modulus of smoothness of a function $f \in C[0,1]$:

$$\omega_2(\delta, f) = \sup_{0<h<\delta} \max_{x\in[0,1]} |f(x+h) + f(x-h) - 2f(x)|,$$

where $h \le x \le 1-h$ and $0 < \delta < 1$.

Theorem 2. *The system $\Phi\{p_n\}$ with $p_n \le N$, $n = 1, 2, \dots$, is a basis in the space $C[0,1]$. Moreover, if*

$$f(x) = \sum_{k=0}^{\infty} a_k \varphi_k, \qquad f(x) \in C[0,1],$$

then the following estimates hold::

$$|a_k(f)| = |a_{n,r}^{(s)}(f)| \le C\omega\biggl(\frac{1}{m_{n+1}}, f\biggr), \qquad k = 0, 1, 2, \dots, \tag{12}$$

$$\biggl\| f(x) - \sum_{k=0}^{l} a_k(f)\varphi_k(x) \biggr\|_{C[0,1]} \le \omega_2\biggl(\frac{1}{m_n}, f\biggr), \qquad l = m_n + r(p_{n+1} - 1). \tag{13}$$

Proof. It follows from (4), (5) that the equality

$$\sum_{m=1}^{l} a_m \varphi_m(x) \equiv 0$$

on $[0,1]$ for $l = 1, 2, \dots$ holds only for $a_n = 0$, i.e., the functions of the system $\Phi\{p_n\}$ are linearly independent. For

$$l = m_n + r(p_{n+1} - 1), \qquad n = 0, 1, \dots, \qquad r = 0, 1, \dots, m_n - 1,$$

the definition of such functions $\varphi_m(x)$, $m = 0, 1, \dots$, and their linear independence imply the following properties:

a) the space $G_l(\Phi)$ of polynomials in the system $\Phi\{p_n\}$ of the form

$$p_n(x) = \sum_{m=0}^{l} a_m \varphi_m(x)$$

is of dimension $l + 1$ and coincides with the space L_l defined by

$$L_1 = \{f \in C[0,1] : f''(x) = 0 \text{ for } x \in (0,1)\},$$

$$L_l = \biggl\{ f \in C[0,1] : f''(x) = 0 \text{ for } x \in \biggl(\bigcup_{k=0}^{rp_{n+1}-1} \delta_{n+1,k} \biggr) \cup \biggl(\bigcup_{k=r+1}^{m_n-1} \delta_{n,k} \biggr) \biggr\},$$

where $\delta_{n,k} = (k/m_n, (k+1)/m_n)$;

b) the partial sums

$$S_l(f,x) = \sum_{m=0}^{l} a_m(f)\varphi_m(x)$$

satisfy the relation

$$S_l(f,x) = f(x)$$

for $x \in \pi_l$, where

$$\pi_1 = \{0, 1\}, \qquad \pi_l = \biggl\{ \frac{k}{m_{n+1}} \biggr\}_{k=0}^{rp_{n+1}} \cup \biggl\{ \frac{k}{m_n} \biggr\}_{k=r+1}^{m_n}.$$

Now it follows from (4), (5), a), and b) that, for an arbitrary function $f(x) \in C[0,1]$, there exists a unique series

$$\sum_{m=0}^{\infty} a_m(f)\varphi_m(x)$$

whose partial sums $S_l(f,x)$ are uniformly convergent to $f(x)$ on the sequence of values

$$l = m_n + r(p_{n+1}-1), \qquad n = 0,1,2,\dots, \quad r = 0,1,\dots,m_n-1.$$

Therefore, it suffices to prove that there exists a constant M for which the following inequalities hold:

$$\Bigg\|\sum_{s=1}^{k} a_{n,r}^{(s)}(f)\varphi_{n,r}^{(s)}(x)\Bigg\|_{C(0,1)} \le M\Bigg\|\sum_{s=1}^{p_{n+1}-1} a_{n,r}^{(s)}(f)\varphi_{n,r}^{(s)}(x)\Bigg\|_{C(0,1)} \tag{14}$$

for all $k = 1,2,\dots,p_{n+1}-1$. But, for a bounded generating sequence, $\{p_n\}$ is collectively bounded, because, in view of (2), we have the estimate

$$\|\varphi_{n,r}^{(s)}(x)\|_{C(0,1)} \le p_{n+1}-1, \tag{15}$$

while Theorem 1 implies estimate (12). Now from (12) and (15) we obtain (14), thus establishing the basis property of the function system in $C[0,1]$.

Let us pass to the proof of estimate (13). Suppose we are given a function $f(x) \in C(0,1)$ and an interval $I = [\alpha,\beta] \subset [0,1]$. On the closed interval I, consider the function

$$u(x) = f(x) - \left\{ f(\alpha) + \frac{f(\beta)-f(\alpha)}{\beta-\alpha}(x-\alpha) \right\}.$$

Obviously, $u(\alpha) = u(\beta) = 0$. If $u(x) \ne 0$ on $[\alpha,\beta]$, then let us find a point $x_0 \in [\alpha,\beta]$ such that

$$|u(x_0)| = \max_{x\in(\alpha,\beta)} |u(x)|.$$

Without loss of generality, we assume that

$$|\alpha - x_0| \le |\beta - x_0|.$$

Then the point $x_1 = \alpha + 2|\alpha - x_0|$ belongs to I. Taking the second difference, we have

$$|2u(x_0) - u(\alpha) - u(x_1)| = |2u(x_0) - u(x_1)| = |u(x_0) + (u(x_0) - u(x_1))| \ge |u(x_0)|. \tag{16}$$

By the linearity of the function

$$f(\alpha) + \frac{f(\beta)-f(\alpha)}{\beta-\alpha}(x-\alpha),$$

from (16) we obtain

$$|u(x_0)| \le |2u(x_0) - u(\alpha) - u(x_1)| = |2f(x_0) - f(\alpha) - f(x_1)| \le \omega_2\left(\frac{\beta-\alpha}{2}, f\right). \tag{17}$$

Since (see a) and b)) the function

$$S_l(f,x) = \sum_{k=0}^{l} a_k(f)\varphi_k(x)$$

for $l = m_n + r(p_{n+1}-1)$ is a polygonal line with vertices at the points

$$\pi_l = \left\{\frac{k}{m_{n+1}}\right\}_{k=0}^{rp_n+1} \cup \left\{\frac{k}{m_n}\right\}_{k=r+1}^{m_n}$$

and satisfies the condition

$$S_l(f,x) = f(x)$$

for $x \in \pi_l$, it follows from (17) that

$$\left|f(x) - \sum_{k=0}^{l} a_k(f)\varphi_k(x)\right| \le \max\left\{\max_{0\le k\le rp_{n+1}}\left[\max_{x\in\delta_{n+1,k}}\left|f(x) - \sum_{k=0}^{l} a_k(f)\varphi_k(x)\right|\right],\right.$$
$$\left.\max_{r+1\le k\le m_n-1}\left[\max_{x\in\delta_{n,k}}\left|f(x) - \sum_{k=0}^{l} a_k(f)\varphi_k(x)\right|\right]\right\}$$
$$\le \max\left\{\omega_2\left(\frac{1}{2m_{n+1}}, f\right), \omega_2\left(\frac{1}{2m_n}, f\right)\right\} \le \omega_2\left(\frac{1}{2m_n}, f\right).$$

Thus, estimate (13) is proved. ☐

Theorem 3. *In order that the series* (3) *with* $\sup p_n = N < \infty$ *be an expansion of the function* $f(x) \in \operatorname{Lip}\alpha$, $0 < \alpha < 1$*, it is necessary and sufficient that the following relation holds:*

$$|a_k| \le ck^{-\alpha}, \qquad k = 1, 2, \dots . \tag{18}$$

Proof. If $f(x) \in \operatorname{Lip}\alpha$, then estimate (12) implies

$$|a_k| \le ck^{-\alpha}, \qquad k = 1, 2, \dots .$$

Suppose that we are given a series of the form (3) whose coefficients satisfy (18). Then this series is absolutely and uniformly convergent, because, for $n = 0, 1, 2, \dots$,

$$\sum_{k=m_n+1}^{m_n+1} |a_k\varphi_k(x)| \le N^2 \max_{m_n<k\le m_{n+1}} |a_k| \le \frac{N^2c}{m_n^{\alpha}}$$

and, therefore, $f(x)$ is the sum of its expansion in the system $\Phi\{p_n\}$. Moreover, for $x, y \in [0, 1]$,

$$|f(x) - f(y)| \le \sum_{k=0}^{\infty} |\alpha_k||\varphi_k(x) - \varphi_k(y)|$$
$$= |a_1||x - y| + \sum_{k=0}^{\infty}\sum_{r=0}^{m_n-1}\sum_{s=1}^{p_{n+1}-1} |a_{n,r}^{(s)}||\varphi_{n,r}^{(s)}(x) - \varphi_{n,r}^{(s)}(y)|.$$

Choose a number n_0 so that

$$\frac{N}{m_{n_0+1}} < |x - y| \le \frac{N}{m_{n_0}},$$

and use the fact that, for all $n = 1, 2, \dots$, $r = 0, 1, \dots, m_n - 1$, $s = 1, 2, \dots, p_n + 1 - 1$, and $x, y \in [0, 1]$ the following inequality holds:

$$|\varphi_{n,r}^{(s)}(x) - \varphi_{n,r}^{(s)}(y)| \le \min\{p_{n+1}, m_{n+1}|x - y|\},$$

while the number of nonzero terms in the sum

$$\sum_{r=0}^{m_{n-1}}\sum_{s=1}^{p_{n+1}-1} |a_{n,r}^{(s)}||\varphi_{n,r}^{(s)}(x) - \varphi_{n,r}^{(s)}(y)|$$

is not greater than $2(p_{n+1} - 1)$ for all $x, y \in [0, 1]$.

Further, we have

$$|f(x) - f(y)| \le |a_1||x - y| + \sum_{n=1}^{\infty} 2(p_{n+1} - 1)\max_{r,s} |a_{n,r}^{(s)}| \min\{p_{n+1}, m_{n+1}|x - y|\}$$

$$\begin{aligned}
&\le |a_1||x-y| + 2N^2 c\sum_{n=1}^{n_0}\frac{1}{m_n^{\alpha}}(p_{n+1}-1)m_{n+1}|x-y| \\
&\qquad + 2N^3 c\sum_{n=n_0+1}^{\infty}\frac{(p_{n+1}-1)}{m_n^{\alpha}} \\
&\le |a_1||x-y| + C_1|x-y|\sum_{n=1}^{n_0} m_n^{1-\alpha} + C_2\sum_{n=n_0+1}^{\infty}\frac{1}{m_n^{\alpha}} \le K|x-y|^{\alpha}.
\end{aligned}$$

Theorem 3 is proved. □

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