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On fourth order accuracy stable difference scheme for a multi-point overdetermined elliptic problem

In this paper fourth order of accuracy difference scheme for approximate solution of a multi-point elliptic overdetermined problem in a Hilbert space is proposed. The existence and uniqueness of the solution of the difference scheme are obtained by using the functional operator approach. Stability, almost coercive stability, and coercive stability estimates for the solution of difference scheme are established. These theoretical results can be applied to construct a stable highly accurate difference scheme for approximate solution of multi-point overdetermined boundary value problem for multidimensional elliptic partial differential equations.

Keywords: overdetermined elliptic problem, multi-point condition, high order difference scheme, difference scheme, inverse, source identification problem, well-posedness, stability, coercive stability, almost coercive stability.

Introduction

Methods of solutions of nonlocal and source identification boundary value problems for partial differential equations have been widely investigated by several researchers (see [1–21] and references therein). Construction of highly accurate difference schemes (DSs) for problems of this type is important, especially for their specific theoretical and practical aspects and also usefulness in wide applications [5, 8] and bibliography herein).

Let H be a Hilbert space, A be a self-adjoint positive definite operator (SAPDO) and I be identity operator.

In paper [10] to find an element $p \in H$ and function $v \in C^2([0,T], H) \cap C([0,T], D(A))$ the following multi-point elliptic overdetermined problem

$$\begin{cases} -v_{tt}(t) + Av(t) = g(t) + p, 0 < t < T, \\ v(0) = \phi, v(T) = \sum\limits_{i=1}^{q} \beta_i v(\lambda_i) + \eta, v(\lambda_0) = \zeta \end{cases} \tag{1}$$

was investigated. Here $q \in N$, $\lambda_0, \lambda_i \in (0,T)$, $\beta_i \in R, \beta_i \geq 0, i = 1, ..., q$ are known numbers, $\zeta, \phi, \eta \in D(A)$, $g \in C^2([0,T], H) \cap C([0,T], D(A))$ are given elements and function, respectively. Moreover,

$$\lambda_1 < \lambda_2 < ... < \lambda_q, \beta = \sum_{i=1}^{q} \beta_i \leq 1.$$

In paper [10] the first and second order of accuracy stable DSs were proposed. The objective of this work is to study the fourth order of ADS for multi-point elliptic overdetermined problem (1) in an arbitrary Hilbert space H with a SAPDO A.

Let $[\cdot]$ be the greatest integer function and

$$l_i = \left[\frac{\lambda_i}{\tau}\right], \mu_i = \frac{\lambda_i}{\tau} - l_i,$$

$$\mu_{i,1} = \frac{1}{12}\mu_i - \frac{1}{12}\mu_i^3, \mu_{i,2} = -\frac{2}{3}\mu_i + \frac{1}{2}\mu_i^2 + \frac{1}{6}\mu_i^3,$$

$$\mu_{i,3} = 1 - \mu_i^2, \mu_{i,4} = \frac{2}{3}\mu_i + \frac{1}{2}\mu_i^2 - \frac{1}{6}\mu_i^3,$$

$$\mu_{i,5} = -\mu_{i,1}, i = 0, 1, 2, 3, 4, 5.$$

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Introduce the following notations

$$C = A + \frac{\tau^2}{12}A, D = \frac{1}{2}\left(\tau C + \left(4C + \tau^2 C^2\right)^{\frac{1}{2}}\right),$$

$$P = (I + \tau D)(2I + \tau D)^{-1} D^{-1}, R = (I + \tau D)^{-1}.$$

Stability estimates

Lemma 1. The following estimates hold [5]:

$$\begin{aligned} &\left\|\exp\left(k\tau A^{\frac{1}{2}}\right) R^k\right\|_{H\to H} \leq M(\delta)\tau^{-k}, \left\|R^k\right\|_{H\to H} \leq M(1+\delta\tau)^{-k}, k \geq 1,\\ &\left\|\left(I - R^{2N}\right)^{-1}\right\|_{H\to H} \leq M(\delta),\ k\tau\left\|DR^k\right\|_{H\to H} \leq M(\delta),\\ &\left\|D^{\beta}\left(R^{k+r} + R^k\right)\right\|_{H\to H} \leq M(\delta)\frac{(r\tau)^{\alpha}}{(k\tau)^{\alpha+\beta}},\\ &1 \leq k \leq k+r \leq N, 0 \leq \alpha, \beta \leq 1, \delta > 0. \end{aligned} \tag{2}$$

Lemma 2. For $1 \leq l_i \leq N-1$, $1 \leq l_0 \leq N-1$, the operator ([10; 861]

$$\Delta = \left(I - R^{2N}\right)\left(I - R^{l_0}\right)\left(I - \sum_{i=1}^{q} k_i R^{N-l_i}\right)\left(I - \sum_{i=1}^{q} k_i R^{N-l_0+l_i}\right) \tag{3}$$

has a bounded inverse Δ^{-1} such that

$$\left\|\Delta^{-1}\right\|_{H\to H} \leq M(\delta).$$

Let us take $1 \leq l_i \leq N-1, 0 \leq i \leq q$. Denote by

$$\begin{aligned} J_1 = &-\left(I - R^{2N}\right)\left\{\mu_{0,1}\left(R^{l_0-2} - R^{2N-l_0+2}\right) - \mu_{0,2}\left(R^{l_0-1} - R^{2N-l_0+1}\right)\right.\\ &+\mu_0^2\left(R^{l_0} - R^{2N-l_0}\right) - \mu_{0,4}\left(R^{l_0+1} - R^{2N-l_0-1}\right)\\ &-\mu_{0,5}\left(R^{l_0+2} - R^{2N-l_0-2}\right) + \sum_{i=1}^{q} k_i\left[-\mu_{i,1}\left(R^{N-l_i+2} - R^{N+l_i-2}\right)\right.\\ &-\mu_{i,2}\left(R^{N-l_i+1} - R^{N+l_i-1}\right) + \mu_i^2\left(R^{N-l_i} - R^{N+l_i}\right)\\ &\left.\left.-\mu_{i,2}\left(R^{N-l_i-1} - R^{N+l_i+1}\right) - \mu_{i,5}\left(R^{N-l_i-2} - R^{N+l_i+2}\right)\right]\right\} \end{aligned} \tag{4}$$

$$\begin{aligned} J_2 = &-\left(I - R^{2N}\right)\left\{\sum_{i=1}^{q} k_i\left\{\mu_{0,1}\mu_{i,5}\left(R^{N-l_0+l_i+4} - R^{N+l_0-l_i-4}\right)\right.\right.\\ &+\left(\mu_{0,1}\mu_{i,4} + \mu_{0,2}\mu_{i,5}\right)\left(R^{N-l_0+l_i+3} - R^{N+l_0-l_i-3}\right)\\ &+\left(\mu_{0,1}\mu_{i,3} + \mu_{0,2}\mu_{i,4} + \mu_{0,3}\mu_{i,5}\right)\left(R^{N-l_0+l_i+2} - R^{N+l_0-l_i-2}\right)\\ &\left.+\left(\mu_{0,1}\mu_{i,2} + \mu_{0,2}\mu_{i,3} + \mu_{0,3}\mu_{i,4} + \mu_{0,4}\mu_{i,5}\right)\left(R^{N-l_0+l_i+1} - R^{N+l_0-l_i-1}\right)\right\} \end{aligned} \tag{5}$$

$$\begin{aligned} J_3 = &-\left(I - R^{2N}\right)\sum_{i=1}^{q} k_i\left\{\left(\mu_{0,1}\mu_{i,1} + \mu_{0,2}\mu_{i,2} + \mu_0^2\mu_i^2 - \mu_0^2 - \mu_i^2\right.\right.\\ &\left.\left.+\mu_{0,4}\mu_{i,4} + \mu_{0,5}\mu_{i,5}\right)\right\}\left(R^{N-l_0+l_i} - R^{N+l_0-l_i}\right) \end{aligned} \tag{6}$$

$$J_4 = -\left(I - R^{2N}\right)\left\{\sum_{i=1}^{q} k_i\left\{(\mu_{0,2}\mu_{i,1} + \mu_{0,3}\mu_{i,2} + \mu_{0,4}\mu_{i,3} + \mu_{0,5}\mu_{i,4})\right.\right.$$
$$\times\left(R^{N-l_0+l_i-1} - R^{N+l_0-l_i+1}\right) + (\mu_{0,3}\mu_{i,1} + \mu_{0,4}\mu_{i,2} + \mu_{0,5}\mu_{i,3}) \tag{7}$$
$$\times\left(R^{N-l_0+l_i-2} - R^{N+l_0-l_i+2}\right) + (\mu_{0,4}\mu_{i,1} + \mu_{0,5}\mu_{i,2})$$
$$\left.\left.\times\left(R^{N-l_0+l_i-3} - R^{N+l_0-l_i+3}\right) + \mu_{0,5}\mu_{i,1}\left(R^{N-l_0+l_i-4} - R^{N+l_0-l_i+4}\right)\right\}\right\}.$$

Lemma 3. Let the operators $\Delta, J_1, J_2, J_3, J_4$ be defined by (3), (4), (5), (6), (7), correspondingly. Then, the operator

$$G = \Delta + J_1 + J_2 + J_3 + J_4 \tag{8}$$

has a bounded inverse G^{-1} such that

$$\left\|G^{-1}\right\|_{H\to H} \le M(\delta). \tag{9}$$

Proof. We have

$$G^{-1} - \Delta^{-1} = G^{-1}\Delta^{-1}K, \tag{10}$$

where

$$K = J_1 + J_2 + J_3 + J_4.$$

Applying (2), it can be showed that the estimates

$$\|J_i\|_{H\to H} \le M\tau, i = 1, 2, 3, 4$$

hold for constant M which does not depend on τ.

Consequently,

$$\|K\|_{H\to H} \le M\tau. \tag{11}$$

By using (10), (11) and triangle inequality, we can get

$$\left\|G^{-1}\right\|_{H\to H} \le \left\|\Delta^{-1}\right\|_{H\to H} + \left\|\Delta^{-1}\right\|_{H\to H}\left\|G^{-1}\right\|_{H\to H}\|K\|_{H\to H}$$
$$\le M(\delta) + \left\|G^{-1}\right\|_{H\to H} M(\delta)M\tau$$

for any small positive number τ. Therefore, estimate (9) is valid.

Let $[0,T]_\tau = \{t_k = k\tau, 0 \le k \le N, N\tau = T\}$ be space of grid points and $v_k = v(t_k), 0 \le k \le N$.

Denote by $C(H)$ and $C_{0T}^{\alpha,\alpha}(H)$ the corresponding Banach spaces of H-valued grid functions $\{w_k\}_0^N$ with norms

$$\left\|\{w_k\}_1^{N-1}\right\|_{C(H)} = \max\nolimits_{0\le k\le N-1}\|w_k\|_H,$$

$$\left\|\{w_k\}_1^{N-1}\right\|_{C_{0T}^{\alpha,\alpha}(H)} = \left\|\{w_k\}_1^{N-1}\right\|_{C(H)} + \sup\nolimits_{0\le k<k+\tau\le N-1}(k\tau + n\tau)^\alpha n\tau^{-\alpha}(T-k\tau)^\alpha\|w_{k+n} - w_k\|_H,$$

respectively.

Applying the fourth order of approximation for function v at point λ_i, $i = 0, 1, ..., q$

$$v(\lambda_i) = \mu_{i,1}v_{l_i-2} + \mu_{i,2}v_{l_i-1} + \mu_{i,3}\, v_{l_i} + \mu_{i,4}v_{l_i+1} + \mu_{i,5}v_{l_i+2}$$

and fourth order of accuracy approximation of differential equation, one can get the next DS

$$\begin{cases} -\tau^{-2}(v_{k+1} - 2v_k + v_{k-1}) + Cv_k = \theta_k + p, \\ \psi_k = g(t_k) + \frac{\tau^2}{12}\left(\frac{g(t_{k+1}) - 2g(t_k) + g(t_{k-1})}{\tau^2} + Ag(t_k)\right), \\ t_k = k\tau, 1 \le k \le N-1, N\tau = T, v_0 = \phi, \\ \mu_{0,1}v_{l_0-2} + \mu_{0,2}v_{l_0-1} + \mu_{0,3}v_{l_0} + \mu_{0,4}v_{l_0+1} + \mu_{0,5}v_{l_0+2} = \zeta, \\ v_N = \sum_{i=1}^{q} k_i\left\{\mu_{i,1}v_{l_i-2} + \mu_{i,2}v_{l_i-1} + \mu_{i,3}v_{l_i} + \mu_{i,4}v_{l_i+1} + \mu_{i,5}v_{l_i+2}\right\} + \eta \end{cases} \tag{12}$$

for approximately solution of problem (1).

We will find solution of DS (1) by formula

$$v_k = u_k + A^{-1}p, \tag{13}$$

where grid function $\{u_k\}_0^N$ is a solution of the following difference problem:

$$\begin{cases} -\tau^{-2}\left(u_{k+1} - 2u_k + u_{k-1}\right) + Au_k + \frac{\tau^2}{12}A^2u_k = \psi_k, \\ t_k = k\tau, 1 \le k \le N-1, N\tau = T, \\ u_0 - \mu_{0,1}u_{l_0-2} - \mu_{0,2}u_{l_0-1} - \mu_{0,3}u_{l_0} - \mu_{0,4}u_{l_0+1} - \mu_{0,5}u_{l_0+2} = \phi - \zeta, \\ u_N - \sum\limits_{i=1}^{q} k_i\left\{\mu_{i,1}u_{l_i-2} + \mu_{i,2}u_{l_i-1} + \mu_{i,3}u_{l_i} + \mu_{i,4}u_{l_i+1} + +\mu_{i,5}u_{l_i+2}\right\} = \eta. \end{cases} \tag{14}$$

After soving DS (14), unknown element p is defined by

$$p = A\phi - Au_0. \tag{15}$$

Theorem 1. Let $\phi, \zeta, \eta \in D(A)$ and $\{\psi_k\}_1^{N-1} \in C(H)$ be given. Then, the difference problem (12) has a solution $(\{v_k\}_1^{N-1}, p)$ which satisfies the stability estimates in below:

$$\left\|\{v_k\}_1^{N-1}\right\|_{C(H)} \le M(\delta)\left[\|\phi\|_H + \|\zeta\|_H + \|\eta\|_H + \left\|\{\psi_k\}_1^{N-1}\right\|_{C(H)}\right], \tag{16}$$

$$\left\|A^{-1}p\right\|_H \le M(\delta)\left[\|\phi\|_H + \|\zeta\|_H + \|\eta\|_H + \left\|\{\psi_k\}_1^{N-1}\right\|_{C(H)}\right],$$

where $M(\delta)$ is independent from ϕ, ζ, η and $\{\psi_k\}_1^{N-1}$.

Proof. For given u_0 and u_N the solution of difference problem

$$-\tau^2\left(u_{k+1} - 2u_k + u_{k-1}\right) + Au_k = \psi_k, 1 \le k \le N-1 \tag{17}$$

is defined by [5]

$$\begin{aligned} u_k = &\left(I - R^{2N}\right)^{-1}\left[\left(R^k - R^{2N-k}\right)u_0 + \left(R^{N-k} - R^{N+k}\right)u_N - \left(R^{N-k} - R^{N+k}\right)\right. \\ &\left.\times (I + \tau D)(2I + \tau D)^{-1}D^{-1}\sum_{j=1}^{N-1}\left(R^{N-j} - R^{N+j}\right)\psi_j\tau\right] + (I + \tau D) \\ &\times (2I + \tau D)^{-1}D^{-1}\textstyle\sum_{j=1}^{N-1}\left(R^{|k-j|} - R^{k+j}\right)\psi_j\tau. \end{aligned} \tag{18}$$

Applying formula (18) to nonlocal conditions of the difference problem (14), we get a system equation for u_0 and u_N:

$$s_{11}u_0 + s_{12}u_N = S_1, s_{21}u_0 + s_{22}u_N = S_2. \tag{19}$$

Here operators $s_{11}, s_{12}, s_{21}, s_{22}, S_1$ and S_2 are defined by

$$\begin{aligned} s_{11} = &\left(I - R^{2N}\right) - \mu_{0,1}\left(R^{l_0-2} - R^{2N-l_0+2}\right) - \mu_{0,2}\left(R^{l_0-1} - R^{2N-l_0+1}\right) \\ &-\mu_{0,3}\left(R^{l_0} - R^{2N-l_0}\right) - \mu_{0,4}\left(R^{l_0+1} - R^{2N-l_0-1}\right) - \mu_{0,5}\left(R^{l_0+2} - R^{2N-l_0-2}\right), \end{aligned}$$

$$\begin{aligned} s_{12} = &-\left\{\mu_{0,1}\left(R^{N-l_0+2} - R^{N+l_0-2}\right) + \mu_{0,2}\left(R^{N-l_0+1} - R^{N+l_0-1}\right)\right. \\ &\left.+\mu_{0,3}\left(R^{N-l_0} - R^{N+l_0}\right) + \mu_{0,4}\left(R^{N-l_0-1} - R^{N+l_0+1}\right) + \mu_{0,5}\left(R^{N-l_0-2} - R^{N+l_0+2}\right)\right\}, \end{aligned}$$

$$
\begin{aligned}
&s_{21}=\sum_{i=1}^{q}k_i\left\{-\mu_{i,1}\left(R^{l_i-2}-R^{2N-l_i+2}\right)-\mu_{i,2}\left(R^{l_i-1}-R^{2N-l_i+1}\right)\right.\\
&\left.-\mu_{i,3}\left(R^{l_i}-R^{2N-l_i}\right)-\mu_{i,4}\left(R^{l_i+1}-R^{2N-l_i-1}\right)-\mu_{i,5}\left(R^{l_i+2}-R^{2N-l_i-2}\right)\right\},\\
&s_{22}=\sum_{i=1}^{q}k_i\left\{\left(I-R^{2N}\right)-\mu_{i,1}\left(R^{N-l_i+2}-R^{N+l_i-2}\right)-\mu_{i,2}\left(R^{l_i-1}-R^{2N-l_i+1}\right)\right.\\
&\left.-\mu_{i,3}\left(R^{l_i}-R^{2N-l_i}\right)-\mu_{i,4}\left(R^{N-l_i-1}-R^{N+l_i+1}\right)-\mu_{i,5}\left(R^{N-l_i-2}-R^{N+l_i+2}\right)\right\},
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
&S_1=\left(I-R^{2N}\right)(\phi-\zeta)+\left[\tfrac{1}{12}\mu_0-\tfrac{1}{12}\mu_0^3\right]\left\{-\left(R^{N-l_0+2}-R^{N+l_0-2}\right)\right.\\
&\left.\times P\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)\theta_j\tau+\left(I-R^{2N}\right)P\sum_{j=1}^{N-1}\left(R^{|l_0-2-j|}-R^{l_0-2+j}\right)\theta_j\tau\right\}\\
&+\left[-\tfrac{8}{12}\mu_0+\tfrac{1}{2}\mu_0^2+\tfrac{1}{6}\mu_0^3\right]\left\{-\left(R^{N-l_0+1}-R^{N+l_0-1}\right)\right.\\
&\left.\times P\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)\theta_j\tau+\left(I-R^{2N}\right)P\sum_{j=1}^{N-1}\left(R^{|l_0-1-j|}-R^{l_0-1+j}\right)\theta_j\tau\right\}\\
&+\left[1-\mu_0^2\right]\left\{-\left(R^{N-l_0}-R^{N+l_0}\right)P\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)\theta_j\tau+\left(I-R^{2N}\right)\right.\\
&\left.\times P\sum_{j=1}^{N-1}\left(R^{|l_0-j|}-R^{l_0+j}\right)\theta_j\tau\right\}\\
&+\left[\tfrac{8}{12}\mu_0+\tfrac{1}{2}\mu_0^2-\tfrac{1}{6}\mu_0^3\right]\left\{-\left(R^{N-l_0-1}-R^{N+l_0+1}\right)\right.\\
&\left.\times P\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)\theta_j\tau+\left(I-R^{2N}\right)P\sum_{j=1}^{N-1}\left(R^{|l_0+1-j|}-R^{l_0+1+j}\right)\theta_j\tau\right\}\\
&+\left[-\tfrac{1}{12}\mu_0+\tfrac{1}{12}\mu_0^3\right]\left\{-\left(R^{N-l_0-2}-R^{N+l_0+2}\right)\right.\\
&\left.\times P\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)\theta_j\tau+\left(I-R^{2N}\right)P\sum_{j=1}^{N-1}\left(R^{|l_0+2-j|}-R^{l_0+2+j}\right)\theta_j\tau\right\},
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
&S_2=\left(I-R^{2N}\right)\eta+\sum_{i=1}^{q}k_i\left\{\mu_{i,1}\left[-\left(R^{N-l_i+2}-R^{N+l_i-2}\right)\times\right.\right.\\
&\left.\times P\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)\psi_j\tau+\left(I-R^{2N}\right)P\sum_{j=1}^{N-1}\left(R^{|l_i-2-j|}-R^{l_i-2+j}\right)\psi_j\tau\right]+\\
&+\mu_{i,2}\left[-\left(R^{N-l_i+1}-R^{N+l_i-1}\right)P\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)\psi_j\tau+\right.\\
&\left.+\left(I-R^{2N}\right)P\sum_{j=1}^{N-1}\left(R^{|l_i-1-j|}-R^{l_i-1+j}\right)\psi_j\tau\right]+
\end{aligned}
$$

$$+\mu_{i,3}\left[-\left(R^{N-l_i}-R^{N+l_i}\right)P\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)\psi_j\tau+\left(I-R^{2N}\right)\times\right.$$

$$\left.\times P\sum_{j=1}^{N-1}\left(R^{|l_i-j|}-R^{l_i+j}\right)\psi_j\tau\right]+\mu_{i,4}\left[-\left(R^{N-l_i-1}-R^{N+l_i+1}\right)\right.$$

$$\left.\times P\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)\psi_j\tau+\left(I-R^{2N}\right)P\sum_{j=1}^{N-1}\left(R^{|l_i+1-j|}-R^{l_i+1+j}\right)\psi_j\tau\right]+$$

$$+\mu_{i,5}\left[-\left(R^{N-l_i-2}-R^{N+l_i+2}\right)P\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)\psi_j\tau+\right. \tag{22}$$

$$\left.\left.+\left(I-R^{2N}\right)P\sum_{j=1}^{N-1}\left(R^{|l_i+2-j|}-R^{l_i+2+j}\right)\psi_j\tau\right]\right\}.$$

Determinant operator $G=s_{11}s_{22}-s_{12}s_{21}$ of the system equations (19) can be rewritten as (8). Consequently, according to Lemma 3, the operator G has bounded inverse G^{-1}. So, the system of equations (19) has a unique solution:

$$u_0=G^{-1}\left(S_1s_{22}-S_2s_{21}\right),u_N=G^{-1}s_{11}S_2-s_{12}S_1. \tag{23}$$

Thus, difference problem (14) has a unique solution $\{u_k\}_0^N$ which is defined by formula (18) with corresponding $s_{11},s_{12},s_{21},s_{22},S_1,S_2,u_0,\ u_N$ by (20)–(23).

For the solution of problem (17) the following inequality [5]

$$\left\|\{u_k\}_0^{N-1}\right\|_{C(H)}\le M\left[\left\|\{\psi_k\}_1^{N-1}\right\|_{C(H)}+\|Ru_0\|_H+\|Ru_N\|_H\right] \tag{24}$$

is valid. By virtue triangle, Cauchy-Schwarz inequalities and (2) one can obtain

$$\max\left\{\|S_1\|_{C(H)},\|S_2\|_{C(H)}\right\}\le M(\delta)\left(\|\phi\|_H+\|\zeta\|_H+\left\|\{\psi_k\}_1^{N-1}\right\|_{C(H)}\right).$$

Applying Cauchy-Schwarz and triangle inequalities to (19) and by using (2), (9), we have

$$\|Ru_0\|_H\le M(\delta)\left[\|\phi\|_H+\|\zeta\|_H+\|\eta\|_H+\left\|\{\psi_k\}_1^{N-1}\right\|_{C(H)}\right],$$

$$\|Ru_N\|_H\le M(\delta)\left[\|\phi\|_H+\|\zeta\|_H+\|\eta\|_H+\left\|\{\psi_k\}_1^{N-1}\right\|_{C(H)}\right].$$

So, by using (24) we get

$$\left\|\{u_k\}_1^{N-1}\right\|_{C(H)}\le M(\delta)\left[\|\phi\|_H+\|\zeta\|_H+\|\eta\|_H+\left\|\{\psi_k\}_1^{N-1}\right\|_{C(H)}\right]. \tag{25}$$

Finally, by virtue (15) and (23) we can establish

$$\left\|A^{-1}p\right\|_H\le M\left(\delta\right)\left[\|\phi\|_H+\|\zeta\|_H+\|\eta\|_H+\left\|\{\psi_k\}_1^{N-1}\right\|_{C(H)}\right].$$

Now, from (25), (13) and triangle inequality one can get inequality (16).

Then, DS (12) has a solution $(\{v_k\}_1^{N-1},p)$, which satisfies stability estimates given in the below theorems.

Theorem 2. Let $\phi,\zeta,\eta\in D(A)\cap D(C)$ and $\{\psi_k\}_1^{N-1}\in C(H)$ be given. Then, for solution $(\{v_k\}_1^{N-1},p)$ of DS (12) almost coercive stability estimate hold:

$$\left\|\left\{\frac{(v_{k+1}-2v_k+v_{k-1})}{\tau^2}\right\}_1^{N-1}\right\|_{C(H)}+\left\|\{Cv_k\}_1^{N-1}\right\|_{C(H)}+\|p\|_H$$
$$\le M(\delta)\left\{\min\left[\ln\left(\tfrac{1}{\tau}\right),1+|\ln\|D\|_{H\to H}|\right]\left\|\{\psi_k\}_1^{N-1}\right\|_{C(H)}+\|C\phi\|_H+\|C\zeta\|_H+\|C\eta\|_H\right\},$$

where $M(\delta)$ does not depend on ϕ, ζ, η and $\{\psi_k\}_1^{N-1}$.

Theorem 3. Let $\phi, \zeta, \eta \in D(A) \cap D(C)$ and $\{\psi_k\}_1^{N-1} \in C_{0T}^{\alpha,\alpha}(H)(0 < \alpha < 1)$ be given. Then, for solution $(\{v_k\}_1^{N-1}, p)$ of difference problem (12) coercive stability estimate

$$\left\| \left\{ \tau^{-2} \left(v_{k+1} - 2v_k + v_{k-1} \right) \right\}_1^{N-1} \right\|_{C_{0T}^{\alpha,\alpha}(H)} + \left\| \left\{ C v_k \right\}_1^{N-1} \right\|_{C_{0T}^{\alpha,\alpha}(H)} + \|p\|_H$$
$$\leq M(\delta) \left[\frac{1}{(1-\alpha)\alpha} \left\| \{\psi_k\}_1^{N-1} \right\|_{C_{0T}^{\alpha,\alpha}(H)} + \|C\phi\|_H + \|C\zeta\|_H + \|C\eta\|_H \right]$$

is true. Here $M(\delta)$ is independent from $\alpha, \phi, \zeta, \eta$ and $\{\psi_k\}_1^{N-1}$.

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Көпнүктелі қайтаанықталған эллипстік есеп үшін төртінші ретті дәлдікті тұрақтылық айырымдық схемасы туралы

Мақалада гильберттік кеңістікте көпнүктелі эллипстік қайтаанықталған есептің жуық шешімін табу үшін төртінші ретті дәлдікті айырымдық схема ұсынылды. Айырымдық схеманың шешімінің бар және жалғыз болуы функционалды-операторлық тәсілді қолдану арқылы алынады. Айырымдық схеманың шешімінің тұрақтылық, дерлік тұрақтылық және коэрцитивті тұрақтылық бағалаулары анықталды. Бұл теориялық нәтижелерді дербес туындылы көп өлшемді эллипстік теңдеулер үшін көпнүктелі қайтаанықталған шеттік есептің жуық шешімін табу үшін тұрақты жоғары дәлдіктегі айырымдық схеманы құру үшін қолдануға болады.

Кілт сөздер: қайтаанықталған эллипстік есеп, көпнүктелі шарт, жоғары ретті айырымдық схема, айырымдық схема, кері, дереккөзді идентификациялау есебі, корректілік, тұрақтылық, коэрцитивті тұрақтылық, дерлік коэрцитивті тұрақтылық.

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Об устойчивой разностной схеме четвертого порядка точности для многоточечной переопределенной эллиптической задачи

В статье предложена разностная схема четвертого порядка точности для приближенного решения многоточечной эллиптической переопределенной задачи в гильбертовом пространстве. Существование и единственность решения разностной схемы получены с использованием функционально-операторного подхода. Установлены оценки устойчивости, почти коэрцитивной устойчивости и коэрцитивной устойчивости решения разностной схемы. Эти теоретические результаты могут быть применены для построения устойчивой высокоточной разностной схемы для приближенного решения многоточечной переопределенной краевой задачи для многомерных эллиптических уравнений в частных производных.

Ключевые слова: переопределенная эллиптическая задача, многоточечное условие, разностная схема высокого порядка, разностная схема, обратная, задача идентификации источника, корректность, устойчивость, коэрцитивная устойчивость, почти коэрцитивная устойчивость.

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