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On Hardy and Bellman Transformations for Orthogonal Fourier Series

N. T. Tleukhanova

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Hardy's and Bellman's results [1, 2] for trigonometric Fourier series are well known: if $f \in L_p[0,\pi]$,

$$f \sim \sum_{k=0}^{\infty} a_k \cos kx, \qquad A_k = \frac{1}{k+1}\sum_{m=0}^{k} a_m, \qquad B_k = \sum_{m=k}^{\infty} \frac{a_m}{m},$$

then the series

$$\sum_{k=0}^{\infty} A_k \cos kx, \qquad \sum_{k=1}^{\infty} B_k \cos kx$$

are Fourier series of certain functions Hf and Bf from $L_p[0,\pi]$ and the following estimates are valid:

$$\|Hf\|_p \le c\|f\|_p, \qquad \|Bf\|_p \le c\|f\|_p.$$

These transformations are called the *Hardy* and *Bellman transformations*, respectively.

In this paper, we generalize these results in three directions: we consider Fourier series with respect to regular orthonormal systems [3] (including all trigonometric and multiplicative systems with bounded generators); a generalization of the Hardy and Bellman transformations is introduced; the boundedness of generalized transformations in anisotropic Lorentz spaces [3] is studied.

Definition 1. An orthonormal system $\Phi = \{\phi_k(x)\}_{k=1}^{\infty}$ is said to be *regular* if there exists a constant B such that

1) for any closed interval e from $[0,1]$ and for $k \in \mathbb{N}$ the following estimate is valid:

$$\left|\int_e \phi_k(x)\,dx\right| \le B \min\left(\mu e, \frac{1}{k}\right);$$

2) for any closed interval w from $\mathbb{N}$ (a finite arithmetical progression with step 1) and for $t \in (0,1]$ the following estimate is valid:

$$\left(\sum_{k\in w} \phi_k(f)\right)^{*}(t) \le B \min\left(|w|, \frac{1}{t}\right),$$

where $(\sum_{k\in w} \phi_k(f))^{*}(t)$ is the nonincreasing rearrangement of the function $\sum_{k\in w} \phi_k(x)$, $|w|$ is the number of elements in the set w.

Suppose that Φ is a regular orthonormal system, and $I = \{I_k\}_{k\in\mathbb{N}}$ is a sequence of finite subsets from $\mathbb{N}$. By J we denote the sequence of sets $\{J_k\}_{k\in\mathbb{N}}$, where $J_k = \{m : k \in I_m\}$. For $f \in L_1[0,1]$, $f \sim \sum_{k=1}^{n} a_k\varphi_k(x)$, and the sequence $I = \{I_k\}_{k\in\mathbb{N}}$, we define the transformations $H(f\,;I)$ and $B(f\,;I)$ as follows:

$$H(f\,;I) \sim \sum_{k=1}^{\infty} \frac{1}{|I_k|}\biggl(\sum_{m\in I_k} a_m\biggr)\varphi_k(x), \tag{1}$$

$$B(f\,;I) \sim \sum_{k=1}^{\infty} \biggl(\sum_{m\in J_k} \frac{a_m}{|I_m|}\biggr)\varphi_k(x). \tag{2}$$

These transformations are called, respectively, the *Hardy* and *Bellman transformations corresponding to the sequences of sets* $I = \{I_k\}_{k\in\mathbb{N}}$. In the case $I_k = \{1,2,\dots,k\}$, $k\in\mathbb{N}$, these transformations are, respectively, the Hardy and Bellman transformations.

By $L_{pq}[0,1]$ we denote the Lorentz space and by l_{pq} the discrete Lorentz space.

Theorem 1. *Suppose that* $2 < p < \infty$, $p' = p/(p-1)$, $1 \le q \le \infty$, $\Phi = \{\varphi_k(x)\}_{k=1}^{\infty}$ *is a regular system and* $I = \{I_k\}_{k=1}^{\infty}$ *is a family of closed intervals in* $\mathbb{N}$ *such that* $|I_k| \ge k$ *(*$|I_k|$ *is the number of elements in* I_k*). Then the Hardy* $H(f,I)$ *and Bellman* $B(f,I)$ *transformations are bounded in the spaces* $L_{pq}[0,1]$ *and* $L_{p'q}[0,1]$*, respectively, i.e.,*

$$\|H(f,I)\|_{L_{pq}} \le c\|f\|_{L_{pq}}, \qquad \|B(f,I)\|_{L_{p'q}} \le c\|f\|_{L_{p'q}}.$$

Theorem 2. *Suppose that* $1 < p \le 2$, $p' = p/(p-1)$, $1 \le q \le \infty$, $\Phi = \{\varphi_k(x)\}_{k=1}^{\infty}$ *is a regular system and* $I = \{I_k\}_{k=1}^{\infty}$ *is a family of bounded sets from* $\mathbb{N}$ *satisfying the conditions* $|I_k| = k$ *and* $I_k \subset I_{k+1}$*. Then the Hardy* $H(f,I)$ *and Bellman* $B(f,I)$ *transformations are bounded in the spaces* $L_{pq}[0,1]$ *and* $L_{p'q}[0,1]$*, respectively.*

Theorem 3. *Suppose that* $2 < p < \infty$, $1 \le q \le \infty$, $\Phi = \{\varphi_k(x)\}_{k=1}^{\infty}$ *is a regular system and* $\{Q_t\}_{t\in(0,1)}$ *is a family of closed intervals from* $[0,1]$ *satisfying the condition* $\mu Q_t \ge t$*. If* $f \sim \sum_{k=1}^{\infty} a_k\varphi_k(x)$ *and* $a = \{a_k\}_{k=1}^{\infty} \in l_{pq}$*, then the sequence of Fourier coefficients* $b = \{b_k\}_{k=1}^{\infty}$ *of the function*

$$F(t) = \frac{1}{\mu Q_t}\int_{Q_t} f(x)\,dx$$

also belongs to the Lorentz space l_{pq} *and the following inequality holds:*

$$\|b\|_{l_{pq}} \le c\|a\|_{l_{pq}}.$$

Theorem 4. *Suppose that* $1 < p < 2$, $1 \le q \le \infty$, $\Phi = \{\varphi_k(x)\}_{k=1}^{\infty}$ *is a regular system. Suppose that* $f \sim \sum_{k=1}^{\infty} a_k\varphi_k(x)$, $a = \{a_k\}_{k=1}^{\infty} \in l_{pq}$*, and* g *is an arbitrary function equimeasurable with* f*. Then the sequence of Fourier coefficients* $b = \{b_k\}_{k=1}^{\infty}$ *of the function*

$$F(t) = \frac{1}{t}\int_0^t g(x)\,dx, \qquad t\in(0,1),$$

also belongs to the Lorentz space l_{pq} *and the following inequality holds:*

$$\|b\|_{l_{pq}} \le c\|a\|_{l_{pq}}.$$

Suppose that $\boldsymbol{p} = (p_1,\dots,p_n)$, $\boldsymbol{q} = (q_1,\dots,q_n)$ are vectors such that if $0 < q_j < \infty$, then $0 < p_j < \infty$, but if $q_j = \infty$, then $0 < p_j \le \infty$, $j = 1,\dots,n$.

The anisotropic space $L_{\boldsymbol{pq}}[0,1]^n$ is defined as the set of measurable functions f for which

$$\|f\|_{L_{\boldsymbol{pq}}} = \left(\int_0^1 \cdots \left(\int_0^1 \left| t_1^{1/p_1} \cdots t_n^{1/p_n} f^{*_1,\dots,*_n}(t_1,\dots,t_n)\right|^{q_1} \frac{dt_1}{t_1}\right)^{q_2/q_1} \cdots \frac{dt_n}{t_n}\right)^{1/q_n} < \infty,$$

where $f^{*_1,\dots,*_n}(t_1,\dots,t_n)$ is the nonincreasing rearrangement of the function f performed successively with respect to the variables $x_1,\dots,x_n$ (assuming the other variables fixed); the expression

$$\left(\int_0^1 \bigl(F(t)\bigr)^q \frac{dt}{t}\right)^{1/q}$$

for $q=\infty$ is understood as $\sup_{t>0} F(t)$.

Suppose that $\Psi_1 = \{\psi_k^1(x)\}_{k=1}^\infty, \dots, \Psi_n = \{\psi_k^n(x)\}_{k=1}^\infty$ are regular systems of functions. We define $\Phi = \{\phi_k(x)\}_{k\in\mathbb{N}^n}$ as follows:

$$\phi_k(x) = \phi_{k_1\dots k_n}(x_1,\dots,x_n) = \psi_{k_1}^1(x_1)\cdots\psi_{k_n}^n(x_n).$$

Suppose that $I_p = \{I_k^p\}_{k\in\mathbb{N}^n}$ is a family of finite subsets $\mathbb{N}$ satisfying the following condition: for $p>2$, the family I_p coincides with the set of closed intervals $\{I_k^p\}_{k\in\mathbb{N}^n}$ such that $|I_k^p| \ge k$, while for $p\le 2$ I_p is a family of sets $\{I_k^p\}_{k\in\mathbb{N}^n}$ such that $I_k^p \subset I_{k+1}^p$ and $|I_k^p| = k$.

Suppose that $1 < p = (p_1,\dots,p_n) < \infty$ and $f \sim \sum_{k\in\mathbb{N}^n} a_k\phi_k(x)$. We define the transformations $B_j(f, I_{p_j})$ and $H_j(f, I_{p_j})$ as follows:

$$H_j(f, I_{p_j}) = \sum_{k_1=1}^\infty \cdots \sum_{k_n=1}^\infty \frac{1}{|I_{k_j}^{p_j}|}\left(\sum_{m_j\in I_{k_j}^{p_j}} a_{k_1,\dots,m_j,\dots,k_n}\right)\phi_{k_1\dots k_n}(x_1,\dots,x_n),$$

$$B_j(f, I_{p_j}) = \sum_{k_1=1}^\infty \cdots \sum_{k_n=1}^\infty \left(\sum_{m_j\in J_{k_j}^{p_j}} \frac{a_{k_1,\dots,m_j,\dots,k_n}}{|I_{m_j}^{p_j}|}\right)\phi_{k_1\dots k_n}(x_1,\dots,x_n).$$

Suppose that $E = \{\varepsilon = (\varepsilon_1,\dots,\varepsilon_n) : \varepsilon_i = 0 \text{ or } \varepsilon_i = 1\}$ are the vertices of the n-dimensional unit cube and $G_p = I_{p_1}\times\cdots\times I_{p_n} = \{I_{k_1}^{p_1}\times\cdots\times I_{k_n}^{p_n}\}_{k\in\mathbb{N}^n}$. Let us define the operator $T_\varepsilon(f, G_p)$, $\varepsilon\in E$, as follows:

$$T_\varepsilon(f\,;G_p) = T_{\varepsilon_n}\cdots T_{\varepsilon_1}(f), \qquad \text{where} \quad T_{\varepsilon_j}(f) = \begin{cases} H_j(f\,;I_{p_j}) & \text{for } \varepsilon_j = 1, \\ B_j(f\,;I_{p_j'}) & \text{for } \varepsilon_j = 0,\ p' = \dfrac{p}{p-1}. \end{cases}$$

Theorem 5. *Suppose that* $1 < \boldsymbol{p} = (p_1,\dots,p_n) < \infty$, $1 \le \boldsymbol{q} = (q_1,\dots,q_n) \le \infty$, E *are the vertices of the unit cube, and* $G_p = I_{p_1}\times\cdots\times I_{p_n}$. *Then for any* $\varepsilon\in E$ *the following inequality is valid:*

$$\|T_\varepsilon(f\,;G_p)\|_{L_{pq}} \le c\|f\|_{L_{pq}}.$$

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E. A. Buketov Karaganda State University

E-mail: nurs@kargu.krg.kz