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## INTEGRATION OF THE LOADED SECOND-ORDER KORTEWEG-DE VRIES EQUATION WITH A FREE TERM INDEPENDENT OF THE SPATIAL VARIABLE

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Solutions in the class of periodic functions for KdV equation were studied in [1]–[3] in various formulations. In the works of [4] the KdV equation with free a term independent of the spatial variable, and in the work of [5], [6] the KdV equation with a loaded term was studied.

In this work, we study the loaded second-order KdV equation with a free term independent of the spatial variable, namely, we consider the following equation

$$q_t = \frac{1}{4}q_{xxxx} - 5q_x q_{xx} - \frac{5}{2}q q_{xxx} + \frac{15}{2}q^2 q_x + \gamma(t) \cdot q|_{x=0} \cdot q_x + f(t), \quad x \in R, t > 0 \quad (1)$$

with initial condition

$$q(x, t)|_{t=0} = q_0(x), \quad (2)$$

where  $\gamma(t) \in C[0, \infty)$  and  $f(t)$  is given real continuous function and  $q_0(x) \in C^5(R^1)$  is given real function. It is required to find a real function  $q(x, t)$ , that is  $\pi$ -periodic in a variable  $x$ :

$$q(x + \pi, t) \equiv q(x, t), \quad x \in R, t > 0 \quad (3)$$

and satisfies the smoothness conditions:

$$q(x, t) \in C_x^5(t > 0) \cap C_t^1(t > 0) \cap C(t \geq 0). \quad (4)$$

**Theorem.** Let  $q(x, t)$  be the solution of problem (1)-(4). Then the boundaries  $\lambda_n(t)$ ,  $n \geq 0$  of the spectrum of the following operator

$$L(\tau, t)y \equiv -y'' + q(x + \tau, t)y = \lambda y, \quad x \in R \quad (5)$$

satisfy the system of equations

$$\dot{\lambda}_n(t) = f(t), \quad n \geq 0, \quad (6)$$

and the spectral parameters  $\xi_n(\tau, t)$ ,  $n \geq 1$  satisfy the analogue of the system of equations of Dubrovin:

$$\frac{\partial \xi_n}{\partial t} = 2(-1)^n \sigma_n(\tau, t) \left[ -\frac{1}{2}q_{\tau\tau}(\tau, t) + \frac{3}{2}q^2(\tau, t) + 2\xi_n^2 q(\tau, t) + 4\xi_n^2 - \gamma(t)q(0, t) \right] h_n(\xi) + f(t)$$

$$h_n(\xi) = \sqrt{(\xi_n - \lambda_{2n-1})(\lambda_{2n} - \xi_n)} \times \sqrt{(\xi_n - \lambda_0) \prod_{\substack{k=1 \\ k \neq n}}^{\infty} \frac{(\lambda_{2k-1} - \xi_n)(\lambda_{2k} - \xi_n)}{(\xi_k - \xi_n)^2}}, \quad n \geq 1, \quad (7)$$

the sign  $\sigma_n(\tau, t)$  changes at each collision of the point  $\xi_n(\tau, t)$  with the boundaries of its gap  $[\lambda_{2n-1}, \lambda_{2n}]$ . Moreover, the following initial conditions are fulfilled:

$$\xi_n(\tau, t)|_{t=0} = \xi_n^0(\tau), \quad \sigma_n(\tau, t)|_{t=0} = \sigma_n^0(\tau), \quad n \geq 1, \quad (8)$$

where  $\xi_n^0(\tau)$ ,  $\sigma_n^0(\tau)$ ,  $n \geq 1$  are the spectral parameters of the Sturm-Liouville equation corresponding to the coefficients  $q_0(x + \tau)$ .

**Remark.** Using the trace formula

$$q(\tau, t) = \lambda_0 + \sum_{k=1}^{\infty} (\lambda_{2k-1} + \lambda_{2k} - 2\xi_k(\tau, t)) \quad (9)$$

$$q^2(\tau, t) - \frac{1}{2}q_{\tau\tau}(\tau, t) = \lambda_0^2 + \sum_{k=1}^{\infty} (\lambda_{2k-1}^2 + \lambda_{2k}^2 - 2\xi_k^2(\tau, t)) \quad (10)$$

system equations of Dubrovin can be rewritten in the “closed” form.

**Corollary 1.** The theorem gives a method for solving problem (1)-(4). First we find the spectral data  $\lambda_n^0, n \geq 0; \xi_n^0(\tau), \sigma_n^0(\tau), n \geq 1$  of the Sturm-Liouville equation

$$-y'' + q_0(x + \tau)y = \lambda y, \quad x \in R.$$

Solving equations (6) with initial conditions  $\lambda_n^0(t)|_{t=0} = \lambda_n^0, n \geq 0$ , we find

$$\lambda_n(t) = \lambda_n^0 + \int_0^t f(s) ds, \quad n \geq 0. \quad (11)$$

Further, solving the Cauchy problem (7), (8) for  $\tau = 0$  we get  $\xi_n(0, t), n \geq 1$ . Then substituting this data into equation (7) and solving the Cauchy problem  $\xi_n(\tau, t)|_{t=0} = \xi_n^0(\tau), \sigma_n(\tau, t)|_{t=0} = \sigma_n^0(\tau), n \geq 1$  for Dubrovin system (7) we find  $\xi_n(\tau, t), n \geq 1$ . Finally, by using the trace formula (9) and (10) we obtain  $q(\tau, t)$ .

**Remark.** Equations (6) show that the spectrum of the Sturm-Liouville operator (5) moves on the axis while keeping the initial structure, that is, the lengths of the gaps do not change.

**Corollary 2.** In [7], there was proved the theorem which states that the lengths of the gaps of the Sturm-Liouville equation with  $\pi$ -periodic real-valued coefficient decrease exponentially if and only if the coefficient is analytic. From this theorem we conclude that if  $q_0(x)$  is real analytical function, then the lengths of the gaps corresponding to this coefficient decrease exponentially. For the coefficient  $q(x, t)$  there correspond the same gaps. Thus the solution  $q(x, t)$  of problem (1)-(4) is real analytical functions on  $x$ .

**Corollary 3.** In [8], a generalization of Borg’s inverse theorem was proved: the number  $\pi / n$  is a period of the coefficient of the Sturm-Liouville equation with  $\pi$ -periodic real-valued coefficient if and only if all the lacunae whose numbers are not divisible by  $n$  are vanished. Here  $n \geq 2$  is a natural number and the lacuna  $(\lambda_{2k-1}, \lambda_{2k})$  has a number  $k$ . Therefore, if  $q_0(x)$  has a period  $\pi / n$  then the solution to problem (1)-(4) is the  $\pi / n$  - periodic function on  $x$ .

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