Applied Mathematics and Computation 399 (2021) 126009

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On a boundary value problem for the heat equation and a singular integral equation associated with it

Meiramkul Amangaliyeva, PhD[a], Muvasharkhan Jenaliyev, Prof[a,*], Sagyndyk Iskakov, PhD[b], Murat Ramazanov, Prof[b]

[a] Department of Differential Equations, Institute of Mathematics and Mathematical Modeling, Pushkin str., 125, Almaty 050010, Kazakhstan
[b] Department of Mathematics and Informational Technologies, Buketov Karaganda State University, Universitetskaya str., 28, Karaganda 100028, Kazakhstan

ARTICLE INFO

Article history:
Received 1 February 2018
Revised 4 December 2020
Accepted 10 January 2021
Available online 30 January 2021

MSC:
35K05
45D05
45Exx
45P05

Keywords:
Heat equation
Degenerate domain
Volterra integral equation
Singular integral equation
Integral operator
Resolvent

ABSTRACT

In this paper we study the solvability of a singular integral equation arising in the theory of boundary value problems for the heat equation in an infinite angular domain. The particular case of the corresponding homogeneous integral equation was investigated earlier in [1, 2] and it was shown that in a weight class of essentially bounded functions it has, along with a trivial solution, a family of non-trivial solutions up to a constant factor. In this paper we study the more general case of a nonhomogeneous integral equation for which a representation of the general solution is found with using the resolvent constructed by us. Estimates of the resolvent and of the solution of the boundary value problem are established.

1. Introduction

In domain $G = \{x, t : 0 < x < kt,\ t > 0\}$ it is considered the boundary value problem

$$\frac{\partial u(x,t)}{\partial t} - a^2 \frac{\partial^2 u(x,t)}{\partial x^2} = g(x,t),\ \ \{x,t\} \in G; \tag{1}$$

$$\frac{\partial u(x,t)}{\partial x}\Big|_{x=0} = u_0(t),\ \ b\frac{\partial u(x,t)}{\partial x}\Big|_{x=kt} + \frac{d\tilde{u}(t)}{dt} = u_1(t); \tag{2}$$

where $\tilde{u}(t) = u(kt,t),\ b = \text{const} > 0,\ k = \text{const} > 0.$

* Corresponding author.
E-mail addresses: jenalika@gmail.com (M. Amangaliyeva), muvasharkhan@gmail.com (M. Jenaliyev), isagyndyk@mail.ru (S. Iskakov), ramamur@mail.ru (M. Ramazanov).

https://doi.org/10.1016/j.amc.2021.126009

We note that problem (1)–(2), is studied in [3] for a finite time interval, where it was noted that the case of a nonhomogeneous boundary value problem "turns out to be useful in the study of certain problems with free boundaries". For example, for a single-phase Stefan problem "under the following assumptions: liquid phase with a positive temperature $u(x,t)$ occupies a segment $0<x<s(t)$, at $x=0$ a positive flux of heat is given, and the free boundary $x=s(t)$ begins at the solid wall $x=0$, i.e. the condition $s(0)=0$ is satisfied". In the work [3] a theorem on the unique solvability of the considered boundary value problem in weight Holder spaces is established.

Previously, under the conditions $b=1,\ k=1$ in [1,2] we have studied the homogeneous and nonhomogeneous integral equations, to which the homogeneous and inhomogeneous boundary value problems (1)–(2) are reduced. For a homogeneous problem it was established that, along with the trivial solution in the class of essentially bounded functions with a given weight, there exist nontrivial solutions up to a constant factor. In [2] the general solution of the inhomogeneous integral equation was found. We also mention the works [4–7], that are close to our subject, in which the boundary value problems are studied for a parabolic equation in domains, that are linearly and nonlinearly degenerate at the initial moment of time. However, in contrast to our work, they study boundary value problems under the condition of a limited time interval. In works [4–7] theorems on unique solvability in anisotropic Sobolev spaces are established. It should be noted that the need to study mathematical problems from the works [1,2,4–10] arises in the questions of describing temperature regimes in high-precision electro-contact devices, the spread of pollutants in liquid and gas areas, etc.

2. Discussion of BVP (1)–(2)

Our approach of research was initially tested on a simple case, as it seemed to us, where $b=1, k=1$ [1,2]. However, our further study showed that the feature of the problem (related to its reduction to a singular integral equation) was preserved.

However, the more general case where the coefficients b and k are only required to be strictly positive, in our view, is important for at least the following two reasons:

(i) When considering a problem with a nonlinear law of motion $x=s(t)$ of a moving part of the boundary, we can apply our results from this paper if we require additionally from the function $s(t)$, for example, the following properties:

(i.1) The function $s(t)$ is differentiable on the interval $(0,\infty)$ and there is a sufficiently small $t^*>0$ such that,

$$s'(t)=\mu>0 \ \text{at} \ t\in(0,t^*); \quad s'(t)\geq 0 \ \text{at} \ t>t^*.$$

(i.2) The motion of the boundary obeys the linear law: $s(t)=\mu t,\ 0<t<t^*$. We have the following boundary value problem

$$\frac{\partial u_1(x,t)}{\partial t}-a^2\frac{\partial^2 u_1(x,t)}{\partial x^2}=g(x,t),\ \{0<x<\mu t,\ 0<t<t^*\}; \tag{1a}$$

$$\frac{\partial u_1(x,t)}{\partial x}|_{x=0}=u_0(t),\ b\frac{\partial u_1(x,t)}{\partial x}|_{x=\mu t}+\frac{d\tilde{u}_1(t)}{dt}=u_1(t); \tag{2a}$$

where $\tilde{u}_1(t)=u_1(\mu t,t)$.

In this case, the equality $\mu=1$ will be a very restrictive and undesirable fact. Continuing obtained boundary value problem (1a)–(2a) to an infinite angle, we obtain a boundary value problem of type (1)–(2), where $k=\mu$. We can apply our results from present paper to solve it. Further, taking from the obtained solution its restriction on the interval $(0,t^*)$, we obtain a required solution to boundary value problem (1a)–(2a) in the domain $\{x,t:\ 0<x<\mu t,\ 0<t<t^*\}$.

(i.3) $s'(t)\geq 0$ on the interval (t^*,∞) and $s(t^*)=\mu t^*$. We have the following boundary value problem in a nondegenerating domain

$$\frac{\partial u_2(x,t)}{\partial t}-a^2\frac{\partial^2 u_2(x,t)}{\partial x^2}=g(x,t),\ \{0<x<s(t),\ t>t^*\}; \tag{1b}$$

$$\frac{\partial u_2(x,t)}{\partial x}|_{x=0}=u_0(t),\ b\frac{\partial u_2(x,t)}{\partial x}|_{x=s(t)}+\frac{d\tilde{u}_2(t)}{dt}=u_1(t),\ t>t^*; \tag{2b}$$

$$u_2(x,t)|_{t=t^*}=u_1(x,t^*),\ 0<x<\mu t^*; \tag{3b}$$

where $\tilde{u}_2(t)=u_2(s(t),t)$ and $u_1(x,t)$ is a solution to boundary value problem (1a)–(2a).

We introduce a new unknown function $v_2(x,t)=\frac{\partial u_2(x,t)}{\partial x}$. Formally, by differentiating equation (1 b) with respect to the variable x, we get:

$$\frac{\partial v_2(x,t)}{\partial t}-a^2\frac{\partial^2 v_2(x,t)}{\partial x^2}=\tilde{g}(x,t),\ 0<x<s(t),\ t>t^*; \tag{1c}$$

$$v_2(x,t)|_{x=0}=v_0(t),\ \left(\frac{\partial v_2(x,t)}{\partial x}+\tilde{a}\,v_2(x,t)\right)|_{x=s(t)}=v_1(t),\ t>t^*, \tag{2c}$$

$$v_2(x,t)|_{t=t^*}=\frac{\partial u_1(x,t^*)}{\partial x},\ 0<x<\mu t^*; \tag{3c}$$

where $\tilde{g}(x,t) \equiv \frac{\partial g(x,t)}{\partial x}$, $\nu_0(t) \equiv u_0(t)$, $\nu_1(t) \equiv \frac{u_1(t)}{a^2}$, $\tilde{a} = \frac{c}{a^2}$, $c = b + s'(t)$.

Using thermal potentials, boundary problem (1c)–(3c) reduces to solving the classical (regular) Volterra integral equation of the second kind, which is uniquely solvable ([11], part 6, 4.1–4.2). By a solution to problem (1c)–(3c), using the inverse transformation $u_2(x,t) = \int\limits_0^x \nu_2(\xi,t)d\xi + C_1$, we will have a solution to problem (1b)–(3b) up to a constant C_1. Note that this does not contradict the formulation of boundary value problem (1)–(2), the solution of which is found up to a constant term.

Further, by gluing together solutions to boundary value problems (1a)–(2a) and (1b)–(3b), we obtain a solution to the original boundary value problem with the nonlinear law of motion $x = s(t)$ of the part of the boundary, where that law satisfies the aforesaid conditions from (i.1)–(i.3).

However, we would like to note the following. For the general nonlinear law of motion $x = s(t)$ of a part of the boundary, our approach is not applicable, since in our work we reduce the obtained singular integral equation to an integral equation with a difference kernel. This is the key point of our research.

(ii) there are some difficulties in applying our approach to the study of boundary problems of heat conduction in degenerating domains with degeneration at the initial moment of time at $b > 0$ and $k > 0$ in comparison with the case when $b = 1$ and $k = 1$. Namely,

(ii.1) in this paper, the weighted space of essentially bounded functions (14) for the solution of integral Eq. (10) differs from the class of solutions (11) for integral Eq. (9) in [2]. Although in these works, at small values of the time variable, the singularity $t^{-1/2}$ of the solution is preserved, however, in the present work, from the solution of Eq. (10) it is additionally required that it decreases at infinity no more slowly than the given exponent, which depends on the parameter k. Moreover, the solution of Eq. (10) should decrease faster as the parameter k increases, which corresponds to the increasement of an angle of the domain of independent variables. The latter is not required in [2].

At the same time, it should be noted that in the presented paper the exponential order of decrease is precisely determined depending on the parameter k.

(ii.2) In this paper, it has been clarified: how does the value of the angle in the domain of independent variables affect the class of solutions of a singular integral equation? From the increasement of value of the parameter k, that corresponds to the increasement of an angle in the domain of independent variables, the decrease rate of the solution of the singular integral equation at $t \to \infty$ should increase. At the same time, it should be noted that in the presented paper the exponential order of decrease is precisely determined depending on the parameter k.

(ii.3) We also note that in the present paper, at finding the resolvent, series (27) arises, which is associated with the expansion of the fractional Newton binomial $(1+\alpha)^{2c_1-1}$, $c_1 = (b+k)/k$ (with binomial coefficients $C_m^{2c_1-1}$), which causes certain difficulties in further calculations. Whereas in our previous work [2] we dealt with a finite sum (21) over the index m, which in [2] greatly simplified the study.

3. Transformation of problem (1)–(2) and reduction of it to an integral equation

We introduce the function $\nu(x,t) = \frac{\partial u(x,t)}{\partial x}$. Formally by differentiating Eq. (1) with respect to the variable x, we get:

$$\frac{\partial \nu(x,t)}{\partial t} - a^2 \frac{\partial^2 \nu(x,t)}{\partial x^2} = \tilde{g}(x,t), \; 0 < x < kt, \; t > 0; \tag{3}$$

$$\nu(x,t)|_{x=0} = \nu_0(t), \; \left(\frac{\partial \nu(x,t)}{\partial x} + \tilde{a}\,\nu(x,t) \right)|_{x=kt} = \nu_1(t), \tag{4}$$

where $\tilde{g}(x,t) \equiv \frac{\partial g(x,t)}{\partial x}$, $\nu_0(t) \equiv u_0(t)$, $\nu_1(t) \equiv \frac{u_1(t)}{a^2}$, $\tilde{a} = \frac{c}{a^2}$, $c = b + k$.

The solution of problem (3)–(4) we are looking for as a sum of thermal potentials ([11], part 6, 4.1–4.2):

$$\begin{aligned} \nu(x,t) = & \frac{1}{2a\sqrt{\pi}} \int\limits_0^t \int\limits_0^\infty \frac{1}{(t-\tau)^{1/2}} \exp\left\{ -\frac{(x-\xi)^2}{4a^2(t-\tau)} \right\} \tilde{g}(\xi,\tau)d\xi d\tau + \\ & + \frac{1}{4a^3\sqrt{\pi}} \int\limits_0^t \frac{x}{(t-\tau)^{3/2}} \exp\left\{ -\frac{x^2}{4a^2(t-\tau)} \right\} \nu(\tau)d\tau + \\ & + \frac{1}{2a\sqrt{\pi}} \int\limits_0^t \frac{1}{(t-\tau)^{1/2}} \exp\left\{ -\frac{(x-k\tau)^2}{4a^2(t-\tau)} \right\} \varphi(\tau)d\tau, \end{aligned} \tag{5}$$

where the functions $\nu(t)$ and $\varphi(t)$ are unknown and must be defined.

Satisfying solution (5) to the first condition in (4), we have:

$$\nu(t) = -\frac{a}{\sqrt{\pi}} \int\limits_0^t \frac{1}{(t-\tau)^{1/2}} \exp\left\{ -\frac{k^2\tau^2}{4a^2(t-\tau)} \right\} \varphi(\tau)d\tau - \tag{6}$$

$$-\frac{a}{\sqrt{\pi}}\int_0^t\int_0^\infty \frac{1}{(t-\tau)^{1/2}}\exp\left\{-\frac{\xi^2}{4a^2(t-\tau)}\right\}\tilde{g}(\xi,\tau)d\xi d\tau+2a^2\nu_0(t). \tag{6}$$

Using representation (5) and equality (6), we obtain the following expression for the solution of problem (3)–(4):

$$\nu(x,t)=\frac{1}{2a\sqrt{\pi}}\int_0^t\frac{1}{(t-\tau)^{1/2}}\left[-\exp\left\{-\frac{(x+k\tau)^2}{4a^2(t-\tau)}\right\}+\right.$$
$$\left.+\exp\left\{-\frac{(x-k\tau)^2}{4a^2(t-\tau)}\right\}\right]\varphi(\tau)d\tau+F_1[x,t;\tilde{g};\nu_0], \tag{7}$$

where

$$F_1[x,t;\tilde{g};\nu_0]\equiv\frac{1}{2a\sqrt{\pi}}\int_0^t\frac{x}{(t-\tau)^{3/2}}\exp\left\{-\frac{x^2}{4a^2(t-\tau)}\right\}\nu_0(\tau)d\tau+$$
$$+\frac{1}{2a\sqrt{\pi}}\int_0^t\int_0^\infty\frac{1}{(t-\tau)^{1/2}}\left[-\exp\left\{-\frac{(x+\xi)^2}{4a^2(t-\tau)}\right\}\right.$$
$$\left.+\exp\left\{-\frac{(x-\xi)^2}{4a^2(t-\tau)}\right\}\right]\tilde{g}(\xi,\tau)d\xi d\tau. \tag{8}$$

Taking into account (7), according to the second boundary condition from (4) we have:

$$\left(\frac{\partial\nu(x,t)}{\partial x}+\frac{c}{a^2}\nu(x,t)\right)_{|_{x=kt}}=$$
$$=\frac{\varphi(t)}{2a^2}+\frac{1}{4a^3\sqrt{\pi}}\int_0^t\frac{k(t+\tau)}{(t-\tau)^{3/2}}\exp\left\{-\frac{k^2(t+\tau)^2}{4a^2(t-\tau)}\right\}\varphi(\tau)d\tau-$$
$$-\frac{1}{4a^3\sqrt{\pi}}\int_0^t\frac{k}{(t-\tau)^{1/2}}\exp\left\{-\frac{k^2(t-\tau)}{4a^2}\right\}\varphi(\tau)d\tau-$$
$$-\frac{c}{2a^3\sqrt{\pi}}\int_0^t\frac{1}{(t-\tau)^{1/2}}\exp\left\{-\frac{k^2(t+\tau)^2}{4a^2(t-\tau)}\right\}\varphi(\tau)d\tau+$$
$$+\frac{c}{2a^3\sqrt{\pi}}\int_0^t\frac{1}{(t-\tau)^{1/2}}\exp\left\{-\frac{k^2(t-\tau)}{4a^2}\right\}\varphi(\tau)d\tau=F_2[kt,t;\tilde{g};\nu_0;\nu_1], \tag{9}$$

where

$$F_2[x,t;\tilde{g};\nu_0;\nu_1]=-\frac{\partial F_1[x,t;\tilde{g};\nu_0]}{\partial x}-\frac{c}{a^2}F_1[x,t;\tilde{g};\nu_0]+\nu_1(t).$$

Using equalities $t+\tau=2t-(t-\tau)$, $(t+\tau)^2=(t-\tau)^2+4t\tau$, from (9) we have the following integral equation for the unknown function $\varphi(t)$:

$$\varphi(t)-\int_0^t K(t,\tau)\varphi(\tau)d\tau=f(t),\ t>0, \tag{10}$$

where

$$K(t,\tau)=k(t,\tau)\exp\left\{-\frac{k^2(t-\tau)}{4a^2}\right\}, \tag{11}$$

$$k(t,\tau)=-\frac{1}{2a\sqrt{\pi}}\left[\frac{2kt}{(t-\tau)^{3/2}}\exp\left\{-\frac{k^2t\tau}{a^2(t-\tau)}\right\}-\right.$$
$$\left.-\frac{2c+k}{(t-\tau)^{1/2}}\exp\left\{-\frac{k^2t\tau}{a^2(t-\tau)}\right\}+\frac{2c-k}{(t-\tau)^{1/2}}\right], \tag{12}$$

$$f(t) = 2a^2 F_2[x, t; \tilde{g}; \nu_0; \nu_1]. \quad (13)$$

Note that kernel $K(t, \tau)$ (11) has the following property:

$$\lim_{t\to 0+} \int_0^t K(t,\tau)d\tau = 1.$$

This means that the method of successive approximations is not applicable to integral Eq. (10). Therefore, in the paper integral Eq. (10) is called a singular equation.

The solution of integral Eq. (10) will be sought in the class

$$\sqrt{t}\,\exp\left\{k^2 t/(4a^2)\right\}\varphi(t) \in L_\infty(\mathbb{R}_+),$$

$$\text{i.e.}\quad \varphi(t) \in L_\infty\left(\mathbb{R}_+;\ \sqrt{t}\,\exp\left\{k^2 t/(4a^2)\right\}\right). \quad (14)$$

Remark 1. The notation $L_\infty(\mathbb{R}_+;\ t^{1/2}\exp\{k^2t/(4a^2)\})$ means that we are dealing with the space of essentially bounded functions on the semi-axis $\mathbb{R}_+ = (0,\infty)$ with weight $t^{1/2}\exp\{k^2t/(4a^2)\}$. Functions of this class for small values of t can admit a singularity of order $t^{-1/2}$, and at infinity they must decrease no slower than the exponential law $\exp\{-k^2t/(4a^2)\}$.

We note that similar Volterra integral equations of the second kind were studied in papers [1,2,8–10].

4. Reducing integral equation (10) to a differential equation in Laplace images

If we introduce new functions:

$$\varphi_1(t) = \varphi(t)\exp\left\{\frac{k^2 t}{4a^2}\right\},\ \ f_1(t) = f(t)\exp\left\{\frac{k^2 t}{4a^2}\right\},$$

then from (10)–(13) we get:

$$\varphi_1(t) - \int_0^t k(t,\tau)\varphi_1(\tau)d\tau = f_1(t),\ \ t > 0. \quad (15)$$

In integral Eq. (15) we make a replacement of the independent variable and introduce a new unknown function

$$t = \frac{1}{t_1},\ \ \tau = \frac{1}{\tau_1},\ \ \varphi_2(t_1) = \frac{1}{\sqrt{t_1}}\varphi_1(1/t_1),\ \ f_2(t_1) = \frac{1}{\sqrt{t_1}} f_1(1/t_1),$$

as a result from (13) and (15) we obtain:

$$\varphi_2(t_1) + \frac{1}{a\sqrt{\pi}}\int_{t_1}^{\infty}\frac{k}{(\tau_1-t_1)^{3/2}}\exp\left\{-\frac{k^2}{a^2(\tau_1-t_1)}\right\}\varphi_2(\tau_1)d\tau_1 -$$
$$-\frac{1}{2a\sqrt{\pi}}\int_{t_1}^{\infty}\frac{1}{(\tau_1-t_1)^{1/2}}\left[(2c+k)\exp\left\{-\frac{1}{a^2(\tau_1-t_1)}\right\}-\right.$$
$$\left.-(2c-k)\right]\frac{1}{\tau_1}\varphi_2(\tau_1)d\tau_1 = f_2(t_1). \quad (16)$$

We note that from the solution of integral Eq. (16), returning to the initial independent variable and the initial unknown function, we can obtain the solution of the initial integral Eq. (10).

To solve initial integral Eq. (16) we will use the Laplace transformation. We have:

$$\left[1+\exp\left\{-\frac{2k}{a}\sqrt{-p}\right\}\right]\hat{\varphi}_2(p) -$$
$$-\frac{1}{2a\sqrt{-p}}\left[(2c+k)\exp\left\{-\frac{2k}{a}\sqrt{-p}\right\} - (2c-k)\right]\int_p^{\infty}\hat{\varphi}_2(q)dq = \hat{f}_2(p). \quad (17)$$

Here we have used the following formulas of the Laplace transformation ([12], p.472; and [13], p.158):

$$\mathcal{L}\left[\int_{t_1}^{\infty} k(t_1-\tau_1)\varphi_2(\tau_1)d\tau_1\right] = \hat{k}(-p)\,\hat{\varphi}_2(p),\quad \mathcal{L}\left[\frac{1}{t_1}\varphi_2(t_1)\right] = \int_p^{\infty}\hat{\varphi}_2(q)dq,$$

We pass from integral Eq. (17) to a differential equation, introducing a new unknown function-image:

$$\hat{\psi}(p) = \int_p^{\infty} \hat{\varphi}_2(q)dq, \quad \text{i.e.,} \quad \hat{\varphi}_2(p) = -\frac{d\hat{\psi}(p)}{dp}, \tag{18}$$

$$\frac{d\hat{\psi}(p)}{dp} + \frac{(2c+k)\exp\left\{-\frac{2k}{a}\sqrt{-p}\right\} - (2c-k)}{2a\sqrt{-p}\left[1+\exp\left\{-\frac{2k}{a}\sqrt{-p}\right\}\right]}\hat{\psi}(p) = -\frac{\hat{f}_2(p)}{1+\exp\left\{-\frac{2k}{a}\sqrt{-p}\right\}}. \tag{19}$$

Remark 2. Proceeding analogously to the results of work [1,2], we obtain a solution of the homogeneous equation (when $\hat{f}_2(p) \equiv 0$), corresponding to (19) in the form:

$$\hat{\psi}(p) = C\frac{\exp\left\{-\frac{2c-k}{a}\sqrt{-p}\right\}}{\left(1+\exp\left\{-\frac{2k}{a}\sqrt{-p}\right\}\right)^{\frac{2c}{k}}}, \tag{20}$$

and correspondingly,

$$\hat{\varphi}_2(p) = C\left[-\frac{2c-k}{2} + \frac{2c+k}{2}\exp\left\{-\frac{2k}{a}\sqrt{-p}\right\}\right] \times \\ \times \frac{\exp\left\{-\frac{2c-k}{a}\sqrt{-p}\right\}}{a\sqrt{-p}\left(1+\exp\left\{-\frac{2k}{a}\sqrt{-p}\right\}\right)^{\frac{2c}{k}+1}}. \tag{21}$$

Further, as in works [1,2] from (19) and (21) we find the original $\psi_{\text{hom}}(t_1)$ and $\varphi_{2\text{hom}}(t)$ and, following the corresponding transformations given at the beginning of this section, we determine the solution $\varphi_{\text{hom}}(t)$:

$$\varphi_{\text{hom}}(t) = \frac{1}{2a\sqrt{\pi}}\sum_{n=0}^{\infty} B_n^{2c_1+1}\left[(2c+k)\varphi_n^{(1)}(t) - (2c-k)\varphi_n^{(2)}(t)\right], \tag{22}$$

where

$$B_n^{2c_1+1} = (-1)^n\frac{(2c_1+1)(2c_1+2)\dots(2c_1+n)}{n!}, \quad c_1 = \frac{c}{k} = \frac{b}{k}+1,$$

$$\begin{cases} \varphi_n^{(1)}(t) = \exp\left\{-\frac{\{[2c+k(2n+1)]^2+k^2\}t}{4a^2}\right\}, \\ \varphi_n^{(2)}(t) = \exp\left\{-\frac{\{[2c+k(2n-1)^2]+k^2\}t}{4a^2}\right\}. \end{cases} \tag{23}$$

Solution (22)–(23) belongs to the class $L_\infty\left(\mathbb{R}_+; \sqrt{t}\exp\left\{k^2t/(4a^2)\right\}\right)$ (14), this follows from relation ([12], p.486) $\lim_{p\to\infty} p\hat{h}(p) = \lim_{t\to+0} h(t)$, where in our case $h(t) = \sqrt{t}\varphi_{\text{hom}}(t)$. And for all $t \geq \varepsilon > 0$ series (22)–(23) is absolutely convergent.

5. Particular solution of nonhomogeneous equation (19). Image of the resolvent

We find its particular solution $\hat{\psi}_{part}(p)$ of equation (19). Using the solution (20) of homogeneous equation, we have:

$$-\hat{\psi}_{part}(p) = \frac{\exp\left\{-\frac{2c-k}{a}\sqrt{-p}\right\}}{\left(1+\exp\left\{-\frac{2k}{a}\sqrt{-p}\right\}\right)^{\frac{2c}{k}}}\int_{-\infty}^{p}\hat{f}_2(q)\exp\left\{\frac{2c-k}{a}\sqrt{-q}\right\} \times \\ \times\left(1+\exp\left\{-\frac{2k}{a}\sqrt{-q}\right\}\right)^{\frac{2c}{k}-1} dq, \tag{24}$$

where

$$\hat{f}_2(q) = \int_0^{\infty} f_2(\tau_1)\exp\{q\tau_1\}d\tau_1, \ \text{Re}\{q\} < 0.$$

We carry out the following transformation of equality (24):

$$-\hat{\psi}_{part}(p) = \int_0^{\infty} f_2(\tau_1)\hat{R}^*(p,\tau_1)d\tau_1, \tag{25}$$

where $\hat{R}^*(p,\tau_1)$ is the image of the resolvent that is determined by the following formula

$$\hat{R}^*(p,\tau_1) = \int_{-\infty}^{p} \exp\{\tau_1 q\} \frac{\exp\left\{-\frac{2c-k}{a}\sqrt{-p}\right\}}{\left(1+\exp\left\{-\frac{2k}{a}\sqrt{-p}\right\}\right)^{\frac{2c}{k}}} \exp\left\{\frac{2c-k}{a}\sqrt{-q}\right\} \times \times\left(1+\exp\left\{-\frac{2k}{a}\sqrt{-q}\right\}\right)^{\frac{2c}{k}-1} dq, \quad (26)$$

Using the following representation:

$$\frac{\exp\left\{-\frac{2c-k}{a}\sqrt{-p}\right\}}{\left(1+\exp\left\{-\frac{2k}{a}\sqrt{-p}\right\}\right)^{\frac{2c}{k}}} = \sum_{n=0}^{\infty} B_n^{2c_1} \exp\left\{-\frac{2c+k(2n-1)}{a}\sqrt{-p}\right\}, \ c_1 = \frac{c}{k},$$

where $c = b + k$, we write down resolvent $\hat{R}^*(p,\tau_1)$ (26) in the form

$$\hat{R}^*(p,\tau_1) = \sum_{m=0}^{\infty} C_m^{2c_1-1} \hat{I}_{\hat{R}^*}(p,m,\tau_1) \sum_{n=0}^{\infty} B_n^{2c_1} \exp\left\{-\frac{2c+k(2n-1)}{a}\sqrt{-p}\right\}, \quad (27)$$

where

$$\hat{I}_{\hat{R}^*}(p,m,\tau_1) = \int_{-\infty}^{p} \exp\left\{\tau_1 q - \frac{k(2m+1)-2c}{a}\sqrt{-q}\right\} dq, \quad (28)$$

$$C_m^{2c_1-1} = \frac{(2c_1-1)(2c_1-2)\dots(2c_1-m)}{m!},$$

$$B_n^{2c_1} = (-1)^n \frac{2c_1(2c_1+1)\dots(2c_1+n-1)}{n!}.$$

To obtain the above series we have used the known decomposition:

$$\frac{1}{(1+z)^{2c_1}} = \sum_{n=0}^{\infty} B_n^{2c_1} z^n, \ \ z = \exp\left\{-\frac{2k}{a}\sqrt{-p}\right\}, \ \ |z| < 1.$$

Replacing the variable $\sqrt{-q} = \eta$ and introducing the notation $\alpha = \frac{k(2m+1)-2c}{a}$, we get for (28):

$$\hat{I}_{\hat{R}^*}(p,m,\tau_1) = 2\int_{\sqrt{-p}}^{\infty} \eta \exp\left\{-\tau_1\eta^2 - \alpha\eta\right\} d\eta.$$

Then integral (28) is represented as the following difference:

$$\hat{I}_{\hat{R}^*}(p,m,\tau_1) = \frac{1}{\tau_1}\exp\left\{\tau_1 p - \frac{k(2m+1)-2c}{a}\sqrt{-p}\right\} - -\frac{\sqrt{\pi}[k(2m+1)-2c]}{2a\tau_1^{3/2}} \exp\left\{\frac{[k(2m+1)-2c]^2}{4a^2\tau_1}\right\} \times \times \operatorname{erfc}\left(\sqrt{-\tau_1 p} + \frac{k(2m+1)-2c}{2a\sqrt{\tau_1}}\right). \quad (29)$$

Using relations (27)–(29), we obtain the following expression for the image of the resolvent:

$$\hat{R}^*(p,\tau_1) = \hat{R}_1^*(p,\tau_1) - \hat{R}_2(p,\tau_1), \quad (30)$$

where

$$\hat{R}_1^*(p,\tau_1) = \frac{1}{\tau_1}\exp\{\tau_1 p\} + \frac{1}{\tau_1}\sum_{\substack{m,n=0,\\ m+n\neq 0}}^{\infty} C_m^{2c_1-1} B_n^{2c_1} \exp\left\{\tau_1 p - (m+n)\frac{2k}{a}\sqrt{-p}\right\}, \quad (31)$$

$$\hat{R}_2(p,\tau_1) = \frac{1}{\tau_1}\frac{\sqrt{\pi}}{2}\sum_{m=0}^{\infty} C_m^{2c_1-1} \frac{k(2m+1)-2c}{a} \hat{I}_1(p,\tau_1,m) \sum_{n=0}^{\infty} B_n^{2c_1} \hat{I}_2(p,\tau_1,n), \quad (32)$$

$$\hat{I}_1(p,\tau_1,m) = \frac{1}{\tau_1^{1/2}} \exp\left\{\frac{[k(2m+1)-2c]^2}{4a^2\tau_1}\right\} \hat{I}_1^*(p,\tau_1,m), \quad (33)$$

$$\hat{I}_1^*(p,\tau_1,m)=\exp\{-\tau_1 p\}\mathrm{erfc}\left(\sqrt{-\tau_1 p}+\frac{k(2m+1)-2c}{2a\sqrt{\tau_1}}\right),$$

$$\hat{I}_2(p,\tau_1,n)=\exp\left\{\tau_1 p-\frac{2c+k(2n-1)}{a}\sqrt{-p}\right\}. \tag{34}$$

6. Constructing the resolvent $R(t,\tau)$

For this purpose first we find the originals of the images $\hat{R}_1^*(p,\tau_1)$ and $\hat{R}_2(p,\tau_1)$. For the image $\hat{R}_1^*(p,\tau_1)$ we have:

$$\hat{R}_1^*(p,\tau_1)\doteqdot\frac{1}{\tau_1}\delta(\tau_1-t_1)+R_1(t_1,\,\tau_1), \tag{35}$$

$$R_1(t_1,\,\tau_1)=\frac{1}{\tau_1}\sum_{\substack{m,n=0,\\ m+n\neq 0}}^{\infty}C_m^{2c_1-1}B_n^{2c_1}\frac{(m+n)k}{a\sqrt{\pi}(\tau_1-t_1)^{3/2}}\exp\left\{-\frac{(m+n)^2k^2}{a^2(\tau_1-t_1)}\right\}. \tag{36}$$

In deriving relations (35)–(36) from (31) we have used the following formulas from ([12], 1965, p.525;[14], 2000, p.921, formula 82, Appendix D):

$$\exp\{-p\tau\}\doteqdot\delta(t-\tau),\ \ \exp\{-k\sqrt{p}\}\doteqdot\frac{k}{2\sqrt{\pi}\,t^{3/2}}\exp\left\{-\frac{k^2}{4t}\right\}.$$

And to get the original of the image $\hat{R}_2(p,\tau_1)$ according to (32)–(34) it is enough to find originals for images $\hat{I}_1(p,\tau_1,m)$, $m=0,1,2,3,\ldots,$ (33) and $\hat{I}_2(p,n)$, $n=0,1,2,\ldots,$ (34).

Lemma 1. *The original of image $\hat{I}_1(p,\tau_1,m)$ (33) is calculated by the formula*

$$\hat{I}_1(p,\tau_1,m)\doteqdot I_1(t_1,\tau_1,m)=\frac{1}{2\pi\sqrt{t_1}(\tau_1+t_1)}\exp\left\{-\frac{[k(2m+1)-2c]^2}{4a^2t_1}\right\}. \tag{37}$$

Proof of Lemma 1.. We introduce the notation $\gamma=\frac{k(2m+1)-2c}{a}$. For further calculations, it will be more convenient for us to pass from the complex plane (p) to the plane $(-p)$. As a result of this, and also applying the Jordan lemma ([12], p.478, 81: XI), we have:

$$\hat{I}_1^*(p,\tau_1,m)\doteqdot I_1^*(t_1,\tau_1,m)=\frac{1}{2\pi i}\int_{-\sigma-i\infty}^{-\sigma+i\infty}\hat{I}_1^*(p,\tau_1,m)\exp\{-pt_1\}dp=$$

$$=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\hat{I}_1^*(-p,\tau_1,m)\exp\{pt_1\}dp=$$

$$\left\| I:\ \begin{array}{l} p=-x,\\ p=xe^{-i\pi},\\ \sqrt{p}=-i\sqrt{x}, \end{array}\quad II:\ \begin{array}{l} p=xe^{i\pi},\\ \sqrt{p}=i\sqrt{x} \end{array}\right\|$$

$$=\lim_{\substack{r\to 0\\ R\to\infty}}\frac{1}{2\pi i}\left\{-\int_R^r\exp\{-(\tau_1+t_1)x\}\mathrm{erfc}\left(-i\sqrt{\tau_1 x}+\frac{\gamma}{2\sqrt{\tau_1}}\right)dx-\right.$$

$$\left.-\int_r^R\exp\{-(\tau_1+t_1)x\}\mathrm{erfc}\left(i\sqrt{\tau_1 x}+\frac{\gamma}{2\sqrt{\tau_1}}\right)dx\right\}=$$

$$=\frac{1}{2\pi i}\int_0^{\infty}\exp\{-(\tau_1+t_1)x\}\left[\mathrm{erfc}\left(-i\sqrt{\tau_1 x}+\frac{\gamma}{2\sqrt{\tau_1}}\right)-\right.$$

$$\left.-\mathrm{erfc}\left(i\sqrt{\tau_1 x}+\frac{\gamma}{2\sqrt{\tau_1}}\right)\right]dx=$$

$$=\frac{\sqrt{\tau_1}}{4\pi(\tau_1+t_1)\sqrt{t_1}}\exp\left\{-\frac{\gamma^2(\tau_1+t_1)}{4\tau_1 t_1}\right\}\mathrm{erfc}\left(-\frac{i\gamma}{2\sqrt{t_1}}\right)+$$

$$+\frac{1}{4\pi\, i(\tau_1+t_1)}\operatorname{erfc}\left(\frac{\gamma}{2\sqrt{\tau_1}}\right)+$$

$$+\frac{\sqrt{\tau_1}}{4\pi\,(\tau_1+t_1)\sqrt{t_1}}\exp\left\{-\frac{\gamma^2(\tau_1+t_1)}{4\tau_1 t_1}\right\}\operatorname{erfc}\left(\frac{i\gamma}{2\sqrt{t_1}}\right)-$$

$$-\frac{1}{4\pi\, i(\tau_1+t_1)}\operatorname{erfc}\left(\frac{\gamma}{2\sqrt{\tau_1}}\right)=$$

$$=\frac{\sqrt{\tau_1}}{4\pi\sqrt{t_1}(\tau_1+t_1)}\exp\left\{-\gamma^2\frac{\tau_1+t_1}{4t_1\tau_1}\right\}\left[\operatorname{erfc}\left(-i\frac{\gamma}{\sqrt{t_1}}\right)+\operatorname{erfc}\left(i\frac{\gamma}{\sqrt{t_1}}\right)\right].$$

□

We note that in deriving the last relation we have used the formula from the book [15], p.30, formula 10).

Finally, taking into account the equality $\operatorname{erfc}\left(-i\frac{\gamma}{\sqrt{t_1}}\right)+\operatorname{erfc}\left(i\frac{\gamma}{\sqrt{t_1}}\right)=2$, we obtain the assertion of Lemma 1.

Now, using the formula from ([14], p.921, formula 82, Appendix D), we find the original of factor $\hat{I}_2(p,\tau_1,n)$ (34). We have

$$\hat{I}_2(p,\tau,n)=\exp\left\{\tau_1 p-\frac{k(2n-1)+2c}{a}\sqrt{-p}\right\}\doteqdot I_2(t_1,\tau,n)=$$

$$=\frac{k(2n-1)+2c}{2a\sqrt{\pi}(\tau_1-t_1)^{3/2}}\exp\left\{-\frac{[k(2n-1)+2c]^2}{4a^2(\tau_1-t_1)}\right\}. \tag{38}$$

We find the convolution $I_1(t_1,\tau,m)\star I_2(t_1,\tau,n)$ (37) and (38) for each $m=0,1,2,3,\ldots$ and $n=0,1,2,3,\ldots$ We have

$$I_1(t_1,\tau,m)\star I_2(t_1,\tau,n)=$$

$$=\frac{k(2n-1)+2c}{4a\pi^{3/2}}\int_{t_1}^{\tau_1}\frac{1}{(\tau_1-\eta)^{3/2}\sqrt{\eta-t_1}[\tau_1+(\eta-t_1)]}\times$$

$$\times\exp\left\{-\frac{[k(2n-1)+2c]^2}{4a^2(\tau_1-\eta)}-\frac{[k(2m+1)-2c]^2}{4a^2(\eta-t_1)}\right\}d\eta=$$

$$\left\|\begin{array}{lll}\frac{\eta-t_1}{\tau_1-\eta}=z^2, & \eta-t_1=(\tau_1-\eta)z^2, & \eta=\frac{\tau_1 z^2+t_1}{1+z^2}\\ d\,\eta=(\tau_1-t_1)\frac{2z}{(1+z^2)^2}dz, & \eta-t_1=(\tau_1-t_1)\frac{z^2}{1+z^2}, & \tau_1-\eta=(\tau_1-t_1)\frac{1}{1+z^2}\end{array}\right\|$$

$$=\frac{k(2n-1)+2c}{2a\pi^{3/2}(\tau_1-t_1)(2\tau_1-t_1)}\exp\left\{-\frac{[k(2n-1)+2c]^2+[k(2m+1)-2c]^2}{4a^2(\tau_1-t_1)}\right\}\times$$

$$\times\int_0^\infty\exp\left\{-\frac{[k(2n-1)+2c]^2}{4a^2(\tau_1-t_1)}z^2-\frac{[k(2m+1)-2c]^2}{4a^2(\tau_1-t_1)}\frac{1}{z^2}\right\}dz+$$

$$+\frac{k(2n-1)+2c}{2a\pi^{3/2}(2\tau_1-t_1)^2}\exp\left\{-\frac{[k(2n-1)+2c]^2+[k(2m+1)-2c]^2}{4a^2(\tau_1-t_1)}\right\}\times$$

$$\times\int_0^\infty\frac{1}{\frac{\tau_1}{2\tau_1-t_1}+z^2}\exp\left\{-\frac{[k(2n-1)+2c]^2}{4a^2(\tau_1-t_1)}z^2-\frac{[k(2m+1)-2c]^2}{4a^2(\tau_1-t_1)}\frac{1}{z^2}\right\}dz=$$

$$=\|z=\sqrt{x};\ dz=\frac{dx}{2\sqrt{x}}\|=$$

$$=\frac{k(2n-1)+2c}{2a\pi^{3/2}(\tau_1-t_1)(2\tau_1-t_1)}\exp\left\{-\frac{[k(2n-1)+2c]^2+[k(2m+1)-2c]^2}{4a^2(\tau_1-t_1)}\right\}\times$$

$$\times\frac{1}{2}\frac{2a\sqrt{\pi}\sqrt{\tau_1-t_1}}{k(2n-1)+2c}\exp\left\{-\frac{2[k(2n-1)+2c][k(2m+1)-2c]}{4a^2(\tau_1-t_1)}\right\}+$$

$$+\frac{k(2n-1)+2c}{2a\pi^{3/2}(2\tau_1-t_1)^2}\exp\left\{-\frac{[k(2n-1)+2c]^2+[k(2m+1)-2c]^2}{4a^2(\tau_1-t_1)}\right\}\times$$

$$\times\frac{1}{2}\int\limits_0^\infty\frac{1}{\sqrt{x}(x+\frac{\tau_1}{2\tau_1-t_1})}\exp\left\{-\frac{[k(2n-1)+2c]^2}{4a^2(\tau_1-t_1)}x-\right.$$

$$\left.-\frac{[k(2m+1)-2c]^2}{4a^2(\tau_1-t_1)}\frac{1}{x}\right\}dx;$$

We calculate the integral

$$I=\frac{1}{2}\int\limits_0^\infty\frac{1}{\sqrt{x}(x+\frac{\tau_1}{2\tau_1-t_1})}\exp\left\{-\frac{[k(2n-1)+2c]^2}{4a^2(\tau_1-t_1)}x-\right.$$

$$\left.-\frac{[k(2m+1)-2c]^2}{4a^2(\tau_1-t_1)}\frac{1}{x}\right\}dx=$$

$$=\frac{1}{2}\frac{\pi\sqrt{2\tau_1-t_1}}{\sqrt{\tau_1}}\exp\left\{\frac{[k(2n-1)+2c]^2}{4a^2(\tau_1-t_1)}\times\frac{\tau_1}{2\tau_1-t_1}+\right.$$

$$\left.+\frac{[k(2m+1)-2c]^2}{4a^2(\tau_1-t_1)}\times\frac{2\tau_1-t_1}{\tau_1}\right\}\times$$

$$\times\operatorname{erfc}\left\{\frac{k(2n-1)+2c}{2a\sqrt{\tau_1-t_1}}\times\frac{\sqrt{\tau_1}}{\sqrt{2\tau_1-t_1}}+\frac{k(2m+1)-2c}{2a\sqrt{\tau_1-t_1}}\times\frac{\sqrt{2\tau_1-t_1}}{\sqrt{\tau_1}}\right\}.$$

Here the following relations are used ([16], p.337, formula 3.325[17], p.277, formula 2.3.16.4). Thus, finally we get

$$I_1(t_1,\tau,m)\star I_2(t_1,\tau,n)=$$

$$=\exp\left\{-\frac{(m+n)^2k^2}{a^2(\tau_1-t_1)}\right\}\times\left\{\frac{1}{2a\pi\sqrt{\tau_1-t_1}(2\tau_1-t_1)}+\right.$$

$$\left.+\frac{k(2n-1)+2c}{4a\sqrt{\pi}\sqrt{\tau_1}(2\tau_1-t_1)^{3/2}}\exp\left\{\gamma_{m,n}^2(t_1,\tau_1)\right\}\operatorname{erfc}(\gamma_{m,n}(t_1,\tau_1))\right\};\quad(39)$$

where

$$\gamma_{m,n}(t_1,\tau_1)=\frac{[k(2n-1)+2c]\sqrt{\tau_1}}{2a\sqrt{\tau_1-t_1}\sqrt{2\tau_1-t_1}}+\frac{[k(2m+1)-2c]\sqrt{2\tau_1-t_1}}{2a\sqrt{\tau_1-t_1}\sqrt{\tau_1}}.$$

From here

$$R_2(t_1,\tau_1)=$$

$$=\frac{1}{4\tau_1(2\tau_1-t_1)}\sum_{m,n=0}^{\infty}C_m^{2c_1-1}B_n^{2c_1}\frac{k(2m+1)-2c}{a}\exp\left\{-\frac{(m+n)^2k^2}{a^2(\tau_1-t_1)}\right\}\times$$

$$\times\left\{\frac{1}{\sqrt{\pi}\sqrt{\tau_1-t_1}}+\frac{k(2n-1)+2c}{2a\sqrt{\tau_1}\sqrt{2\tau_1-t_1}}\exp\left\{\gamma_{m,n}^2\right\}\operatorname{erfc}(\gamma_{m,n})\right\}.$$

Now we can write down the original $\psi_{\text{part}}(t_1)$ of the particular solution of Eq. (19):

$$\psi_{part}(t_1)=t_1^{-1}f_2(t_1)+\int\limits_{t_1}^{\infty}[R_1(t_1,\tau_1)-R_2(t_1,\tau_1)]f_2(\tau_1)d\tau_1.\quad(40)$$

Next, we pass from the function $\psi_{\text{part}}(t_1)$ to the particular solution $\varphi_{2\text{part}}(t_1)$ of Eq. (16):

$$\varphi_{2\text{part}}(t_1)=t_1\psi_{\text{part}}(t_1)=f_2(t_1)+\int\limits_{t_1}^{\infty}t_1[R_1(t_1,\tau_1)-R_2(t_1,\tau_1)]f_2(\tau_1)d\tau_1,\quad(41)$$

and, using the following replacements:

$$\varphi_2(t_1) = t_1^{-1/2}\varphi_1(t_1^{-1}) = t^{1/2}\varphi_1(t) = \varphi_2(t^{-1}),$$

$$\varphi_1(t) = t^{-1/2}\varphi_2(t^{-1}), \quad f_2(t_1) = t_1^{-1/2} f_1(t_1^{-1}) = t^{1/2} f_1(t),$$

$$\varphi(t) = \varphi_1(t)\exp\left\{-\frac{k^2 t}{4a^2}\right\}, \quad f(t) = f_1(t)\exp\left\{-\frac{k^2 t}{4a^2}\right\},$$

we define the particular solutions $\varphi_{1\text{part}}(t)$ of Eq. (15) and $\varphi_{part}(t)$ as the particular solution of Eq. (10):

$$\varphi_{1\text{part}}(t) = f_1(t) + \int_0^t t^{-3/2}\tau^{-3/2}[R_1(t^{-1},\tau^{-1}) - R_2(t^{-1},\tau^{-1})] f_1(\tau)d\tau, \tag{42}$$

$$\varphi_{part}(t) = f(t) + \int_0^t R(t,\tau)\exp\left\{-\frac{k^2(t-\tau)}{4a^2}\right\} f(\tau)d\tau, \tag{43}$$

where

$$R(t,\tau) = \tilde{R}_1(t,\tau) - \tilde{R}_2(t,\tau), \tag{44}$$

$$\tilde{R}_1(t,\tau) = t^{-3/2}\tau^{-3/2}R_1(t^{-1},\tau^{-1}) = \sum_{m,n=0}^{\infty} C_m^{2c_1-1} B_n^{2c_1} \frac{(m+n)k}{a\sqrt{\pi}} R_{1,m,n}(t,\tau), \tag{45}$$

$$R_{1,m,n}(t,\tau) = \frac{\tau}{(t-\tau)^{3/2}}\exp\left\{-(m+n)^2\frac{k^2 t\tau}{a^2(t-\tau)}\right\}, \tag{46}$$

$$\tilde{R}_2(t,\tau) = t^{-3/2}\tau^{-3/2}R_2(t^{-1},\tau^{-1}) =$$

$$= \sum_{m,n=0}^{\infty} C_m^{2c_1-1} B_n^{2c_1} \frac{k(2m+1)-2c}{4a\sqrt{\pi}} R_{2,m,n}(t,\tau), \tag{47}$$

$$R_{2,m,n}(t,\tau) = \left\{\frac{\tau}{\sqrt{t-\tau}(2t-\tau)} + \frac{[k(2n-1)+2c]\sqrt{\pi}}{2a} \times \frac{\tau}{(2t-\tau)^{3/2}} \times \right.$$
$$\left. \times\sqrt{\tau}\exp\left\{\gamma^2(m,n,t,\tau)\right\}\text{erfc}\{\gamma(m,n,t,\tau)\}\right\}\exp\left\{-(m+n)^2\frac{k^2 t\tau}{a^2(t-\tau)}\right\}, \tag{48}$$

$$\gamma(m,n,t,\tau) = \frac{\sqrt{\tau}}{2a\sqrt{t-\tau}} \times \frac{2k(m+n)t + (t-\tau)[k(2m+1)-2c]}{\sqrt{2t-\tau}}.$$

7. Estimate of the resolvent

We show that the following theorem holds.

Theorem 1. *Resolvent $R(t,\tau)$ (44)–(48) admits the estimate*

$$|R(t,\tau)| \le C\frac{\tau}{(t-\tau)^{3/2}}\exp\left\{-\frac{k^2 t\tau}{a^2(t-\tau)}\right\}, \quad 0 < \tau < t < +\infty. \tag{49}$$

The proof of Theorem 1 follows from the assertions of the following Lemmas 2 and 3.

Lemma 2. *Resolvent $\tilde{R}_1(t,\tau)$ (45)–(46) satisfies the estimate:*

$$|\tilde{R}_1(t,\tau)| < C_1\frac{\tau}{(t-\tau)^{3/2}}\exp\left\{-\frac{k^2 t\tau}{a^2(t-\tau)}\right\}, \quad 0 < \tau < t < +\infty. \tag{50}$$

The assertion of Lemma 2 follows from the properties of alternating series.

Lemma 3. *Resolvent $\tilde{R}_2(t,\tau)$ (47)–(48) satisfies the estimate:*

$$|\tilde{R}_2(t,\tau)| < C_2 \frac{\tau}{(t-\tau)^{3/2}} \exp\left\{-\frac{k^2 t\tau}{a^2(t-\tau)}\right\}, \quad 0<\tau<t<+\infty. \tag{51}$$

Proof of Lemma 3... We note that summands of resolvent $R_{2,m,n}(t,\tau)$ (48) have representations:

$$R_{2,m,n}(t,\tau) = R^{(1)}_{2,m,n}(t,\tau) + R^{(2)}_{2,m,n}(t,\tau), \tag{52}$$

where

$$R^{(1)}_{2,m,n}(t,\tau) = \frac{k(2m+1)-2c}{4} \times \frac{\tau}{(2t-\tau)\sqrt{t-\tau}} \exp\left\{-(m+n)^2 \frac{k^2 t\tau}{a^2(t-\tau)}\right\}, \tag{53}$$

$$R^{(2)}_{2,m,n}(t,\tau) = \frac{[k(2m+1)-2c][k(2n-1)+2c]}{8a} \times \frac{\tau\sqrt{\tau}}{(2t-\tau)^{3/2}} \times$$
$$\times \exp\left\{\gamma^2(m,n,t,\tau)\right\} \operatorname{erfc}\{\gamma(m,n,t,\tau)\} \exp\left\{-(m+n)^2 \frac{k^2 t\tau}{a^2(t-\tau)}\right\}. \tag{54}$$

□

Summands $R^{(1)}_{2,m,n}(t,\tau)$ (53) are estimated by summands $R_{1,m,n}(t,\tau)$ (46). This follows from the following inequality for their coefficients:

$$\frac{\tau}{(2t-\tau)\sqrt{t-\tau}} < \frac{\tau}{(t-\tau)^{3/2}}, \quad 0<\tau<t<+\infty.$$

To obtain an estimate of summand $R^{(2)}_{2,m,n}(t,\tau)$ (54) we represent it in the form:

$$R^{(2)}_{2,m,n}(t,\tau) = \frac{[k(2m+1)-2c][k(2n-1)+2c]}{8a} \times \frac{\tau}{(2t-\tau)^{3/2}} \times$$
$$\times \exp\left\{-(m+n)^2 \frac{k^2 t\tau}{a^2(t-\tau)}\right\} D(m,n,t,\tau),$$

where

$$D(m,n,t,\tau) = \sqrt{\tau} \exp\left\{[\gamma(m,n,t,\tau)]^2\right\} \operatorname{erfc}(\gamma(m,n,t,\tau)).$$

For any values of $|\gamma(m,n,t,\tau)|$, it is obvious that for $D(m,n,t,\tau)$ the following estimate that is uniform with respect to m,n,t and τ is valid:

$$|D(m,n,t,\tau)| \le C < +\infty.$$

To complete the proof of Lemma 3, it remains to take into account the following inequality:

$$\frac{\tau}{(2t-\tau)^{3/2}} < \frac{\tau}{(t-\tau)^{3/2}}, \quad 0<\tau<t<+\infty.$$

As a result, we obtain that the expressions $R^{(2)}_{2,m,n}(t,\tau)$ are also estimated, respectively, by summands $R_{1,m,n}(t,\tau)$.
Thus, the validity of assertion (49) of Theorem 1 follows from Lemmas 2 and 3.

8. Main result on the solvability of integral equation (10)

Thus, we have established the following main result on the solvability of integral Eq. (10).

Theorem 3. *For any right-hand side $f(t) \in L_\infty\left(\mathbb{R}_+;\ \sqrt{t}\exp\left\{k^2 t/(4a^2)\right\}\right)$ from class (14) integral Eq. (10) has the general solution $\varphi(t) \in L_\infty(\mathbb{R}_+;\ \sqrt{t}\exp\left\{k^2 t/(4a^2)\right\})$:*

$$\varphi(t) = f(t) + \int_0^t R(t,\tau) f(\tau)\,d\tau + C\varphi_{\text{hom}}(t), \ C = \text{const}, \tag{55}$$

where $\varphi_{\text{hom}}(t)$ is defined in section 4, and for resolvent $R(t,\tau)$ (44) the following estimate holds (49)

$$|R(t,\tau)| \le C \frac{\tau}{(t-\tau)^{3/2}} \exp\left\{-\frac{k^2 t\tau}{a^2(t-\tau)}\right\}, \quad 0<\tau<t<+\infty.$$

9. Solution of BVP (1)–(2). Main result

The solution $v(x,t)$ of boundary value problem (3)–(4) is determined according to formulas (7)–(8), (22)–(23), (43)–(48) and (55), and the solution of the initial boundary value problem (1)–(2) will have the form:

$$u(x,t) = \tilde{u}(x,t;C) + C_1, \quad \text{where} \quad \tilde{u}(x,t;C) = \int_0^x v(\xi,t;C)d\xi, \tag{56}$$

since its solution is determined to an accuracy of constant term C_1, where $v(x,t)$ is defined according to formula (7)–(8) and constant C is defined in (55).

To establish the boundedness of solution $u(x,t)$ (56) of boundary value problem (1)–(2) we need to study the properties of solution $v(x,t)$ (7)–(8), (22)–(23) and (43)–(48) of problem (3)–(4). Since $\varphi(t) \in L_\infty(R_+; \sqrt{t}\exp\{k^2t/(4a^2)\})$, then its necessary to evaluate and prove the boundedness of the integral $I(x,t)$ for all $\{x,t\} \in G$:

$$I(x,t) = \frac{1}{2a\sqrt{\pi}} \int_0^t I_1(x,t,\tau) \frac{\exp\left\{-\frac{k^2\tau}{4a^2}\right\}}{\sqrt{\tau}} d\tau, \tag{57}$$

where

$$I_1(x,t,\tau) =$$

$$= \int_0^x \frac{1}{(t-\tau)^{1/2}} \left[-\exp\left\{ -\frac{(x_1+k\tau)^2}{4a^2(t-\tau)} \right\} + \exp\left\{ -\frac{(x_1-k\tau)^2}{4a^2(t-\tau)} \right\} \right] dx_1 =$$

$$= a\sqrt{\pi} \left[-\mathrm{erf}\left(\frac{x+k\tau}{2a\sqrt{t-\tau}} \right) + \mathrm{erf}\left(\frac{x-k\tau}{2a\sqrt{t-\tau}} \right) \right]. \tag{58}$$

Further, substituting the value of integral $I_1(x,t,\tau)$ (58) into (57) we obtain

$$I(x,t) = \frac{1}{2} \int_0^t \left[-\mathrm{erf}\left(\frac{x+k\tau}{2a\sqrt{t-\tau}} \right) + \mathrm{erf}\left(\frac{x-k\tau}{2a\sqrt{t-\tau}} \right) \right] \frac{\exp\left\{-\frac{k^2\tau}{4a^2}\right\}}{\sqrt{\tau}} d\tau. \tag{59}$$

Hence we obtain an estimate for integral (59):

$$I(x,t) \le 4a \int_0^{k\sqrt{t}/(2a)} \exp\{-\xi^2\} d\xi = 2a\sqrt{\pi}\,\mathrm{erf}\left(\frac{k\sqrt{t}}{2a} \right). \tag{60}$$

Thus, we have established uniform boundedness of integral (57) with respect to $\{x,t\} \in G$, i.e we have shown that solution $\tilde{u}(x,t;C)$ (56) of boundary value problem (1)–(2) belongs to the class $L_\infty(G)$. Note that the solution $\tilde{u}(x,t;C)$ is determined to an accuracy to a constant term C_1, i.e. formula

$$u(x,t) = \tilde{u}(x,t;C) + C_1,$$

defines the general solution of boundary value problem (1)–(2). Estimation (60) also allows us to get its order of smallness for any $\{x,t\} \in G$, i.e. the inclusion

$$\tilde{u}(x,t;C) \in L_\infty(G;\ t^{-1/2})$$

is true. This follows from the asymptotic behavior of the function $\mathrm{erf}\left(\frac{k\sqrt{t}}{2a}\right)$ for small values of variable t, and for large values of the variable t this fact follows from boundedness property of solution $\tilde{u}(x,t)$ on G and boundedness of the expression $t^{-1/2}\,\tilde{u}(x,t;C)$.

We formulate the main result of the work.

Theorem 4. *Boundary value problem (1)–(2) has a nontrivial solution* $u(x,t) = \tilde{u}(x,t;C) + C_1$, *where* $\tilde{u}(x,t;C) \in L_\infty(G;\ t^{-1/2})$, *$C$ and C_1 are constants.*

10. Conclusion

It follows from the results of this paper that the Stefan problem considered in [3] can have a nonunique solution in some weight class of essentially bounded functions. However, if this class is restricted, the Stefan problem will have the unique solution. The latter is consistent with the result of the work [3]. Apparently, this fact should be taken into account, for example, for approximately solving the Stefan problem.

Acknowledgements

This research is financially supported by the grant projects AP08855372 (2020–2022) and AP08956033 (2020–2021) from the Ministry of Science and Education of the Republic of Kazakhstan.

References

[1] M. Jenaliyev, M. Ramazanov, On a homogeneous parabolic problem in an infinite corner domain, AIP Conf. Proc. 1759 (020085-1–020085-6) (2016), doi:10.1063/1.4959699.
[2] M. Jenaliyev, S. Iskakov, M. Ramazanov, The resolvent and the particular solution of one singular integral equation, AIP Conf. Proc. (050009-1–050009-10) (2017) 1880, doi:10.1063/1.5000646.
[3] V.A. Solonnikov, A. Fasano, One-dimensional parabolic problem arising in the study of some free boundary problems [ob odnomernoj parabolicheskoj zadache, voznikajushhej pri izuchenii nekotoryh zadach so svobodnymi granicami] (in russian), Zap. Nauchn. Sem. POMI. 269 (2000) 322–338.
[4] A. Kheloufi, B.K. Sadallah, On the regularity of the heat equation solution in non-cylindrical domains: two approaches, Appl. Math. Comput. 218 (2011) 1623–1633.
[5] A. Kheloufi, R. Labbas, B.K. Sadallah, On the resolution of a parabolic equation in a nonregular domain of $\mathbb{R}^3$, Differ. Equat. Appl. 2 (2) (2010) 251–263.
[6] A. Kheloufi, Existence and uniqueness results for parabolic equations with robin type boundary conditions in a non-regular domain of $\mathbb{R}^3$, Appl. Math. Comput. 220 (2013) 756–769.
[7] A. Kheloufi, B.K. Sadallah, Resolution of a high-order parabolic equation in conical time-dependent domains of $\mathbb{R}^3$, Arab J. Math. Sci. 22 (2016) 165–181.
[8] M.M. Amangaliyeva, M.T. Jenaliyev, M.T. Kosmakova, M.I. Ramazanov, About Dirichlet boundary value problem for the heat equation in the infinite angular domain, Bound. Value Prob. 2014 (213) (2014) 21.
[9] M.M. Amangaliyeva, M.T. Dzhenaliev, M.T. Kosmakova, M.I. Ramazanov, On one homogeneous problem for the heat equation in an infinite angular domain, Sib. Math. J. 56 (6) (2015) 982–995.
[10] M.T. Jenaliyev, M.M. Amangaliyeva, M.T. Kosmakova, M.I. Ramazanov, On a Volterra equation of the second kind with 'incompressible' kernel, Adv. Differ. Equ. (2015) (2015) 14–71.
[11] A.N. Tikhonov, A.A. Samarskii, Equations of the Mathematical Physics [uravnenija matematicheskoj fiziki] (in Russian), Nauka, Moscow, 1972, p. 735.
[12] M.A. Lavrent'ev, B.V. Shabat, Methods of the Theory of Functions of a Complex Variable [metody teorii funkcij kompleksnogo peremennogo] (in Russian), GIFML, Moscow, 1958, p. 679.
[13] M.L. Krasnov, Integral Equations an Introduction to the Theory [integralnye uravneniya vvedenie v teoriyu] (in Russian), Nauka, Moscow, 1975, p. 703.
[14] G.A. Korn, T.M. Korn, Mathematical Handbook for Scientists and Engineers, Dover publications Inc. Mineola, New York, 2000, p. XVII+1130.
[15] A.P. Prudnikov, Y.A. Brychkov, O.I. Marichev, Integrals and Series [integraly i rjady] (in russian), 2, FizMatLit, Moscow, 2003, p. 664.
[16] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series, and Products, Elsevier Inc., AP, New York,, 2007. XLVIII+1171
[17] A.P. Prudnikov, Y.A. Brychkov, O.I. Marichev, Integrals and Series [integraly i rjady] (in Russian), 1, FizMatLit, Moscow, 2002, p. 632.