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Existence and smoothness of solutions of a singular differential equation of hyperbolic type

This paper investigates the question of the existence of solutions to the semiperiodic Dirichlet problem for a class of singular differential equations of hyperbolic type. The problem of smoothness of solutions is also considered, i.e., maximum regularity of solutions. Such a problem will be interesting when the coefficients are strongly growing functions at infinity. For the first time, a weighted coercive estimate was obtained for solutions to a differential equation of hyperbolic type with strongly growing coefficients.

Keywords: resolvent, hyperbolic type equation, maximal regularity, unbounded domain.

1 Introduction. Formulation of results

Considered on the strip

$$\overline{\Omega} = \{(x, y) : -\pi < x < \pi, -\infty < y < \infty\}$$

next problem:

$$(L + \lambda I)u = u_{xx} - u_{yy} + a(y)u_x + c(y)u + \lambda u = f(x, y) \in L_2(\Omega), \tag{1}$$

$$u(-\pi, y) = u(\pi, y), u_x(-\pi, y) = u_x(\pi, y), -\infty < y < \infty. \tag{2}$$

Further, we assume that the coefficients $a(y)$, $c(y)$ satisfy the conditions:

i) $|a(y)| \geq \delta_0 > 0$, $c(y) \geq \delta > 0$ are continuous functions in $R(-\infty, \infty)$;

ii) $\mu_0 = \sup\limits_{|y-t|\leq 1} \frac{a(y)}{a(t)} < \infty$, $\mu = \sup\limits_{|y-t|\leq 1} \frac{c(y)}{c(t)} < \infty$;

iii) $c(y) \leq c_0 \cdot c^2(y)$ $for\ all\ y \in R, c_0 > 0$ is a constant number.

Here it has to be noted that $a(y)$ and $c(y)$ can be unlimited functions at infinity.

The existence and uniqueness, as well as the qualitative behavior, of solutions for differential equations of hyperbolic type, were studied in [1–14]. In these works, Darboux and Goursat problems and the Cauchy problem, periodic and some boundary value problems for differential equations of hyperbolic type with constant or variable bounded coefficients were examined.

In this paper, in the space $L_2(\Omega)$, we study questions about the existence, uniqueness of solutions, and also the smoothness of solutions to a periodic problem without initial conditions [13] for a differential equation of hyperbolic type with strongly increasing coefficients at infinity.

In our previous paper [14], we studied a differential operator of hyperbolic type in the space $L_2(R^2)$.

In contrast to [14], in this paper, on a strip, we consider the so-called periodic problem without initial conditions. Here we note that in the future, this work will allow us to study questions about the compactness of the resolvent, about estimates for the singular (s-numbers) and eigenvalues of a differential operator of hyperbolic type corresponding to problem (1)–(2).

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Definition 1. We say that function $u(x,y) \in L_2(\Omega)$ is a strong solution to problem (1)–(2), if there is a sequence $\{u_n\} \subset C^{\infty}_{0,\pi}(\Omega)$ such that

$$\|u_n - u\|_{L_2(\Omega)} \to 0, \ \ \|(L+\lambda I)u_n - f\|_{L_2(\Omega)} \to 0 \ \ for \ \ n \to \infty,$$

where $C^{\infty}_{0,\pi}(\Omega)$ is the set consisting of infinitely differentiable finite functions with respect to variable y and satisfying conditions (2) with respect to variable x.

Theorem 1. Let the condition $i)$ be fulfilled. Then for $\lambda \geq 0$ for any $f(x,y) \in L_2(\Omega)$ there is a unique strong solution to the problem (1)–(2), and the equality is true

$$u(x,y) = (L+\lambda I)^{-1} f = \sum_{n=-\infty}^{\infty} (l_n + \lambda I)^{-1} f_n(y) \cdot e^{inx},$$

where $f(x,y) \in L_2(\Omega)$, $f(x,y) = \sum\limits_{n=-\infty}^{\infty} f_n(y) \cdot e^{inx}$, $f_n(y) =< f(x,y), e^{inx} >$ are fourier coefficients, $i^2 = -1$, $< \cdot, \cdot >$ is the scalar product in $L_2(\Omega)$,

$$(l_n + \lambda I)u = -u''(y) + (-n^2 + ina(y) + c(y) + \lambda)u(y), u \in D(l_n).$$

Theorem 2. Let the condition $i)$ be fulfilled. Then for $\lambda \geq 0$ for any $f(x,y) \in L_2(\Omega)$ there is a unique strong solution to the problem (1)–(2), and the equality is true

$$\|u_{xx} - u_{yy}\|_2 + \|u_y\|_2 + \|a(y)u_x\|_2 + \|c(y)u\|_2 \leq c \cdot \|f\|_2,$$

where $c > 0$ is constant number.

2 Proof of theorems 1-2

Using the Fourier method, we reduce problem (1)–(2) to the study of the following differential operator with negative discrete parameter n $(n = 0, \pm 1, \pm 2, ...)$:

$$(l_n + \lambda I)u = -u''(y) + (-n^2 + ina(y) + c(y) + \lambda)u(y), u \in D(l_n),$$

where $D(l_n)$ is the domain of the operator l_n.

Consider two cases:

I. Let be $(n = 0)$. In this case, the operator l_0 is the Sturm–Liouville operator.

This operator has been studied thoroughly in [15–21].

II. Let be $n \neq 0$. In this case, it is easy to see that the first term in the coefficient $(-n^2 + ina(y) + c(y) + \lambda)$ tends to $-\infty$, i.e. $-n^2 \to -\infty$.

In this case, the l_n operator is not a semi-bounded operator. Consequently, the methods that have been worked out for the Sturm–Liouville operator $L = -\frac{d^2}{dx^2} + q(x)u$ turn out to be poorly adapted to the study of the Sturm–Liouville operator with a negative parameter.

Let us take $\{\varphi_j\}$ the set of non-negative functions from $C^{\infty}_0(R)$ such that

$$\sum_j \varphi_j^2 \equiv 1, \sup \ p \ \varphi_j \subset \triangle_j, \ \bigcup_j \triangle_j = R,$$

where $\triangle_j = (j-1, j+1)$, $j = 0, \pm 1, \pm 2, ...$ the multiplicity of the intersection of which is not higher than three. The existence of such a covering follows from the results of [22].

Continue $a(y)$, $c(y)$ from $\triangle_j$ for all R . The resulting functions will be denoted by $a_j(y)$ and $c_j(y)$.

These functions are bounded and periodic functions. Denote by $(l_{n,j,\alpha}+\lambda I)$ the closure of operator

$$(l_{n,j,\alpha}+\lambda I)u=-u''(y)+(-n^2+in(a_j(y)+\alpha)+c_j(y)+\lambda)u$$

defined on $C_0^\infty(R)$. We introduced the real number α to evaluate the norm of the operator $D_y(l_{n,j,\alpha}+\lambda I)^{-1}$ in the space $L_2(\Omega)$, where $D_y=\frac{\partial}{\partial y}$. The sign of the number α is chosen as follows: $\alpha\cdot b(y)>0$ at $y\in R$.

In the course of the proof using Lemma 3, we will get rid of this number.

Lemma 1. Suppose that the coefficients of the operator $l_{n,j,\alpha}+\lambda I$ satisfy condition i). Then for $\lambda\geq 0$:

1) for the differential operator $l_{n,j,\alpha}+\lambda I$ at $\lambda\geq 0$, there is a bounded inverse operator $(l_{n,j,\alpha}+\lambda I)^{-1}$ defined at all $L_2(R)$.

2) the resolvent of the operator $l_{n,j,\alpha}+\lambda I$ satisfies the following estimates:

a) $\|(l_{n,j,\alpha}+\lambda I)^{-1}\|_{2\to 2}\leq\frac{c}{(\delta+\lambda)^{\frac{1}{2}}}$, $c>0$ — constant number independent of n,j,α;

b) $\|\frac{d}{dy}(l_{n,j,\alpha}+\lambda I)^{-1}\|_{2\to 2}\leq\frac{c}{(\delta+\lambda)^{\frac{1}{4}}}$, $c>0$ — constant number independent of n,j,α;

c) $\|(l_{n,j,\alpha}+\lambda I)^{-1}\|_{2\to 2}\leq\frac{1}{|n|\cdot|a(\tilde{y}_j)|}$, $n\neq 0$, $c>0$ — constant number independent of n,j,α;

d) $\|(l_{n,j,\alpha}+\lambda I)^{-1}\|_{2\to 2}\leq\frac{2\cdot c}{c(\overline{y}_j)+\lambda}$, $c>0$ — constant number independent of n,j,α;

where $\|\cdot\|_{2\to 2}$ — is the norm of the operator $(l_{n,j,\alpha}+\lambda I)^{-1}$ in space $L_2(R)$, $|a(\tilde{y}_j)|=\min\limits_{y\in\overline{\triangle}_j}|a(y)|$, $|c(\overline{y}_j)|=\min\limits_{y\in\overline{\triangle}_j}|c(y)|$.

Lemma 1 is proved using functionals $<(l_{n,j,\alpha}+\lambda I)u,u>$, $<(l_{n,j,\alpha}+\lambda I)u,-inu>$ $(n=0,\pm1,\pm2,...)$ and repeating the calculations and arguments for these functionals, which were used in the proof of Lemma 2.1 [22] and Lemmas 4–6 [23].

Now, consider the differential operator

$$(l_{n,\alpha}+\lambda I)u=-u''+(-n^2+in(a(y)+\alpha)+c(y)+\lambda)\cdot u,$$

which is a closure in $L_2(R)$ of the following operator originally defined on $C_0^\infty(R)$:

$$(l_{n,\alpha}+\lambda I)u=-u''+(-n^2+in(a(y)+\alpha)+c(y)+\lambda)\cdot u.$$

We introduce the operator

$$K_{\lambda,\alpha}f=\sum_{\{j\}}\varphi_j(l_{n,j,\alpha}+\lambda I)^{-1}\varphi_j f,\, f\in L_2(R).$$

The following lemma is proved with the help of calculations and arguments that were used in the proof of Theorem 1.1–1.3 in [22] and Theorem 1 in [23].

Lemma 2. Suppose that the coefficients of the operator $l_{n,j,\alpha}+\lambda I$ satisfy condition i). Then there is a number $\lambda_0>0$ such that for the operator $l_{n,j,\alpha}+\lambda I$ for $\lambda\geq\lambda_0$ there is a resolvent and the equality

$$(l_{n,j,\alpha}+\lambda I)^{-1}f=K_{\lambda,\alpha}(I-M_{\lambda,\alpha})^{-1}f,\, f\in L_2(R)$$

holds, where $M_{\lambda,\alpha}f=\sum\limits_{\{j\}}\varphi_j''(l_{n,j,\alpha}+\lambda I)^{-1}\varphi_j f+2\sum\limits_{\{j\}}\varphi_j'\frac{d}{dy}(l_{n,j,\alpha}+\lambda I)^{-1}\varphi_j f,\ f\in L_2(R)$.

Lemma 3. Suppose that the coefficients of the operator $l_{n,j,\alpha}+\lambda I$ satisfy condition i). Then there is a number $\lambda_0>0$ such that for the operator $l_n+\lambda I$ $(n=0,\pm1,\pm2,...)$ for $\lambda\geq\lambda_0$ there is a resolvent and the equality

$$(l_n+\lambda I)^{-1}f=(l_{n,\alpha}+\lambda I)^{-1}(I-M_{\lambda,\alpha})^{-1}f,\, f\in L_2(R)$$

holds, where $M_{\lambda,\alpha} = in\alpha(l_{n,\alpha}+\lambda I)^{-1}$ $(n = 0, \pm 1, \pm 2, ...)$ and the operator's norm $M_{\lambda,\alpha}$: $\|M_{\lambda,\alpha}\|_{2\to 2} < 1$.

Using the method used in the proof of Lemma 9 in [23], we obtain the proof of Lemma 3.

Proof of Theorem 1. Using the scalar product $< (L + \lambda I)u, u >$ for all $u \in D(L)$ and taking into account the condition $i)$, we obtain that

$$\|(L + \lambda I)u\|_2 \geq c \cdot \|u\|_2,$$

where $c(\delta) > 0$ is a constant number. Further, repeating the calculations and arguments used in the proof of Theorem 1 in [23], we obtain the proof of Theorem 1.

Proof of Theorem 2. Taking into account conditions $ii)$–$iii)$, and also using the method used in the proof of Theorems in [24–26], we obtain the proof of Theorem 2.

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Гиперболалық типтегі сингулярлық дифференциалдық теңдеудің шешімдерінің бар болуы және тегістігі

Мақалада гиперболалық типті сингулярлық дифференциалдық теңдеулер класы үшін жартылай периодтық Дирихле есебінің шешімдерінің бар екендігі туралы мәселе зерттелген. Сонымен қатар шешімдердің тегістігі туралы мәселе, яғни шешімдердің максималды регулярлығы қарастырылған. Коэффициенттері шексіздікте жылдам өсетін функциялар болғанда мұндай есеп қызықты болады. Осы жұмыста бірінші рет коэффициенттері жылдам өсетін гиперболалық типті дифференциалдық теңдеудің шешімдері үшін салмақты коэрцитивті бағалаулар алынған.

Кілт сөздер: резольвента, гиперболалық типтес теңдеу, максималды регулярлық, шексіз облыс.

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Существование и гладкость решений сингулярного дифференциального уравнения гиперболического типа

В статье изучен вопрос о существовании решений полупериодической задачи Дирихле для одного класса сингулярных дифференциальных уравнений гиперболического типа. Также рассмотрена задача о гладкости решений, т.е. максимальная регулярность решений. Данная задача будет интересной, когда коэффициенты являются сильно растущими функциями на бесконечности. По-видимому, в настоящей работе впервые получена весовая коэрцитивная оценка решений дифференциального уравнения гиперболического типа с сильно растущими коэффициентами.

Ключевые слова: резольвента, уравнение гиперболического типа, максимальная регулярность, бесконечная область.

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