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Ternary semigroups of topological transformations

A ternary semigroup is a nonempty set with a ternary operation which is associative. The purpose of the present paper is to give a characterization of open sets of finite-dimensional Euclidean spaces by ternary semigroups of pairs of homeomorphic transformations and extend to ternary semigroups certain results of L.M. Gluskin concerned with semigroups of homeomorphic transformations of finite-dimensional Euclidean spaces.

Keywords: Euclidean n-space, ternary semigroup, homeomorphic transformations.

Introduction

Lehmer [1] investigated certain triple systems called triplexes, Santiago and Sri Bala [2] developed regular and completely regular ternary semigroups. Dutta, Kar and Maity studied intra-regular ternary semigroups [3]. Wagner studied generalized heaps and generalized groups [4]. Gluskin showed that semigroups of topological transformations of bounded closed sets on Euclidean n-spaces define those sets exactly up to homeomorphism [5]. Mustafaev studied semiheaps of homeomorphic maps of open and closed sets of Euclidean n-spaces [6]. In this paper we study some properties of ternary semigroups of topological maps between open sets of Euclidean n-spaces.

A ternary semigroup is a nonempty set T together with a ternary operation $[abc]$ satisfying the associative law $[[abc]\,de] = [a\,[bcd]\,e] = [ab\,[cde]]$ for every $a, b, c, d, e \in T$. Any semigroup can be made into a ternary semigroup by defining the ternary product to be $[abc] = abc$. A nonempty subset L of a ternary semigroup T is called a left (right, lateral) *ideal* of T, if $[TTL] \subseteq L$ $([LTT] \subseteq L, [TLT] \subseteq L)$. A nonempty subset A of a ternary semigroup T is called a *two sided ideal* of T if it is a left and right ideal of T. A nonempty subset A of a ternary semigroup T is called an *ideal* of T if it is a left, right and lateral ideal of T. If the intersection K of all the ideals of a ternary semigroup T is not empty, we shall call K the *kernel* of T. A ternary semigroup is called (left, right) simple if it does not contain any proper (left, right) ideals [7]. A ternary semigroup is simple if it does not have nontrivial homomorphisms, that is, if each of its homomorphisms is either an isomorphism or a mapping onto a ternary semigroup consisting of one element. A zero "0"of a ternary semigroup T is an element such that for all $a, b \in T$, $[0ab] = [a0b] = [ab0] = 0$. An equivalence relation ρ on a ternary semigroup T is said to be a left congruence if $(a, b) \in \rho \Longrightarrow ([sta], [stb]) \in \rho$ for all $a, b, s, t \in T$. Similarly, ρ is a right congruence if $(a, b) \in \rho \Longrightarrow ([ast], [bst]) \in \rho$ for all $a, b, s, t \in T$ and a lateral congruence if $(a, b) \in \rho \Longrightarrow ([sat], [sbt]) \in \rho$ for all $a, b, s, t \in T$. An equivalence relation ρ on a ternary semigroup T is said to be a congruence if $(a, a') \in \rho, (b, b') \in \rho, (c, c') \in \rho \Longrightarrow ([abc], [a'b'c']) \in \rho$ for all $a, a', b, b', c, c' \in T$. An equivalence relation ρ on a ternary semigroup T is a congruence if and only if it is a left, a right and a lateral congruence on T [8].

Let X and Y be two nonempty sets and let $F(X, Y)$ be the set of all pairs of functions (γ, η), where $\gamma : X \to Y$ and $\eta : Y \to X$. The set $F(X, Y)$ is a ternary semigroup with respect to the ternary operation

$$[(\gamma_1, \eta_1)(\gamma_2, \eta_2)(\gamma_3, \eta_3)] = (\gamma_1\eta_2\gamma_3, \eta_1\gamma_2\eta_3),$$

where $(\gamma_1\eta_2\gamma_3)\,x = \gamma_1(\eta_2(\gamma_3(x)))$ and $(\eta_1\gamma_2\eta_3)\,y = \eta_1(\gamma_2(\eta_3(y)))$.

Let S be a ternary semigroup and a be any element of S. The set $SSa \cup a$ is a left ideal of S and is called the principal left ideal of S generated by a. Consider the following symmetric and reflexive relation on the set S defined by

$$\sigma_t : x\sigma_t y \leftrightarrow x,\ y \in SSa \cup a,\ (x, y \in S).$$

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$\overline{\sigma_t}$ denotes the transitive closure of σ_t. Each class L_ν of $\overline{\sigma_t}$ is the union of some principal left ideals of S and therefore is a left ideal of S. The partition

$$S = \cup L_\nu \tag{1}$$

of the ternary semigroup S into classes of $\overline{\sigma_t}$ is the representation of S as the union of the pairwise disjoint left ideals. We say that (1) is the most fractional partition of S into pairwise disjoint left ideals.

Characterization of open sets of Euclidean n-spaces by ternary semigroups

Let Ω_1 and Ω_2 be two open sets of a finite-dimensional Euclidean space. $H_i(\Omega_i, \Omega_j)$ denotes the set of all homeomorphic maps from Ω_i to Ω_j, where $i, j = 1, 2, (i \neq j)$. Let $K_i(\Omega_i, \Omega_j)$ denote the set of all $a \in H_i(\Omega_i, \Omega_j)$ for which there is an n-sized element $E_a \subset \Omega_j$ (a set homeomorphic to some closed n-ball) and a closed set $F_a \subset \Omega_j$ such that $a\Omega_i \subset F_a \subset IntE_a$, where $i, j = 1, 2$ $(i \neq j)$. Let $OH = OH(\Omega_1, \Omega_2) = H_1(\Omega_1, \Omega_2) \times H_2(\Omega_2, \Omega_1)$ be the set of all pairs of homeomorphic maps (a, b), where $a \in H_1(\Omega_1, \Omega_2)$, $b \in H_2(\Omega_2, \Omega_1)$. The set OH is a ternary semigroup with respect to the ternary operation

$$[(a_1, b_1)(a_2, b_2)(a_3, b_3)] = (a_1 b_2 a_3, b_1 a_2 b_3).$$

Clearly, the set $K = K(\Omega_1, \Omega_2) = K_1(\Omega_1, \Omega_2) \times K_2(\Omega_2, \Omega_1)$ is a ternary subsemigroup and even an ideal of the ternary semigroup OH.

Theorem 1. Let R and R' be finite-dimensional Euclidean spaces. Let Ω_1 and Ω_2 be open sets of a finite-dimensional Euclidean space R and Ω_1' and Ω_2' be open sets of a finite-dimensional Euclidean space R'. The ternary semigroups $K(\Omega_1, \Omega_2)$ and $K(\Omega_1', \Omega_2')$ are isomorphic if and only if the spaces Ω_i and Ω_i' are homeomorphic $(i = 1, 2)$.

Proof. Let Ω_1 and Ω_2 be open subsets of a finite-dimensional Euclidean space R and let Ω_1' and Ω_2' be open subsets of a finite-dimensional Euclidean space R'. Suppose that $\xi_1 : \Omega_1 \to \Omega_1'$ is a homeomorphism of Ω_1 onto Ω_1' and $\xi_2 : \Omega_2 \to \Omega_2'$ is a homeomorphism of Ω_2 onto Ω_2'. Then, the mapping $\varphi_{\xi_1,\xi_2} : K(\Omega_1, \Omega_2) \to K(\Omega_1', \Omega_2')$ defined by

$$\varphi_{\xi_1,\xi_2}(a, b) = \left(\xi_2 a \xi_1^{-1}, \xi_1 b \xi_2^{-1}\right)$$

is an isomorphism from $K(\Omega_1, \Omega_2)$ onto $K(\Omega_1', \Omega_2')$. The proof of the necessary condition follows from Lemmas 1–5.

Throughout this paper the symbol φ denotes an isomorphism $\varphi : K(\Omega_1, \Omega_2) \to K(\Omega_1', \Omega_2')$ unless otherwise stated.

Lemma 1. Let $(a_1, b_1), (a_2, b_2) \in K(\Omega_1, \Omega_2)$ such that

$$(a_2, b_2)(\Omega_1, \Omega_2) \subseteq (a_1, b_1)(\Omega_1, \Omega_2).$$

Then,

$$\varphi(a_2, b_2)(\Omega_1', \Omega_2') \subset \overline{\varphi(a_1, b_1)(\Omega_1', \Omega_2')}.$$

Proof. Let $(a_1, a_2), (b_1, b_2)$ be any two elements in $K(\Omega_1, \Omega_2)$ such that

$$(a_2, b_2)(\Omega_1, \Omega_2) \subseteq (a_1, b_1)(\Omega_1, \Omega_2).$$

If

$$[\varphi(a_1, b_1)(x_1', y_1')\varphi(a_1, b_1)] = [\varphi(a_1, b_1)(x_2', y_2')\varphi(a_1, b_1)] \tag{2}$$

is valid for some $(x_1', y_1'), (x_2', y_2') \in K(\Omega_1', \Omega_2')$, then there exist elements $(x_1, y_1), (x_2, y_2) \in K(\Omega_1, \Omega_2)$ such that $\varphi(x_1, y_1) = (x_1', y_1')$ and $\varphi(x_2, y_2) = (x_2', y_2')$. Therefore

$$[\varphi(a_1, b_1)\varphi(x_1, y_1)\varphi(a_1, b_1)] = [\varphi(a_1, b_1)\varphi(x_2, y_2)\varphi(a_1, b_1)].$$

From this it follows

$$\varphi[(a_1, b_1)(x_1, y_1)(a_1, b_1)] = \varphi[(a_1, b_1)(x_2, y_2)(a_1, b_1)]$$

and since φ is an isomorphism of $K(\Omega_1, \Omega_2)$ onto $K(\Omega_1', \Omega_2')$ we have

$$[(a_1, b_1)(x_1, y_1)(a_1, b_1)] = [(a_1, b_1)(x_2, y_2)(a_1, b_1)].$$

Then,

$$[(a_2,b_2)(x_1,y_1)(a_2,b_2)] = [(a_2,b_2)(x_2,y_2)(a_2,b_2)]$$

or

$$[\varphi(a_2,b_2)\varphi(x_1,y_1)\varphi(a_2,b_2)] = [\varphi(a_2,b_2)\varphi(x_2,y_2)\varphi(a_2,b_2)]$$

or

$$[\varphi(a_2,b_2)(x_1',y_1')\varphi(a_2,b_2)] = [\varphi(a_2,b_2)(x_2',y_2')\varphi(a_2,b_2)].$$

Since the last equality is valid for every $(x_1',y_1'),(x_2',y_2') \in K(\Omega_1',\Omega_2')$ satisfying (2), we have $\varphi(a_2,b_2)(\Omega_1',\Omega_2') \subset \overline{\varphi(a_1,b_1)(\Omega_1',\Omega_2')}$.

The following two lemmas are immediate consequences of Lemma 1.

Lemma 2. Let $(a_1,b_1),(a_2,b_2) \in K(\Omega_1,\Omega_2)$. If

$$(a_1,b_1)(\Omega_1,\Omega_2) \cap (a_2,b_2)(\Omega_1,\Omega_2) \neq \varnothing,$$

then

$$\overline{\varphi(a_1,b_1)(\Omega_1',\Omega_2')} \cap \overline{\varphi(a_2,b_2)(\Omega_1',\Omega_2')} \neq \varnothing.$$

Lemma 3. Let $(a_1,b_1),(a_2,b_2),(a_3,b_3) \in K(\Omega_1,\Omega_2)$. The equation

$$[(a_2,b_2)(a_1,b_1)(x_2,y_2)] = (a_3,b_3)$$

has a solution for $(x_2,y_2) \in K(\Omega_1,\Omega_2)$ if and only if there exist n-sized elements T_1 and T_2 such that

$$T_1 \subset b_2a_1\Omega_1, T_2 \subset a_2b_1\Omega_2, a_3\Omega_1 \subset IntT_2, b_3\Omega_2 \subset IntT_1.$$

Let $(\alpha,\beta) \in \Omega_1 \times \Omega_2$. We say that an infinite sequence $\{(a_i,b_i)\}_{i=1}^{\infty}$ of elements $(a_i,b_i) \in K(\Omega_1,\Omega_2)$ has a limit (α,β), if the following conditions are satisfied:

a) $(\cap_{i=2}^{\infty} b_i\Omega_2, \cap_{i=2}^{\infty} a_i\Omega_1) = (\alpha,\beta)$,

b) for every i there exists an element (x_{i+1},y_{i+1}) such that

$$[(a_{i+1},b_{i+1})(a_i,b_i)(x_{i+1},y_{i+1})] = (a_{i+2},b_{i+2}).$$

A sequence $\{(a_i,b_i)\}_{i=1}^{\infty}$ of elements $(a_i,b_i) \in K(\Omega_1,\Omega_2)$ converging to the point $(\alpha,\beta) \in \Omega_1 \times \Omega_2$ can be built, for example, as follows. Suppose that $E_1 \subset \Omega_1$ is a closed n-ball centered at α and $T_1 \subset \Omega_2$ is a closed n-ball centered at β. There exists $(a_1,b_1) \in K(\Omega_1,\Omega_2)$ such that

$$\alpha \in b_1\Omega_2 \subset IntE_1,\ \ \beta \in a_1\Omega_1 \subset IntT_1.$$

Suppose now that E_2 is a closed n-ball in $b_1\Omega_2$ and centered at α and T_2 is a closed n-ball in $a_1\Omega_1$ and centered at β. Then, there exists an element $(a_2,b_2) \in K(\Omega_1,\Omega_2)$ such that

$$\alpha \in b_2\Omega_2 \subset IntE_2,\ \ \beta \in a_2\Omega_1 \subset IntT_2.$$

Let $\alpha_1 = (b_2a_1)^{-1}(\alpha)$ and $\beta_1 = (a_2b_1)^{-1}(\beta)$. Let $A_1 \subset \Omega_1$ be a closed n-ball centered at α_1 and $B_1 \subset \Omega_2$ be a closed n-ball centered at β_1. Clearly,

$$\alpha \in b_2a_1A_1 \cap E_2, \beta \in a_2b_1B_1 \cap T_2.$$

Let $E_3 \subset Int(b_2a_1A_1 \cap E_2)$ be a closed n-ball centered at α, and let $T_3 \subset Int(a_2b_1B_1 \cap T_2)$ be a closed n-ball centered at β. Then, there exists an element $(a_3,b_3) \in K(\Omega_1,\Omega_2)$ such that

$$\alpha \in b_3\Omega_2,\ \ \ \beta \in a_3\Omega_1$$

and

$$b_3\Omega_2 \subset IntE_3,\ \ \ a_3\Omega_1 \subset IntT_3.$$

By Lemma 3, the point $x_2 = (a_2b_1)^{-1}a_3, y_2 = (b_2a_1)^{-1}b_3$ is the solution of the equation

$$[(a_2,b_2)(a_1,b_1)(x_2,y_2)] = (a_3,b_3).$$

Assume now that the first n terms of the sequence are already found. Denote $\alpha_{n-1} = (b_n a_{n-1})^{-1}(\alpha)$ and $\beta_{n-1} = (a_n b_{n-1})^{-1}(\beta)$. Suppose that $A_{n-1} \subset \Omega_1$ is a closed n-ball centered at α_{n-1} and $B_{n-1} \subset \Omega_2$ is a closed n-ball centered at β_{n-1}. Clearly,

$$\alpha \in b_n a_{n-1} A_{n-1} \cap E_n, \beta \in a_n b_{n-1} B_{n-1} \cap T_n.$$

Let $E_{n+1} \subset Int\,(b_n a_{n-1} A_{n-1} \cap E_n)$ be a closed n-ball centered at α, and let $T_{n+1} \subset Int\,(a_n b_{n-1} B_{n-1} \cap T_n)$ be a closed n-ball centered at β. Then, there exists an element $(a_{n+1}, b_{n+1}) \in K(\Omega_1, \Omega_2)$ such that

$$\alpha \in b_{n+1}\Omega_2, \beta \in a_{n+1}\Omega_1$$

and

$$b_{n+1}\Omega_2 \subset IntE_{n+1}, a_{n+1}\Omega_1 \subset IntT_{n+1}.$$

By Lemma 3, the point $x_n = (a_n b_{n-1})^{-1} a_{n+1}, y_n = (b_n a_{n-1})^{-1} b_{n+1}$ is the solution of the equation

$$[(a_n, b_n)(a_{n-1}, b_{n-1})(x_n, y_n)] = (a_{n+1}, b_{n+1}).$$

This sequence satisfies condition (b) and condition (a), if the sequences of radii of E_n and T_n converge to zero.

Lemma 4. If the ternary semigroups $K(\Omega_1, \Omega_2)$ and $K(\Omega'_1, \Omega'_2)$ are isomorphic, then there exist a bijective map f from $\Omega_1 \times \Omega_2$ onto $\Omega'_1 \times \Omega'_2$ and bijective maps ξ_i from Ω_i onto Ω'_i $(i = 1, 2)$ such that $f(\alpha, \beta) = (\xi_1\alpha, \xi_2\beta)$ for every $(\alpha, \beta) \in \Omega_1 \times \Omega_2$.

Proof. Let (α, β) be any point in $\Omega_1 \times \Omega_2$ and let $\{(a_i, b_i)\}_{i=1}^{\infty}$ be a sequence of elements $(a_i, b_i) \in B(\Omega_1, \Omega_2)$ converging to the point (α, β). Denote $\varphi(a_i, b_i)$ by (a'_i, b'_i). The sequence $\{(a'_i, b'_i)\}_{i=1}^{\infty}$ converges to the point (α', β'). Define a map $f : (\Omega_1, \Omega_2) \to (\Omega'_1, \Omega'_2)$ by $f(\alpha, \beta) = (\alpha', \beta')$. The point (α', β') does not depend on the choice of the sequence $\{(a_i, b_i)\}_{i=1}^{\infty}$ of elements $(a_i, b_i) \in B(\Omega_1, \Omega_2)$ converging to the point (α, β). The map f is one-to-one and there are one-to-one maps ξ_i from Ω_i onto Ω'_i for $(i = 1, 2)$ such that $\forall (\alpha, \beta) \in \Omega_1 \times \Omega_2, f(\alpha, \beta) = (\xi_1\alpha, \xi_2\beta)$.

Lemma 5. 1) The following implication holds

$$(\alpha, \beta) \in (a, b)(\Omega_1, \Omega_2) \to f(\alpha, \beta) = (\xi_1\alpha, \xi_2\beta) \in \overline{\varphi(a, b)(\Omega'_1, \Omega'_2)}$$

for any $(\alpha, \beta) \in \Omega_1 \times \Omega_2$ and $(a, b) \in K(\Omega_1, \Omega_2)$.

2) The map f is a homeomorphism from $\Omega_1 \times \Omega_2$ onto $\Omega'_1 \times \Omega'_2$ and therefore the map ξ_i is a homeomorphism from Ω_i onto Ω'_i for $i = 1, 2$.

Theorem 2. The ternary semigroup $K(\Omega_1, \Omega_2)$ is a minimal ideal (the kernel) of the ternary semigroup $OH(\Omega_1, \Omega_2)$.

Theorem 3. Let Ω_1 and Ω_2 be open subsets of a finite-dimensional Euclidean space R and Ω'_1 and Ω'_2 be open subsets of a finite-dimensional Euclidean space R'. The ternary semigroups $OH(\Omega_1, \Omega_2)$ and $OH(\Omega'_1, \Omega'_2)$ are isomorphic if and only if the spaces Ω_i and Ω'_i are homeomorphic $(i = 1, 2)$.

Properties of the ternary semigroup of topological maps

Let G be a group and let $A = G \cup \{0\}$ be a zero adjoint semigroup. Let I, Λ be non-empty sets and let P be a $\Lambda \times I$ matrix over A such that every row and every column of P contain at least one non-zero entry. The set $S = A \times I \times \Lambda$ with ternary multiplication

$$[(a; i, \lambda)(b; j, \mu)(c; k, \nu)] = (ap_{\lambda j}bp_{\mu k}c; i, \nu)$$

is a ternary semigroup with zero $0 = (0, i, \lambda)$. Let (A, i, λ) denote a subset of S consisting of all triples (a, i, λ), where $a \in A$ and i, λ are fixed elements, then

$$S = \bigcup_{i \in I, \lambda \in \Lambda} (A, i, \lambda).$$

From the definition of the ternary operation it follows that S is the union of its nonzero minimal right ideals R_i and S is the union of its nonzero minimal left ideals L_λ, where

$$R_i = \bigcup_{\lambda \in \Lambda} (A, i, \lambda), \quad L_\lambda = \bigcup_{i \in I} (A, i, \lambda).$$

The ternary semigroup S does not contain any proper two sided ideals, in particular, S does not contain any proper ideals. We denote S by $M^0(G, I, \Lambda, P)$.

Let R and R' be finite-dimensional Euclidean spaces, Ω_1 and Ω_2 be open subsets of a finite-dimensional Euclidean space R and let Ω_1' and Ω_2' be open subsets of a finite-dimensional Euclidean space R'. Suppose that $\xi_1 : \Omega_1 \to \Omega_1'$ is a homeomorphism of Ω_1 onto Ω_1' and $\xi_2 : \Omega_2 \to \Omega_2'$ is a homeomorphism of Ω_2 onto Ω_2'. Then, the mapping $\varphi_{\xi_1,\xi_2} : K(\Omega_1, \Omega_2) \to K(\Omega_1', \Omega_2')$ defined by

$$\varphi_{\xi_1,\xi_2}(a, b) = \left(\xi_2 a \xi_1^{-1}, \xi_1 b \xi_2^{-1}\right)$$

is a homeomorphism from $K(\Omega_1, \Omega_2)$ onto $K(\Omega_1', \Omega_2')$.

Introduce the following symmetric and reflexive relation σ_r in the ternary semigroup $OH(\Omega_1, \Omega_2) : (a_1, b_1)$, $(a_2, b_2) \in \sigma_r$ if and only if $(a_1, b_1) = (a_2, b_2)$ or $(a_1, b_1), (a_2, b_2) \in R_{(a,b)}$ for some $(a, b) \in OH(\Omega_1, \Omega_2)$, where $R_{(a,b)}$ is a right ideal of $OH(\Omega_1, \Omega_2)$ generated by (a, b). Since the relation σ_r is stable, its transitive closure $\overline{\sigma_r}$ is a congruence on $OH(\Omega_1, \Omega_2)$. Each equivalence class R_α of $\overline{\sigma_r}$ either consists of one element of $OH(\Omega_1, \Omega_2)$ not contained in $K(\Omega_1, \Omega_2)$ or is a right ideal of $OH(\Omega_1, \Omega_2)$. Then,

$$K(\Omega_1, \Omega_2) = \underset{\alpha \in I}{\cup} R_\alpha$$

is the most fractional partition of $K(\Omega_1, \Omega_2)$ into pairwise disjoint union of the distinct right ideals of $OH(\Omega_1, \Omega_2)$.

Let A_i be some component of the set Ω_1, B_μ be some component of the set Ω_2 and $R_{i\mu}$ be a subset of the ternary semigroup $K(\Omega_1, \Omega_2)$ consisting of (a, b) such that $a\Omega_1 \subset B_\mu, b\Omega_2 \subset A_i$. If (a, b) is any element of $R_{i\mu}$ and $(x_1, y_1), (x_2, y_2)$ are the elements of the ternary semigroup $OH(\Omega_1, \Omega_2)$, then from $a\Omega_1 \subset B_\mu, b\Omega_2 \subset A_i$ it follows that

$$a y_1 x_2 \Omega_1 \subset B_\mu, \quad b x_1 y_2 \Omega_2 \subset A_i.$$

Thus, $(a y_1 x_2, b x_1 y_2) \in R_{i\mu}$ and $R_{i\mu}$ is a right ideal of $OH(\Omega_1, \Omega_2)$. Consequently, the partition

$$K(\Omega_1, \Omega_2) = \underset{i \in I, \mu \in M}{\cup} R_{i\mu}$$

is the presentation of $K(\Omega_1, \Omega_2)$ as the pairwise disjoint union of the distinct right ideals of $OH(\Omega_1, \Omega_2)$.

Lemma 6. If E_i is any closed ball contained in Ω_i, and α_i, β_i are any points in $IntE_i$, then there is $(b_1, b_2) \in K(\Omega_1, \Omega_2)$ such that

$$\alpha_i, \beta_i \in b_j \Omega_j, \quad b_j \Omega_j \subset IntE_i,$$

where $i, j = 1, 2$ $(i \neq j)$.

Proof. Let (a_1, a_2) be an arbitrary element in $K(\Omega_1, \Omega_2)$, C_i be a closed ball in $a_j \Omega_j$ and A_i, B_i, D_i be any closed balls such that

$$\Omega_i \subset IntE_i, \quad B_i \subset IntD_i, \quad D_i \subset IntE_i, \quad \alpha_i, \beta_i \in IntB_i.$$

Futher, let f_i be the homeomorphisms of A_i onto D_i such that $f_i(C_i) = B_i$. Then $(b_1, b_2) = (f_2 a_1, f_1 a_2)$ is the requiered element of $K(\Omega_1, \Omega_2)$.

Lemma 7. The partition

$$K(\Omega_1, \Omega_2) = \underset{i \in I, \mu \in M}{\cup} R_{i\mu}$$

is the most fractional partition of $K(\Omega_1, \Omega_2)$ into pairwise disjoint right ideals of $OH(\Omega_1, \Omega_2)$.

Proof. It is sufficient to show that for any $i \in I, \mu \in M$ the condition

$$(a_0, b_0), (a, b) \in R_{i\mu}$$

implies

$$(a_0, b_0), (a, b) \in \overline{\sigma_r}.$$

Let's prove this for the maps b_0 and b from Ω_2 into A_i, where A_i is a component of Ω_1. Let $\xi \in b_0 \Omega_2, \quad \xi' \in b\Omega_2$. Since $\xi, \xi' \in A_i$, they can be connected by a simple arc l contained in A_i. We have $d(F_r(A_i), l) = m > 0$, where $F_r(A_i)$ is a boundary of A_i. Then, it can be found finite covering of l with open balls of radius $r < m$ centered on l. Denote these balls by E_{2k} $(k = 1, 2, ..., s)$, numerated in order of their centers positions on l. Choose points

$$\xi_k \in E_{2k} \cap E_{2k+2}, (k = 1, 2, ..., s - 1)$$

and denote $\xi_0 = \xi, \xi_s = \xi'$. Since $\overline{E_{2k}} \subset A_i$, there are closed balls D_{2k} in $\overline{E_{2k}}$ such that $\xi_{k-1}, \xi_k \in IntD_{2k}$. According to Lemma 1 there exists a homeomorphism b_{2k} from Ω_2 to Ω_1 such that $b_{2k}\Omega_2 \subset IntD_{2k}$ and

$$\xi_{k-1}, \xi_k \in b_{2k}\Omega_2, (k = 1, 2, ..., s-1).$$

Denote $b = b_{2s+2}$. There are closed sets E_{2i+1} and E'_{2i+1} centered at ξ_i such that $E_{2i+1} \subset E'_{2i+1} \subset b_{2i}\Omega_2 \cap \cap b_{2i+2}, (k = 1, 2, ..., s)$. By Lemma 1, there exists a homeomorphism b_{2k+1} from Ω_2 to Ω_1 such that $b_{2i+1}\Omega_2 \subset IntE_{2i+1}$. Let E be a closed n-ball containing the set Ω_1 and let $\widetilde{E}$ be a closed n-ball contained in Ω_1. If f_{2i+1} is a homeomorphism from E onto E'_{2i+1} for which $f_{2i+1}\left(\widetilde{E}\right) = E_{2i+1}$, then

$$\begin{aligned} b_{2i+1} &= b_{2i} \circ b_{2i}^{-1} f_{2i+1} \circ f_{2i+1}^{-1} b_{2i+1}, \\ b_{2i+1} &= b_{2i+2} \circ b_{2i+2}^{-1} f_{2i+1} \circ f_{2i+1}^{-1} b_{2i+1}, \end{aligned}$$

where $k = 0, 1, 2, ..., s$. Analogously, it can be shown that the following equations hold for some homeomorphism g_{2i+1}

$$\begin{aligned} a_{2i+1} &= a_{2i} \circ a_{2i}^{-1} g_{2i+1} \circ g_{2i+1}^{-1} a_{2i+1}, \\ a_{2i+1} &= a_{2i+2} \circ a_{2i+2}^{-1} g_{2i+1} \circ g_{2i+1}^{-1} a_{2i+1}, \end{aligned}$$

where $j = 0, 1, 2, ..., s'$ and $a_{2s'+2} = a$. Suppose that $s < s'$. Thus, $(a_{i-1}, b_{i-1}), (a_i, b_i) \in \overline{\sigma_r}$ for $i = 1, 2, ..., 2s+2$ and

$$(a_{j-1}, b_{2s+2}), (a_j, b_{2s+2}) \in \overline{\sigma_r} fori = 1, 2, ..., 2s+2.$$

This means that $(a_0, b_0), (a, b) \in \overline{\sigma_r}$.

Analogously, it can be shown that

$$K(\Omega_1, \Omega_2) = \underset{i \in I, \mu \in M}{\cup} R_{i\mu}$$

is the most fractional partition of $K(\Omega_1, \Omega_2)$ into pairwise disjoint right ideals.

Theorem 4. The quotient $OH(\Omega_1, \Omega_2)/\overline{\sigma_r}$ is a ternary semigroup with the minimal ideal $K(\Omega_1, \Omega_2)/\overline{\sigma_r}$.

Proof. Since $K(\Omega_1, \Omega_2)$ is an ideal of $OH(\Omega_1, \Omega_2)$ the quotient $K(\Omega_1, \Omega_2)/\overline{\sigma_r}$ is an ideal of $OH(\Omega_1, \Omega_2)/\overline{\sigma_r}$. It follows from Lemma 2 that the elements of $K(\Omega_1, \Omega_2)/\overline{\sigma_r}$ are the classes $R_{i\lambda}$. Let $G = \{e\}$ be the unit group, and let P be a $\Lambda \times I$ matrix over G. Denote by T the completely simple ternary semigroup over $G = \{e\}$. To each element $R_{i\lambda}$ of $K(\Omega_1, \Omega_2)/\overline{\sigma_r}$ assign an element $(e; i, \lambda)$ of T. The map f is the isomorphism from $K(\Omega_1, \Omega_2)/\overline{\sigma_r}$ onto T. Indeed,

$$f([R_{i\lambda} R_{j\mu} R_{k\nu}]) = f(R_{i\nu}) = (e; i, \nu)$$

$$= [(e; i, \lambda)(e; j, \mu)(e; k, \nu)] = [f(R_{i\lambda}) f(R_{j\mu}) f(R_{k\nu})].$$

Theorem 5. The ternary semigroup $K(\Omega_1, \Omega_2)/\overline{\sigma_r}$ is a topological invariant of the pair (Ω_1, Ω_2).

Proof. Let Ω_1 and Ω_2 be open subsets of a finite-dimensional Euclidean space R^n and let Ω'_1 and Ω'_2 be open subsets of a finite-dimensional Euclidean space R^m. If $\xi_i : \Omega_i \to \Omega'_i \, (i = 1, 2)$ is a homeomorphism, then the mapping $f : K(\Omega_1, \Omega_2) \to K(\Omega'_1, \Omega'_2)$ defined by

$$f(a, b) = \left(\xi_2 a \xi_1^{-1}, \xi_1 b \xi_2^{-1}\right)$$

is an isomorphism from $K(\Omega_1, \Omega_2)$ onto $K(\Omega'_1, \Omega'_2)$. In the case of ξ_1 the component $A_i \subset \Omega_1$ is mapped onto the component $A'_{i'} \subset \Omega'_1$, in the case of ξ_2 the component $B_\lambda \subset \Omega_2$ is mapped onto the component $B'_{\lambda'} \subset \Omega'_2$. Therefore $K(\Omega_1, \Omega_2)/\overline{\sigma_r}$ is isomorphic to $K(\Omega'_1, \Omega'_2)/\overline{\sigma_r}$.

The following three theorems are immediate consequences of Lemma 7.

Theorem 6. The spaces Ω_1 and Ω_2 are connected if and only if the ternary semigroup $K(\Omega_1, \Omega_2)$ cannot be represented as the pairwise disjoint union of its distinct right ideals.

Theorem 7. If the quotient $K(\Omega_1, \Omega_2)/\overline{\sigma_r}$ is finite and its order is a prime number, then one of the spaces Ω_1 and Ω_2 is connected.

Theorem 8. Let Ω_1 and Ω_2 be open subsets of a finite-dimensional Euclidean space. The space Ω_2 is connected if and only if the quotient $K(\Omega_1, \Omega_2)/\overline{\sigma_r}$ is a ternary semigroup of left zeros, the space Ω_1 is connected if and only if the quotient $K(\Omega_1, \Omega_2)/\overline{\sigma_r}$ is a ternary semigroup of right zeros.

Theorem 9. If at least one of the ternary semigroups $R_{i\mu}$ in the partition $K(\Omega_1, \Omega_2) = \underset{i \in I, \mu \in M}{\cup} R_{i\mu}$ is simple, then the spaces Ω_1 and Ω_2 are connected.

Proof. Introduce the following relation $\rho_{i\mu}$ in the ternary semigroup $R_{i\mu} : \forall (\varphi_1, \phi_1)$, $(\varphi_2, \phi_2) \in R_{i\mu}$, $((\varphi_1, \phi_1), (\varphi_2, \phi_2)) \in \rho_{i\mu}$ if and only if $\forall \xi \in A_i, \eta \in B_\mu, \varphi_1(\xi) = \varphi_2(\xi), \phi_1(\eta) = \phi_2(\eta)$. The relation $\rho_{i\mu}$ is a congruence on $R_{i\mu}$. Indeed, if $(\varphi_1, \phi_1), (\varphi_2, \phi_2)$ are any two elements of $R_{i\mu}$ such that $((\varphi_1, \phi_1), (\varphi_2, \phi_2)) \in \rho_{i\mu}$ and $(x_1, y_1), (x_2, y_2) \in R_{i\mu}$, then

$$\begin{gathered}(\varphi_1, \phi_1) \overset{\rho_{i\mu}}{\sim} (\varphi_2, \phi_2) \to \{\forall \xi \in A_i, \eta \in B_\mu, (x_i, y_i) \in R_{i\mu} \\ \varphi_1 y_2 x_1(\xi) = \varphi_2 y_2 x_1(\xi), y_1 x_2 \phi_1(\eta) = y_1 x_2 \phi_2(\eta)\} \to \\ \forall (x_i, y_i) \in R_{i\mu}, \\ ([(x_1, y_1)(x_2, y_2)(\varphi_1, \phi_1)], [(x_1, y_1)(x_2, y_2)(\varphi_2, \phi_2)]) \in \rho_{i\mu}.\end{gathered}$$

Analogously, it can be shown that

$$\begin{gathered}([(x_1, y_1)(\varphi_1, \phi_1)(x_2, y_2)], [(x_1, y_1)(\varphi_2, \phi_2)(x_2, y_2)]) \in \rho_{i\mu}, \\ ([(\varphi_1, \phi_1)(x_1, y_1)(x_2, y_2)], [(\varphi_2, \phi_2)(x_1, y_1)(x_2, y_2)]) \in \rho_{i\mu}.\end{gathered}$$

Consider the mapping $f(\varphi, \phi) = (\varphi', \phi')$ from $R_{i\mu}$ to $K(A_i, B_\mu)$ such that

$$\varphi|_{A_i} = \varphi', \phi|_{B_\mu} = \phi'.$$

Clearly, the mapping f is a homomorphism of $R_{i\mu}$ to $K(A_i, B_\mu)$. Now, let the ternary semigroup $R_{i\mu}$ be simple and let at least one of the spaces Ω_1 or Ω_2 be connected. Suppose that Ω_1 is disconnected but Ω_2 is connected. The elements $(\varphi_1, \phi_1), (\varphi_2, \phi_2)$ can be found in $R_{i\mu}$ such that $((\varphi_1, \phi_1), (\varphi_2, \phi_2)) \in \rho_{i\mu}$ (φ_1, φ_2 can map Ω_1 to two disjoint balls $E_1, E_2 \subset \Omega_2$ and because of this $\varphi_1(\xi) \neq \varphi_2(\xi), \xi \in \Omega_1$). Besides, there are more than one element in each class of $\rho_{i\mu}$. Indeed, if φ is a homeomorphism of Ω_1 to the closed ball $E_1 \subset \Omega_2$ and g is a homeomorphism of Ω_1 to the closed ball $E_2 \subset \Omega_2$ such that $E_1 \cap E_2 = \emptyset, E_1, E_2 \subset E_3$, where E_3 is a closed ball in Ω_2, then the mapping

$$\chi(\alpha) = \begin{cases} \varphi(\alpha), \text{ if } \alpha \in A_i, \\ g(\alpha), \text{ if } \alpha \notin A_i, \end{cases}$$

where A_i is a component of Ω_1, is a homomorphism of Ω_1 to Ω_2. Clearly, $\chi \neq \varphi$ and $\chi|_{A_i} = \varphi|_{A_i}$. Therefore $((\chi, \phi), (\varphi, \phi)) \in \rho_{i1}$. Here ϕ denotes some homeomorphism of Ω_2 into the interior of some n-element containd in Ω_1 and ρ_{i1} is some congruence on R_{i1} $(B_1 = \Omega_2)$. Then it follows that f is a homomorphism from R_{i1} onto $f(R_{i1}) \subset K(A_i, \Omega_2)$, which contains more than one element. But f is not an isomorphism, which contradicts the simplicity of R_{i1}.

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Топологиялық түрлендірудің тернарлы жартылайтоптары

Тернарлы жартылайтоп — бұл ассоциативті тернарлы операциясы бар босемес жиын. Мақаланың мақсаты — ақырлы өлшемді евклид кеңістіктерінің ашық жиынтығын гомеоморфты қайта құру жұптарының тернарлы жартылайтоптарымен сипаттау және Л.М. Глушкиннің ақырлы өлшемді евклид кеңістіктерінің гомеоморфты түрлендірудің жартылайтоптарына қатысты кейбір нәтижелерін теріс жартылайтоптарға тарату.

Кілт сөздер: Евклид n-кеңістік, тернарлы жартылайтоп, гомеоморфты түрлендіру.

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Тернарные полугруппы топологических преобразований

Тернарная полугруппа — это непустое множество с ассоциативной тернарной операцией. Цель настоящей статьи — охарактеризовать открытые множества конечномерных евклидовых пространств тернарными полугруппами пар гомеоморфных преобразований и распространить на тернарные полугруппы некоторые результаты Л.М. Глушкина, касающиеся полугрупп гомеоморфных преобразований конечномерных евклидовых пространств.

Ключевые слова: евклидово n-пространство, тернарная полугруппа, гомеоморфные преобразования.

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