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NONLINEAR INTEGRAL-DIFFERENTIAL EQUATION WITH ZERO OPERATOR OF DIFFERENTIAL PART AND RAPIDLY OSCILLATING COSINE

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In this work, the regularization method of S.A. Lomov (1; 2) is generalized for problems for an integro-differential equation with a rapidly changing kernel and with a right-hand side depending on a rapidly oscillating cosine

$$\varepsilon\frac{dy}{dt} = \int\limits_0^t e^{\frac{1}{\varepsilon}\int\limits_s^t \mu(\theta)d\theta} K(t,s)y(s,\varepsilon)ds + h(t)\cos\frac{\beta(t)}{\varepsilon} + \varepsilon f\left(y,t\right),\ y(0,\varepsilon) = y^0,\ t\in[0,T]. \qquad (1)$$

Problem (1) is considered under the following conditions:

1) $\mu(t), \beta'(t), h(t) \in \mathrm{C}^\infty([0,T],R),\ K(t,s) \in \mathrm{C}^\infty(0 \le s \le t \le T, R)$;

2) $\mu(t) < 0, \beta'(t) > 0\ \ \forall t \in [0,T]$;

3) $f(y,t)$ is a polynomial, i.e. $f(y,t) = \sum\limits_{m=0}^{N} f_m(t)y^m$ with the coefficients $f_m(t) \in \quad \in C^\infty\left([0,T],R\right), m = \overline{0,N},\ \ N < \infty$.

We introduce a new unknown function

$$z = \int\limits_0^t e^{\frac{1}{\varepsilon}\int\limits_s^t \mu(\theta)d\theta} K(t,s)y(s,\varepsilon)ds.$$

Differentiating it by t, we will have

$$\frac{dz}{dt}=K(t,t)y+\frac{\mu(t)}{\varepsilon}\int\limits_0^t e^{\frac{1}{\varepsilon}\int\limits_s^t \mu(\theta)d\theta}K(t,s)y(s,\varepsilon)ds+\int\limits_0^t e^{\frac{1}{\varepsilon}\int\limits_s^t \mu(\theta)d\theta}\frac{\partial K(t,s)}{\partial t}y(s,\varepsilon)ds \quad \Leftrightarrow$$

$$\Leftrightarrow \varepsilon\frac{dz}{dt}=\mu(t)z+\varepsilon K(t,t)y+\varepsilon\int_0^t e^{\frac{1}{\varepsilon}\int\limits_s^t \mu(\theta)d\theta}\frac{\partial K(t,s)}{\partial t}y(s,\varepsilon)ds.$$

Instead of (1), we get the system

$$\begin{array}{l}\varepsilon\frac{dw}{dt}=A(t)w+\varepsilon A_1(t)w+\varepsilon\int\limits_0^t e^{\frac{1}{\varepsilon}\int\limits_s^t \mu(\theta)d\theta}G(t,s)w(s,\varepsilon)ds+\\ +H(t,\varepsilon)+\varepsilon F(w,t),\quad w(0,\varepsilon)=w^0\equiv\{y^0,0\}\end{array}\qquad(2)$$

where $w=\{y,z\}, F(w,t)=\{f(y,t),0\}, H(t,\varepsilon)=\{h(t)\cos\frac{\beta(t)}{\varepsilon},0\}$, and the matrices $A(t)$, $A_1(t), G(t,s)$ are of the form

$$A(t)=\begin{pmatrix}0&1\\0&\mu(t)\end{pmatrix}, A_1(t)=\begin{pmatrix}0&0\\K(t,t)&0\end{pmatrix}, G(t,s)=\begin{pmatrix}0&0\\\frac{\partial K(t,s)}{\partial t}&0\end{pmatrix}.$$

For convenience, let us denote $\lambda_1(t)=-i\beta'(t)$, $\lambda_2(t)=+i\beta'(t)$, $\lambda_3(t)=\mu(t)$. It should be noted that in the study of problem (2.1), resonances can arise between the spectral values $\lambda_1(t)\equiv-i\beta'(t)$, $\lambda_2(t)\equiv+i\beta'(t)$ and $\lambda_3(t)\equiv\mu(t)$, i.e. for all $t\in[0,T]$, the identities ($m=(m_1,m_2,m_3)$ – multi-index, $|m|=m_1+m_2+m_3$)

$$\begin{array}{l}(m,\lambda(t))\equiv m_1\lambda_1(t)+m_2\lambda_2(t)+m_3\lambda_3(t)\equiv0, |m|\geq2,\\ (m,\lambda(t))\equiv m_1\lambda_1(t)+m_2\lambda_2(t)+m_3\lambda_3(t)\equiv\lambda_j(t), |m|\geq2, j=\overline{1,3}\end{array}\qquad(2_0)$$

hold. However, due to the fact that $\lambda_1(t)$ and $\lambda_2(t)$ is a purely imaginary functions, but $\lambda_3(t)$ is a real function, we obtain the following sets of resonant multi-intexes:

$$\begin{array}{l}\Gamma_1=\{(s+1,s,0)\,,s\geq1\}\cup\{(s+1,s,1)\,,s\geq0\}\,,\\ \Gamma_2=\{(s,s+1,0)\,,s\geq1\}\cup\{(s,s+1,1)\,,s\geq0\}\end{array}$$

with multi-indexes from Γ_1 satisfying the relation $(m,\lambda(t))\equiv\lambda_1(t)\,,|m|\geq2$, and multiindexes from Γ_2 satisfying the relation $(m,\lambda(t))\equiv\lambda_2(t)\,,|m|\geq2$ (here s is an integer).

We introduce regularizing variables (see (1))

$$\tau_j=\frac{1}{\varepsilon}\int\limits_0^t\lambda_j(\theta)d\theta\equiv\frac{\psi_j(t)}{\varepsilon}, j=\overline{1,3}$$

and consider the following problem:

$$\begin{array}{l}\varepsilon\frac{\partial\tilde{w}}{\partial t}+\sum\limits_{j=1}^3\lambda_j(t)\frac{\partial\tilde{w}}{\partial\tau_j}-A(t)\tilde{w}-\varepsilon\int\limits_0^t e^{\frac{1}{\varepsilon}\int\limits_s^t\lambda_3(\theta)d\theta}G(t,s)\tilde{w}(s,\frac{\psi(s)}{\varepsilon},\varepsilon)ds-\\ -\varepsilon A_1(t)\tilde{w}=H(t,\tau)+\varepsilon F(\tilde{w},t),\quad \tilde{w}(t,\tau\;,\sigma,\varepsilon)|_{t=0,\tau=0}=w^0\end{array}\qquad(3)$$

where $H(t,\tau) = \{\frac{h(t)}{2}(e^{\tau_1}\sigma_1 + e^{\tau_2}\sigma_2), 0\}$, $\sigma_1 = e^{-\frac{i}{\varepsilon}\beta(0)}$, $\sigma_2 = e^{+\frac{i}{\varepsilon}\beta(0)}$, for the function $\tilde{w} = \tilde{w}(t,\tau,\sigma,\varepsilon)$, where $\tau = (\tau_1,\tau_2,\tau_3)$, $\psi = (\psi_1,\psi_2,\psi_3)$. It is clear that if $\tilde{w} = \tilde{w}(t,\tau,\sigma,\varepsilon)$ is the solution to problem (3), then the vector function $w = \tilde{w}\left(t, \frac{\psi(t)}{\varepsilon}, \sigma, \varepsilon\right)$ is an exact solution to problem (2); therefore, the problem (3) is extended with respect to the problem (2). However, the problem (3) cannot be considered completely regularized, since the integral operator

$$J\tilde{w} \equiv J\left(\tilde{w}\left(t,\tau,\varepsilon\right)\big|_{t=s,\tau=\frac{\psi(s)}{\varepsilon}}\right) = \varepsilon\int_0^t e^{\frac{1}{\varepsilon}\int_s^t \lambda_3(\theta)d\theta} G(t,s)\tilde{w}\left(s, \frac{\psi(s)}{\varepsilon}, \varepsilon\right) ds.$$

is not regularized. To regularize it, we introduce a class $M_\varepsilon = U|_{\tau=\frac{\psi(t)}{\varepsilon}}$,that is asymptotically invariant with respect to the operator J (see (1), p. 62]). In this case, we take U as the space of vector functions representable by sums of the form

$$\begin{gathered} w(t,\tau,\sigma) = w_0(t,\sigma) + \sum_{j=1}^{3} w_j(t,\sigma)e^{\tau_j} + \sum_{2\le|m|\le N_w}^{*} w^{(m)}(t,\sigma)e^{(m,\tau)}, \\ w_j(t,\sigma), w^{(m)}(t,\sigma) \in C^\infty\left([0,T], C^2\right), j = \overline{0,3}, \quad 2 \le |m| \le N_w \end{gathered} \tag{4}$$

where $(*)$ over the sum means that there are no terms with $|m| \ge 2$, satisfying the resonance relations (2_0). Let us show that the class M_ε, is asymptotically invariant with respect to the operator J. To do this, it is necessary to show that the image $Jw(t,\tau)$ on functions of the form (4) can be represented as a series

$$Jw(t,\tau) = \sum_{k=0}^{\infty} \varepsilon^k \left(\sum_{j=1}^{3} w_k(t)e^{\tau_j} + w_k^{(0)}(t)\right)\Bigg|_{\tau=\frac{\psi(t)}{\varepsilon}}$$

converging asymptotically to Jw (as $\varepsilon \to +0$) uniformly with respect to $t \in [0,T]$. Substituting (4) in $Jw(t,\tau)$, we have

$$\begin{gathered} Jw(t,\tau) = e^{\frac{1}{\varepsilon}\int_0^t \lambda_3(\theta)d\theta} \int_0^t G(t,s)w_0(s)e^{-\frac{1}{\varepsilon}\int_0^s \lambda_3(\theta)d\theta} ds + \\ + \sum_{j=1}^{2} e^{\frac{1}{\varepsilon}\int_0^t \lambda_3(\theta)d\theta} \int_0^t G(t,s)w_j(s)e^{\frac{1}{\varepsilon}\int_0^s (\lambda_j(\theta)-\lambda_3(\theta))d\theta} ds + e^{\frac{1}{\varepsilon}\int_0^t \lambda_3(\theta)d\theta} \int_0^t G(t,s)w_3(s)ds + \\ + \sum_{|m|\ge 2}^{*} e^{\frac{1}{\varepsilon}\int_0^t \lambda_3(\theta)d\theta} \int_0^t G(t,s)w^{(m)}(s)e^{\frac{1}{\varepsilon}\int_0^s [(m,\lambda(\theta))-\lambda_3(\theta)]d\theta} ds. \end{gathered}$$

By applying the operation of integration by parts, we will have

$$\begin{gathered} Jw(t,\tau) = e^{\frac{1}{\varepsilon}\int_0^t \lambda_3(\theta)d\theta} \int_0^t G(t,s)w_3(s)ds + \sum_{\nu=0}^{\infty} \varepsilon^{\nu+1}(-1)^\nu \left[\left(I_0^\nu(G(t,s)w_0(s))\right)_{s=t} - \right. \\ \left. -\left(I_0^\nu(G(t,s)w_0(s))\right)_{s=0} e^{\frac{1}{\varepsilon}\int_0^t \lambda_3(\theta)d\theta}\right] + \sum_{j=1}^{2}\sum_{\nu=0}^{\infty} (-1)^\nu \varepsilon^{\nu+1} \left[\left(I_j^\nu(G(t,s)w_1(s))\right)_{s=t} e^{\frac{1}{\varepsilon}\int_0^t \lambda_j(\theta)d\theta} - \right. \end{gathered}$$

$$-\left(I_j^{\nu}(G(t,s)w_1(s))\right)_{s=0} e^{\frac{1}{\varepsilon}\int\limits_0^t \lambda_3(\theta)d\theta}\Bigg] +$$
$$+\sum_{2\le|m|\le N_w}^{*}\sum_{\nu=0}^{\infty}(-1)^{\nu}\varepsilon^{\nu+1}\Bigg[\left(I_m^{\nu}(G(t,s)w^{(m)}(s))\right)_{s=t} e^{\frac{1}{\varepsilon}\int\limits_0^t (m,\lambda(\theta))d\theta} -$$
$$-\left(I_m^{\nu}(G(t,s)w^{(m)}(s))\right)_{s=0} e^{\frac{1}{\varepsilon}\int\limits_0^t \lambda_3(\theta)d\theta}\Bigg].$$

It is easy to show (see, for example, (3), pp. 291-294) that this series converges asymptotically for $\varepsilon \to +0$ to (uniformly in $t \in [0,T]$). This means that the class M_ε is asymptotically invariant (for $\varepsilon \to +0$) with respect to the operator J.

Then the image $Jw(t,\tau)$ can be written in the form

$$Jw(t,\tau) = R_0 w(t,\tau) + \sum_{m=0}^{\infty}\varepsilon^{m+1}R_{m+1}w(t,\tau)$$

where $\tau = \frac{\psi(t)}{\varepsilon}$. Let us now extend the operator J to series of the form

$$\tilde{w}(t,\tau,\varepsilon) = \sum_{k=0}^{\infty} w_k(t,\tau) \tag{5}$$

with coefficients $w_k(t,\tau) \in U, k \ge 0$.

Definition. A formal extension $\tilde{J}$ of the operator J on series of the form (5) is the operator

$$\tilde{J}\tilde{w}(t,\tau,\varepsilon) \stackrel{def}{=} \sum_{\nu=0}^{\infty}\varepsilon^{\nu}\sum_{s=0}^{\nu}R_{\nu-s}w_s(t,\tau). \tag{6}$$

Despite the fact that the extension $\tilde{J}$ of the operator J is defined formally, it is quite possible to use it in constructing an asymptotic solution of finite order of ε. Now it is easy to write out the regularized (with respect to (3)) problem:

$$\begin{array}{c} \varepsilon\frac{\partial\tilde{w}}{\partial t} + \sum\limits_{j=1}^{3}\lambda_j(t)\frac{\partial\tilde{w}}{\partial\tau_j} - A(t)\tilde{w} - \varepsilon A_1(t)\tilde{w} - \varepsilon\tilde{J}\tilde{w} = \\ = H(t,\tau) + \varepsilon F(\tilde{w},t), \quad \tilde{w}(t,\tau\ ,\sigma,\varepsilon)|_{t=0,\tau=0} = w^0 \end{array} \tag{7}$$

where $\tilde{J}$ is the operator (6).

Substituting series (5) into (7) and equating the coefficients at the same degrees of ε, we obtain the following iterative problems:

$$L_0 w_0(t,\tau) \equiv \sum_{j=1}^{3}\lambda_j(t)\frac{\partial w_0}{\partial\tau_j} - A(t)w_0 = \begin{pmatrix}\frac{h(t)}{2}\sigma_1\\0\end{pmatrix}e^{\tau_1} + \begin{pmatrix}\frac{h(t)}{2}\sigma_2\\0\end{pmatrix}e^{\tau_2}, \ \ w_0(0,0) = w^0; \tag{8_0}$$

$$L_0 w_1(t,\tau) = -\frac{\partial w_0}{\partial t} + A_1(t)w_0 + F(w_0,t) + R_0 w_0, \quad w_1(0,0) = 0; \tag{8_1}$$

$$\ldots$$

$$L_0 w_k(t,\tau) = -\frac{\partial w_{k-1}}{\partial t} + A_1(t)w_{k-1} + P_k(w_0,\dots,w_{k-1},t) + R_0 w_{k-1} + \\ + R_1 w_{k-2} + \dots + R_k w_0, \quad w_k(0,0)=0,\ k \geq 1 \qquad (8_k)$$

where $P_k(w_0,\dots,w_{k-1},t)$ is some polynomial of $w_0,\dots,w_{k-1}$, linear with respect to w_{k-1}.

Let us introduce a scalar product (at each $t \in [0,T]$) in the space U :

$$< y,z > \equiv < y_0(t) + \sum_{i=1}^{n} y_i(t) e^{\tau_i} + \sum_{2\leq|m|\leq N_z}^{*} y^m(t) e^{(m,\tau)}, \\ z_0(t) + \sum_{i=1}^{n} z_i(t) e^{\tau_i} + \sum_{2\leq|m|\leq N_w}^{*} z^m(t) e^{(m,\tau)} > \equiv \\ \equiv (y_0(t),\, z_0(t)) + \sum_{i=1}^{n} (y_i(t), z_i(t)) + \sum_{2\leq|m|\leq \min(N_z,N_w)}^{*} (y^m(t), z^m(t))$$

where the usual scalar product in the complex space ^{2} is denoted by $(*,*)$.

Each of the iterative systems has the form

$$L_0 w(t,\tau) \equiv \sum_{j=1}^{3} \lambda_j(t) \frac{\partial w}{\partial \tau_j} - A(t)w = P(t,\tau) \qquad (9)$$

where $P(t,\tau) = P_0(t) + \sum_{j=1}^{3} P_j(t)e^{\tau_j} + \sum_{N_P\geq|m|\geq 2}^{*} P^{(m)}(t)e^{(m,\tau)} \in U.$

Theorem. Let conditions 1) – 3) be satisfied and $P(t,\tau) \in U$. For system (9) to have a solution in U it is necessary and sufficient that

$$< P(t,\tau),\, \chi_j(t) e^{\tau_j} > \equiv 0 \Leftrightarrow (P_j(t), \chi_j(t)) = 0, \ \ j = 0,3, \ \forall t \in [0,T]. \qquad (10)$$

where $\chi_0(t), \chi_3(t)$ are the eigenfunctions of the matrix $A^*(t)$ corresponding to the eigenvalues $\bar{\lambda}_0(t) \equiv 0,\ \bar{\lambda}_3(t) \equiv \mu(t)$, respectively.

Remark. If conditions (10) are satisfied, that system (9) has the following solution in the space

$$w(t,\tau) = \alpha_0(t)\varphi_0(t) + \frac{(P_0(t),\chi_3(t))}{-\lambda_3(t)}\varphi_3(t) + \left[\frac{(P_1(t),\chi_0(t))}{\lambda_1(t)}\varphi_0(t) + \frac{(P_1(t),\chi_3(t))}{\lambda_1(t)-\lambda_3(t)}\varphi_3(t)\right] e^{\tau_1} + \\ + \left[\frac{(P_2(t),\chi_0(t))}{\lambda_2(t)}\varphi_0(t) + \frac{(P_2(t),\chi_3(t))}{\lambda_2(t)-\lambda_3(t)}\varphi_3(t)\right] e^{\tau_2} + \\ + \left[\alpha_3(t)\varphi_3(t) + \frac{(P_3(t),\chi_0(t))}{\lambda_3(t)}\varphi_0(t)\right] e^{\tau_3} + \sum_{N_P\geq|m|\geq 2}^{*} \left\{\left[(m,\lambda(t))I - A(t)\right]^{-1} P^{(m)}(t)\right\} e^{(m,\tau)}$$

where $\alpha_j(t) \in C^{\infty}([0,T],C)$ are arbitrary functions, $j = 0,3$.

We will not prove the unique solvability of iterative problems. Note that when solving two consecutive iterative problems (8_k) and (8_{k+1}), the solution to the problem (8_k) in the space U will be found uniquely.

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REDUCTION THEOREMS FOR THE DISCRETE NONLINEAR OPERATOR ON THE CONES OF MONOTONE SEQUENCES

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Let $\{u_k\}$, $\{a_n\}$ and $\{b_n\}$ be given non-negative sequences. Let $p, q \in (0,\infty)$. We will investigate the following inequalities:

$$\left(\sum_{n=1}^{\infty}\left(\sup_{n\le i<\infty} u_i \sum_{k=1}^{i} x_k\right)^q a_n\right)^{\frac{1}{q}} \le C\left(\sum_{n=1}^{\infty} x_n^p b_n\right)^{\frac{1}{p}}, \tag{1}$$

$$\left(\sum_{n=1}^{\infty}\left(\sup_{1\le i<n} u_i \sum_{k=1}^{i} x_k\right)^q a_n\right)^{\frac{1}{q}} \le C\left(\sum_{n=1}^{\infty} x_n^p b_n\right)^{\frac{1}{p}} \tag{2}$$

for non-negative, non-increasing sequences $x = \{x_n\}$ and the constant $C > 0$ is independent of x.

Theorem 1. Let $0 < q \le \infty,\ 1 < p < \infty$. Assume that $\{a_n\}$ and $\{b_n\}$ are given non-negative weight sequences. Then the inequality (1) holds for all non-negative, non-increasing sequences $\{x_n\}$ if and only if the following inequality:

$$\left(\sum_{n=1}^{\infty}\left(\sup_{n\le i<\infty} u_i \sum_{k=1}^{i}\left(\sum_{j=k}^{\infty} y_j\right)\right)^q a_n\right)^{\frac{1}{q}} \le C\left(\sum_{n=1}^{\infty} y_n^p B_{n+1}^{p-1} B_n b_{n+1}^{1-p}\right)^{\frac{1}{p}}, \tag{3}$$

holds for all non-negative sequences $\{y_n\}$, where $B_n = \sum_{k=1}^{n} b_k, \quad n \in \mathbb{N}$.

Theorem 2. Let $0 < q \le \infty,\ 1 < p < \infty$. Assume that $\{a_n\}$ and $\{b_n\}$ are given non-negative weight sequences.Then the inequality (2) holds for all non-negative, non-increasing sequences $\{x_n\}$ if and only if for any $\alpha > 0$ the following inequality:

$$\left(\sum_{n=1}^{\infty}\left(\sup_{1\le i<n} u_i \sum_{i=n}^{\infty}\frac{1}{B_i^{\alpha+1}}\left(\sum_{k=1}^{i} B_k^{\alpha+1} y_k\right)\right)^q a_n\right)^{\frac{1}{q}} \le C\left(\sum_{n=1}^{\infty} y_n^p B_n^p b_n^{1-p}\right)^{\frac{1}{p}}, \tag{4}$$