

## On Orders of Approximation of Function Classes in Lorentz Spaces with Anisotropic Norm

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**Abstract**—In this paper, we study the anisotropic Lorentz space of periodic functions. We establish a sharp estimate of the order of approximation for the Besov class by trigonometric polynomials in Lorentz spaces with anisotropic norm.

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Key words: *anisotropic Lorentz space, Besov class, approximation of function classes, trigonometric polynomial, periodic function, Lebesgue space, Hölder's inequality.*

Suppose that  $\mathbb{R}^m$  is the  $m$ -dimensional Euclidean space of points  $\bar{x} = (x_1, \dots, x_m)$  with real coordinates,  $I^m = [0, 2\pi]^m$  is the  $m$ -dimensional cube, and we are given

$$\bar{q} = (q_1, \dots, q_m), \quad \bar{\theta} = (\theta_1, \dots, \theta_m), \quad q_j, \theta_j \in [1, +\infty), \quad j = 1, \dots, m.$$

By  $L_{\bar{q}, \bar{\theta}}(I^m)$  we denote the anisotropic Lorentz space of Lebesgue measurable  $(2\pi)$ -periodic functions  $f(\bar{x})$  for which the norm

$$\|f\|_{\bar{q}, \bar{\theta}} = \left[ \int_0^{2\pi} t_m^{\frac{\theta_m}{q_m}-1} \left[ \dots \left[ \int_0^{2\pi} (f^{*1, \dots, *m}(t_1, \dots, t_m))^{\theta_1} t_1^{\frac{\theta_1}{q_1}-1} dt_1 \right]^{\frac{\theta_2}{q_2}} \dots \right]^{\frac{\theta_{m-1}}{q_{m-1}}} dt_m \right]^{\frac{1}{\theta_m}}$$

is finite; here  $f^{*1, \dots, *m}(t_1, \dots, t_m)$  is a nonincreasing rearrangement of the function  $|f(\bar{x})|$  with respect to each variable  $x_j$  for fixed values of the other variables (first, with respect to  $x_1$ , then with respect to  $x_2$  etc.; see [1], [2]).  $L_{\bar{q}, \infty}(I^m)$  is the Marcinkiewicz space of functions for which

$$\|f\|_{\bar{q}, \infty} = \sup_{\bar{t} \in I^m} \prod_{j=1}^m t_j^{\frac{1}{q_j}} f^{*1, \dots, *m}(t_1, \dots, t_m) < +\infty.$$

The Lebesgue space  $L_{\bar{p}}(I^m)$ ,  $p_j \in [1, +\infty)$ ,  $j = 1, \dots, m$ , with mixed norm consists of all Lebesgue measurable functions  $f(\bar{x})$  for which (see [3], [4])

$$\|f\|_{\bar{p}} = \left[ \int_0^{2\pi} \left[ \dots \left[ \int_0^{2\pi} |f(\bar{x})|^{p_1} dx_1 \right]^{\frac{p_2}{p_1}} \dots \right]^{\frac{p_m}{p_{m-1}}} dx_m \right]^{\frac{1}{p_m}} < +\infty.$$

$L_{\bar{p}}^0(I^m)$  is the set of all functions  $f \in L_{\bar{p}}(I^m)$  such that

$$\int_0^{2\pi} f(\bar{x}) dx_j = 0, \quad j = 1, \dots, m.$$

Suppose we are given a vector  $\bar{r} = (r_1, \dots, r_m)$ ,  $r_j > 0$ ,  $j = 1, \dots, m$ . Consider the Nikol'skii class  $H_{\bar{p}}^{\bar{r}}$  consisting of all functions  $f \in L_{\bar{p}}(I^m)$  for which

$$\Omega_{\bar{t}}(f, \bar{t})_{\bar{p}} \leq \prod_{j=1}^m t_j^{r_j}, \quad \bar{t} \in [0, 1]^m,$$

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where  $l_j > r_j$ ,  $j = 1, \dots, m$ , and  $\Omega_{\bar{\tau}}(f, \bar{t})_{\bar{p}}$  is the mixed modulus of smoothness of the function  $f \in L_{\bar{p}}(I^m)$  (see [3]).

In what follows,

$$\delta_{\bar{s}}(f, \bar{x}) = \sum_{\bar{n} \in \rho(\bar{s})} a_{\bar{n}}(f) e^{i\langle \bar{n}, \bar{x} \rangle},$$

where

$$\langle \bar{y}, \bar{x} \rangle = \sum_{j=1}^m y_j x_j, \quad \rho(\bar{s}) = \{\bar{k} = (k_1, \dots, k_m) \in \mathbb{Z}^m : 2^{s_j-1} \leq |k_j| < 2^{s_j}, j = 1, \dots, m\},$$

and the  $a_{\bar{n}}(f)$  are the Fourier coefficients of a function  $f \in L_1(I^m)$  in the multiple system  $\{e^{i\langle \bar{n}, \bar{x} \rangle}\}$ .

Suppose we are given a vector  $\bar{r} = (r_1, \dots, r_m)$ . Set  $\bar{\gamma} = \bar{r}/r_1$ ,  $r_j > 0$ ,  $j = 1, \dots, m$ , and

$$Q_n^{\bar{\gamma}} = \bigcup_{\langle \bar{s}, \bar{\gamma} \rangle \leq n} \rho(\bar{s}), \quad T(Q_n^{\bar{\gamma}}) = \left\{ t(\bar{x}) = \sum_{\bar{k} \in Q_n^{\bar{\gamma}}} b_{\bar{k}} e^{i\langle \bar{k}, \bar{x} \rangle} \right\};$$

$E_n^{(\bar{\gamma})}(f)_{\bar{q}, \bar{\theta}}$  is the best approximation of a function  $f \in L_{\bar{q}, \bar{\theta}}(I^m)$  by polynomials from the set  $T(Q_n^{\bar{\gamma}})$ .

By  $C(\alpha, \beta, \dots)$  we denote positive values depending on the parameters (given in parentheses), which are, in general, different in different formulas.

For brevity, we write  $A \asymp B$  to indicate that there exist positive constants  $c_1$  and  $c_2$  such that  $c_1 \cdot A \leq B \leq c_2 \cdot A$ .

The theory of embedding of function spaces  $H_{\bar{p}}^{\bar{\alpha}}$  was studied by Nikol'skii [3], Amanov [4], and other mathematicians (see the references in [3], [4]). In Lebesgue spaces with isotropic norm, the orders of approximation of  $H$ -classes by trigonometric polynomials with harmonics from hyperbolic crosses were studied in [5]–[12], while those in spaces with mixed norm were investigated in [13], [14]. In the present paper, we present approximation estimates for classes  $H_{\bar{p}}^{\bar{\alpha}}$  in Lorentz and Marcinkiewicz spaces with anisotropic norm.

**Lemma A** (see [15]). *Suppose we are given positive numbers  $0 < \theta < +\infty$  and  $a_k, b_k$ ,  $k = 1, 2, \dots$ .*

a) *If  $\sum_{k=1}^n b_k \leq C \cdot b_n$ , then*

$$\sum_{n=1}^{\infty} a_n \left( \sum_{k=n}^{\infty} b_k \right)^{\theta} \leq C \cdot \sum_{n=1}^{\infty} a_n b_n^{\theta}.$$

b) *If  $\sum_{k=n}^{\infty} b_k \leq C \cdot b_n$ , then*

$$\sum_{n=1}^{\infty} a_n \left( \sum_{k=1}^n b_k \right)^{\theta} \leq C \cdot \sum_{n=1}^{\infty} a_n \cdot b_n^{\theta}.$$

**Lemma B** (see [3, p. 133], [16]). *Suppose that  $1 \leq p_j < \infty$ ,  $j = 1, \dots, m$ . For any trigonometric polynomial*

$$T_{\bar{n}}(\bar{x}) = \sum_{\bar{k} \leq \bar{n}} a_{\bar{k}} e^{i\langle \bar{k}, \bar{x} \rangle},$$

*the following inequality holds:*

$$\max_{\bar{x} \in I^m} |T_{\bar{n}}(\bar{x})| \leq C \cdot \prod_{j=1}^m n_j^{\frac{1}{p_j}} \cdot \|T_{\bar{n}}\|_{\bar{p}}.$$

Let us now prove some auxiliary assertions.

Set

$$G_e(\bar{n}) = \{\bar{s} = (s_1, \dots, s_m) \in \mathbb{N}^m : s_j \leq n_j, j \in e; \quad s_j > n_j, j \notin e\},$$

where  $e \subset \{1, \dots, m\}$ ;

$$U_{\bar{n}}(f, \bar{x}) = \sum_{e \subset \{1, \dots, m\}} \sum_{\bar{s} \in G_e(\bar{n})} \delta_{\bar{s}}(f, \bar{x}).$$

Let there be given numbers  $\theta_j \in [1, +\infty)$ ,  $j = 1, \dots, m$ . By  $\ell_{\bar{\theta}}(\mathbb{Z}_+^m)$  we denote the space of all numerical sequences  $\{a_{\bar{s}}\}_{\bar{s} \in \mathbb{Z}_+^m}$  for which

$$\|\{a_{\bar{s}}\}\|_{\ell_{\bar{\theta}}(\mathbb{Z}_+^m)} = \left\{ \sum_{s_m=0}^{\infty} \left[ \dots \left\{ \sum_{s_1=0}^{\infty} |a_{\bar{s}}|^{\theta_1} \right\}^{\frac{\theta_2}{\theta_1}} \dots \right]^{\frac{\theta_m}{\theta_{m-1}}} \right]^{\frac{1}{\theta_m}} < +\infty.$$

Set

$$Y^m(n, \bar{\gamma}) = \{\bar{s} = (s_1, \dots, s_m) \in \mathbb{Z}_+^m : \langle \bar{s}, \bar{\gamma} \rangle > n\},$$

where  $\bar{\gamma} = (\gamma_1, \dots, \gamma_m)$ .

**Lemma 1.** Suppose that  $\bar{\theta} = (\theta_1, \dots, \theta_m)$ ,  $1 \leq \theta_j < +\infty$ ,  $j = 1, \dots, m$ ,  $\alpha \in [0, +\infty)$ ,

$$b_{\bar{s}}(n, \bar{\gamma}) = \begin{cases} 2^{-\alpha \langle \bar{s}, \bar{\gamma} \rangle} & \text{for } \bar{s} \in Y^m(n, \bar{\gamma}), \\ 0 & \text{for } \bar{s} \notin Y^m(n, \bar{\gamma}). \end{cases}$$

Then the following inequality holds:

$$\|\{b_{\bar{s}}(n, \bar{\gamma})\}\|_{\ell_{\bar{\theta}}(\mathbb{Z}_+^m)} \leq C \cdot 2^{-n\alpha} \cdot n^{\sum_{j=2}^m \frac{1}{\theta_j}}.$$

**Proof.** Let us prove the lemma by induction on the dimension  $m$ . For  $m = 1$ , the assertion of the lemma is obvious.

Suppose that  $m = 2$ . It follows from the definition of the set  $Y^m(n, \bar{\gamma})$  and the numbers  $b_{\bar{s}}(n, \bar{\gamma})$  that

$$\begin{aligned} & \|\{b_{\bar{s}}(n, \bar{\gamma})\}\|_{\ell_{\bar{\theta}}(\mathbb{Z}_+^2)} \\ &= \left\{ \sum_{s_2 < \frac{n}{\gamma_2}} \left[ \sum_{s_1 > \frac{1}{\gamma_1}(n - \gamma_2 s_2)} |b_{\bar{s}}(n, \bar{\gamma})|^{\theta_1} \right]^{\frac{\theta_2}{\theta_1}} + \sum_{s_2 \geq \frac{n}{\gamma_2}} \left[ \sum_{s_1=0}^{\infty} |b_{\bar{s}}(n, \bar{\gamma})|^{\theta_1} \right]^{\frac{\theta_2}{\theta_1}} \right\}^{\frac{1}{\theta_2}} \\ &= \left\{ \sum_{s_2 < \frac{n}{\gamma_2}} \left[ \sum_{s_1 > \frac{1}{\gamma_1}(n - \gamma_2 s_2)} 2^{-\alpha(s_1 \gamma_1 + s_2 \gamma_2) \theta_1} \right]^{\frac{\theta_2}{\theta_1}} + \sum_{s_2 \geq \frac{n}{\gamma_2}} \left[ \sum_{s_1=0}^{\infty} 2^{-\alpha(s_1 \gamma_1 + s_2 \gamma_2) \theta_1} \right]^{\frac{\theta_2}{\theta_1}} \right\}^{\frac{1}{\theta_2}} \\ &\leq C(\gamma_1, \gamma_2, \theta_1, \theta_2) \cdot 2^{-n\alpha} \cdot \left\{ \sum_{s_2 < \frac{n}{\gamma_2}} 1 + 1 \right\}^{\frac{1}{\theta_2}} \leq C(\gamma_1, \gamma_2, \theta_1, \theta_2) \cdot 2^{-n\alpha} \cdot n^{\frac{1}{\theta_2}}. \end{aligned}$$

Now suppose that the assertion of the lemma is valid for  $m = \nu$ , i.e.,

$$\|\{b_{\bar{s}}(n, \bar{\gamma})\}\|_{\ell_{\bar{\theta}}(\mathbb{Z}_+^{\nu})} \leq C \cdot 2^{-n\alpha} \cdot n^{\sum_{j=2}^{\nu} \frac{1}{\theta_j}}. \tag{1}$$

Let us prove that the assertion of the lemma is also valid for  $m = \nu + 1$ . For brevity, set  $\bar{s}(\nu) = (s_1, \dots, s_{\nu})$ . Then, by the definition of the norm, we have

$$\|\{b_{\bar{s}}(n, \bar{\gamma})\}\|_{\ell_{\bar{\theta}}(\mathbb{Z}_+^{\nu+1})} = \left\{ \sum_{s_{\nu+1}=0}^{\infty} \|\{b_{\bar{s}(\nu), s_{\nu+1}}(n, \bar{\gamma})\}\|_{\ell_{\bar{\theta}}(\mathbb{Z}_+^{\nu})}^{\theta_{\nu+1}} \right\}^{\frac{1}{\theta_{\nu+1}}}$$

$$= \left\{ \sum_{s_{\nu+1} \leq \frac{n}{\gamma_{\nu+1}}} \|\{b_{\bar{s}(\nu), s_{\nu+1}}(n, \bar{\gamma})\}\|_{\ell_{\bar{\theta}}^{\theta_{\nu+1}}(\mathbb{Z}_+^{\nu})} + \sum_{s_{\nu+1} > \frac{n}{\gamma_{\nu+1}}} \|\{b_{\bar{s}(\nu), s_{\nu+1}}(n, \bar{\gamma})\}\|_{\ell_{\bar{\theta}}^{\theta_{\nu+1}}(\mathbb{Z}_+^{\nu})} \right\}^{\frac{1}{\theta_{\nu+1}}}. \quad (2)$$

If  $s_{\nu+1} \leq n/\gamma_{\nu+1}$ , then, by definition,

$$b_{\bar{s}}(n, \bar{\gamma}) = \begin{cases} 0 & \text{for } \sum_{j=1}^{\nu} \gamma_j s_j < n - \gamma_{\nu+1} \cdot s_{\nu+1}, \\ 2^{-\alpha(\bar{s}, \bar{\gamma})} & \text{for } \sum_{j=1}^{\nu} \gamma_j s_j > n - \gamma_{\nu+1} \cdot s_{\nu+1}. \end{cases}$$

Therefore, by the assumption (1), we obtain

$$\begin{aligned} \sum_{s_{\nu+1} \leq \frac{n}{\gamma_{\nu+1}}} \|\{b_{\bar{s}(\nu), s_{\nu+1}}(n, \bar{\gamma})\}\|_{\ell_{\bar{\theta}}^{\theta_{\nu+1}}(\mathbb{Z}_+^{\nu})} &\leq C \cdot 2^{-n\alpha\theta_{\nu+1}} \sum_{s_{\nu+1} \leq \frac{n}{\gamma_{\nu+1}}} (n - s_{\nu+1}\gamma_{\nu+1})^{\theta_{\nu+1} \sum_{j=2}^{\nu} \frac{1}{\theta_j}} \\ &\leq C \cdot 2^{-n\alpha\theta_{\nu+1}} \cdot n^{1 + \sum_{j=2}^{\nu} \frac{\theta_{\nu+1}}{\theta_j}}. \end{aligned} \quad (3)$$

If  $s_{\nu+1} > n/\gamma_{\nu+1}$ , then

$$\sum_{j=1}^{\nu+1} s_j \gamma_j > n \quad \forall (s_1, \dots, s_{\nu}) \in \mathbb{Z}_+^{\nu}.$$

Therefore,

$$\begin{aligned} \sum_{s_{\nu+1} > \frac{n}{\gamma_{\nu+1}}} \|\{b_{\bar{s}}(n, \bar{\gamma})\}\|_{\ell_{\bar{\theta}}^{\theta_{\nu+1}}(\mathbb{Z}_+^{\nu})} &= C(\gamma_1, \dots, \gamma_{\nu}, \theta_1, \dots, \theta_{\nu}) \sum_{s_{\nu+1} > \frac{n}{\gamma_{\nu+1}}} 2^{-s_{\nu+1}\alpha\gamma_{\nu+1}\theta_{\nu+1}} \\ &\leq C(\gamma_1, \dots, \gamma_{\nu}, \theta_1, \dots, \theta_{\nu}) 2^{-n\alpha\theta_{\nu+1}}. \end{aligned} \quad (4)$$

It follows from inequalities (2)–(4) that

$$\|\{b_{\bar{s}}(n, \bar{\gamma})\}\|_{\ell_{\bar{\theta}}^{\theta_{\nu+1}}(\mathbb{Z}_+^{\nu+1})} \leq C(\gamma, \theta) \cdot 2^{-n\alpha} \cdot n^{\sum_{j=2}^{\nu+1} \frac{1}{\theta_j}}.$$

The lemma is proved.  $\square$

**Lemma 2.** Suppose that  $\bar{\theta} = (\theta_1, \dots, \theta_m)$ ,  $1 \leq \theta_j < +\infty$ ,  $j = 1, \dots, m$ ,  $\bar{\gamma} = (\gamma_1, \dots, \gamma_m)$ ,  $\bar{\gamma}' = (\gamma'_1, \dots, \gamma'_m)$  satisfy  $\gamma_1 = \dots = \gamma_{\nu} = \gamma'_1 = \dots = \gamma'_{\nu} = 1 < \gamma'_j < \gamma_j$  for any  $j = \nu + 1, \dots, m$  and

$$b_{\bar{s}}(n, \bar{\gamma}, \bar{\gamma}') = \begin{cases} 2^{-\alpha(\bar{s}, \bar{\gamma})} & \text{if } \bar{s} \in Y^m(n, \bar{\gamma}'), \\ 0 & \text{for } \bar{s} \notin Y^m(n, \bar{\gamma}'), \end{cases} \quad \alpha > 0.$$

Then the following inequality holds:

$$\|\{b_{\bar{s}}(n, \bar{\gamma}, \bar{\gamma}')\}\|_{\ell_{\bar{\theta}}^{\theta_{\nu+1}}(\mathbb{Z}_+^m)} \leq C(\theta, \bar{\gamma}, \bar{\gamma}') \cdot 2^{-n\alpha} \cdot n^{\sum_{j=2}^{\nu} \frac{1}{\theta_j}}.$$

**Proof.** The set  $Y^m(n, \bar{\gamma}')$  can be expressed as

$$\begin{aligned} Y^m(n, \bar{\gamma}') &= \left\{ \bar{s} \in \mathbb{Z}_+^m : \sum_{j=\nu+1}^m s_j \gamma'_j < n \text{ and } \sum_{j=1}^{\nu} s_j \gamma'_j > n - \sum_{j=\nu+1}^m s_j \gamma'_j \right\} \\ &\cup \left\{ \bar{s} \in \mathbb{Z}_+^m : \sum_{j=\nu+1}^m s_j \gamma'_j \geq n, s_1, \dots, s_{\nu} \geq 0 \right\} = Y_1^m(\bar{\gamma}') \cup Y_2^m(\bar{\gamma}'). \end{aligned}$$

We define the numbers  $b_{\bar{s}}^{(1)}(n, \bar{\gamma}, \bar{\gamma}')$  and  $b_{\bar{s}}^{(2)}(n, \bar{\gamma}, \bar{\gamma}')$  as follows:

$$b_{\bar{s}}^{(1)}(n, \bar{\gamma}, \bar{\gamma}') = \begin{cases} 2^{-\alpha(\bar{s}, \bar{\gamma})}, & \bar{s} \in Y_1^m(\bar{\gamma}'), \\ 0, & \bar{s} \in Y_2^m(\bar{\gamma}'), \end{cases} \quad b_{\bar{s}}^{(2)}(n, \bar{\gamma}, \bar{\gamma}') = b_{\bar{s}}(n, \bar{\gamma}, \bar{\gamma}') - b_{\bar{s}}^{(1)}(n, \bar{\gamma}, \bar{\gamma}').$$

Obviously,  $b_{\bar{s}}(n, \bar{\gamma}, \bar{\gamma}') = b_{\bar{s}}^{(2)}(n, \bar{\gamma}, \bar{\gamma}') + b_{\bar{s}}^{(1)}(n, \bar{\gamma}, \bar{\gamma}')$ .

Set

$$Y_3^{m-\nu}(\bar{\gamma}', n) = \{(s_{\nu+1}, \dots, s_m) \in \mathbb{Z}_+^{m-\nu} : \sum_{j=\nu+1}^m s_j \gamma'_j < n\}.$$

Since  $\gamma_1 = \gamma'_1 = \dots = \gamma_\nu = \gamma'_\nu, \gamma'_j < \gamma_j$  for each  $j = \nu + 1, \dots, m$  and  $\alpha > 0$ , by Lemma 1 (with  $m$  replaced by  $\nu$  and  $n$  by  $n - \sum_{j=\nu+1}^m s_j \gamma'_j$ ), we have

$$\begin{aligned} & \| \{b_{\bar{s}}^{(1)}(n, \bar{\gamma}, \bar{\gamma}')\} \|_{\ell_{\bar{\theta}}(\mathbb{Z}_+^m)} \\ &= \left\| \left\{ 2^{-\alpha \sum_{j=\nu+1}^m s_j \gamma_j} \left\| \left\{ 2^{-\alpha \sum_{j=1}^{\nu} s_j \gamma_j} \right\} \right\|_{\ell_{\bar{\theta}}(Y^\nu(\bar{\gamma}', n - \sum_{j=\nu+1}^m s_j \gamma'_j))} \right\} \right\|_{\ell_{\bar{\theta}}(Y_3^{m-\nu}(\bar{\gamma}', n))} \\ &\leq C(\theta, \gamma) 2^{-n\alpha} \left\| \left\{ \left( n - \sum_{j=\nu+1}^m s_j \gamma'_j \right)^{\sum_{j=2}^{\nu} \frac{1}{\theta_j}} 2^{-\alpha \sum_{j=\nu+1}^m s_j (\gamma_j - \gamma'_j)} \right\} \right\|_{\ell_{\bar{\theta}}(Y_3^{m-\nu}(\bar{\gamma}', n))} \\ &\leq C(\theta, \gamma) 2^{-n\alpha} n^{\sum_{j=2}^{\nu} \frac{1}{\theta_j}} \cdot \prod_{j=\nu+1}^m \left\{ \sum_{s_j=0}^{\infty} 2^{-\alpha s_j \theta_j (\gamma_j - \gamma'_j)} \right\}^{\frac{1}{\theta_j}} \\ &= C(\theta, \gamma) 2^{-n\alpha} n^{\sum_{j=2}^{\nu} \frac{1}{\theta_j}}. \end{aligned} \tag{5}$$

Further, by the definition of  $b_{\bar{s}}^{(2)}(n, \bar{\gamma}, \bar{\gamma}')$  for all  $(s_{\nu+1}, \dots, s_m)$  such that  $\sum_{j=\nu+1}^m s_j \gamma'_j > n$ , the following estimate holds:

$$b_{\bar{s}}^{(2)}(n, \bar{\gamma}, \bar{\gamma}') = 2^{-\alpha(\bar{s}, \bar{\gamma})} \leq 2^{-n\alpha} 2^{-\alpha \sum_{j=1}^{\nu} s_j \gamma_j} 2^{-\alpha \sum_{j=\nu+1}^m s_j (\gamma_j - \gamma'_j)}.$$

Therefore, taking into account the fact that the norm is monotonic and also the inequality  $\gamma'_j < \gamma_j, j = \nu + 1, \dots, m$ , we obtain

$$\begin{aligned} & \| \{b_{\bar{s}}^{(2)}(n, \bar{\gamma}, \bar{\gamma}')\} \|_{\ell_{\bar{\theta}}(\mathbb{Z}_+^m)} = 2^{-n\alpha} \left\| \left\{ 2^{-\alpha \sum_{j=1}^{\nu} s_j \gamma_j} \cdot 2^{-\alpha \sum_{j=\nu+1}^m s_j (\gamma_j - \gamma'_j)} \right\} \right\|_{\ell_{\bar{\theta}}(Y_2^m(\bar{\gamma}', n))} \\ &\leq 2^{-n\alpha} \cdot \prod_{j=1}^{\nu} \left\{ \sum_{s_j=0}^{\infty} 2^{-\alpha \theta_j s_j} \right\}^{\frac{1}{\theta_j}} \cdot \prod_{j=\nu+1}^m \left\{ \sum_{s_j=0}^{\infty} 2^{-\alpha \theta_j s_j (\gamma_j - \gamma'_j)} \right\}^{\frac{1}{\theta_j}} = C(\gamma, \theta) \cdot 2^{-n\alpha}. \end{aligned} \tag{6}$$

Now, by (5) and (6), we have

$$\begin{aligned} \| \{b_{\bar{s}}(n, \bar{\gamma}, \bar{\gamma}')\} \|_{\ell_{\bar{\theta}}(\mathbb{Z}_+^m)} &\leq \| \{b_{\bar{s}}^{(1)}(n, \bar{\gamma}, \bar{\gamma}')\} \|_{\ell_{\bar{\theta}}(\mathbb{Z}_+^m)} + \| \{b_{\bar{s}}^{(2)}(n, \bar{\gamma}, \bar{\gamma}')\} \|_{\ell_{\bar{\theta}}(\mathbb{Z}_+^m)} \\ &\leq C(\theta, \gamma) \cdot 2^{-n\alpha} \cdot n^{\sum_{j=2}^{\nu} \frac{1}{\theta_j}}. \end{aligned}$$

The lemma is proved. □

**Remark 1.** For  $\theta_1 = \dots = \theta_m$ , Lemmas 1 and 2 imply Lemmas B and C from [7].

Set

$$\bar{f}(\bar{t}) = \sup_{|E_m| \geq t_m} \frac{1}{|E_m|} \int_{E_m} \dots \sup_{|E_1| \geq t_1} \frac{1}{|E_1|} \int_{E_1} |f(x_1, \dots, x_m)| dx_1 \dots dx_m,$$

where  $|E_j|$  is the Lebesgue measure of the set  $E_j \in [0, 2\pi)$ .

Now, let us prove the main assertions.

**Theorem 1.** Suppose that  $1 \leq p_j < +\infty$ ,  $j = 1, \dots, m$ . Then, for any function  $f \in L_{\bar{p}}(I^m)$ , the following inequality holds:

$$\bar{f}(\bar{t}) \leq C(p, m) \left\{ \prod_{j=1}^m t_j^{-\frac{1}{p_j}} \|f - U_{\bar{n}}(f)\|_{\bar{p}} + \sum_{e \subset \{1, \dots, m\}} \prod_{j \notin e} t_j^{-\frac{1}{p_j}} \sum_{\bar{s} \in G_e(\bar{n})} \prod_{j \in e} 2^{\frac{s_j}{p_j}} \|\delta_{\bar{s}}(f)\|_{\bar{p}} \right\}$$

for all  $t_j \in (2^{-n_j-1}, 2^{-n_j}]$ ,  $n_j = 0, 1, \dots$ ,  $j = 1, \dots, m$ .

**Proof.** Suppose that  $E_j \subset [0, 2\pi]$  are Lebesgue measurable sets. Then, by the property of the integral, we have

$$\int_{E_m} \cdots \int_{E_1} |f(x_1, \dots, x_m)| dx_1 \cdots dx_m \leq \int_{E_m} \cdots \int_{E_1} |f(\bar{x}) - U_{\bar{n}}(f, \bar{x})| dx_1 \cdots dx_m + \int_{E_m} \cdots \int_{E_1} |U_{\bar{n}}(f, \bar{x})| d\bar{x}. \quad (7)$$

Applying Hölder's integral inequality, we obtain

$$\int_{E_m} \cdots \int_{E_1} |f(\bar{x}) - U_{\bar{n}}(f, \bar{x})| d\bar{x} \leq \prod_{j=1}^m |E_j|^{\frac{1}{p'_j}} \|f - U_{\bar{n}}(f)\|_{\bar{p}}, \quad (8)$$

where  $p'_j = p_j/(p_j - 1)$ ,  $j = 1, \dots, m$ .

Suppose that  $e = \{1, \dots, i\} \subset \{1, \dots, m\}$ . Then, applying the inequality for different metrics for the polynomials in the trigonometric system (see Lemma B) for the variables  $x_1, \dots, x_i$ , and Hölder's inequality for the other variables, we obtain

$$\begin{aligned} & \int_{E_m} \cdots \int_{E_1} \left| \sum_{\bar{s} \in G_e(\bar{n})} \delta_{\bar{s}}(f, \bar{x}) \right| d\bar{x} \leq \int_{E_m} \cdots \int_{E_1} \sum_{\bar{s} \in G_e(\bar{n})} |\delta_{\bar{s}}(f, \bar{x})| d\bar{x} \\ &= \int_{E_m} \cdots \int_{E_1} \sum_{s_1=1}^{n_1} \cdots \sum_{s_i=1}^{n_i} \sum_{s_{i+1}=n_{i+1}+1}^{\infty} \cdots \sum_{s_m=n_m+1}^{\infty} |\delta_{\bar{s}}(f, \bar{x})| d\bar{x} \\ &\leq C \cdot \sum_{s_1=1}^{n_1} \cdots \sum_{s_i=1}^{n_i} \sum_{s_{i+1}=n_{i+1}+1}^{\infty} \cdots \sum_{s_m=n_m+1}^{\infty} \prod_{j=1}^i 2^{\frac{s_j}{p_j}} \|\delta_{\bar{s}}(f)\|_{\bar{p}} \cdot \prod_{j=1}^i |E_j| \cdot \prod_{j=i+1}^m |E_j|^{\frac{1}{p'_j}} \\ &= C \cdot \prod_{j \in e} |E_j| \cdot \prod_{j \notin e} |E_j|^{\frac{1}{p'_j}} \sum_{\bar{s} \in G_e(\bar{n})} \prod_{j \in e} 2^{\frac{s_j}{p_j}} \|\delta_{\bar{s}}(f)\|_{\bar{p}}. \end{aligned} \quad (9)$$

For the other sets  $e \subset \{1, \dots, m\}$ , the proof of inequality (9) is similar.

In the definition of the function  $\bar{f}$ , the measures  $|E_j|$  satisfy  $|E_j| \geq t_j$ ,  $j = 1, \dots, m$ . Therefore, from inequalities (7)–(9) we obtain

$$\begin{aligned} \prod_{j=1}^m \frac{1}{|E_j|} \int_{E_m} \cdots \int_{E_1} |f(\bar{x})| d\bar{x} &\leq C \cdot \left\{ \prod_{j=1}^m t_j^{-\frac{1}{p_j}} \|f - U_{\bar{n}}(f)\|_{\bar{p}} \right. \\ &\quad \left. + \sum_{e \subset \{1, \dots, m\}} \prod_{j \notin e} t_j^{-\frac{1}{p_j}} \sum_{\bar{s} \in G_e(\bar{n})} \prod_{j \in e} 2^{\frac{s_j}{p_j}} \|\delta_{\bar{s}}(f)\|_{\bar{p}} \right\}. \end{aligned}$$

Hence, by the definition of the least upper bound of a set, we obtain the assertion of the theorem.  $\square$

**Theorem 2.** Suppose that  $1 \leq p_j < q_j < +\infty$ ,  $1 \leq \theta_j < +\infty$ . If  $f \in L_{\bar{p}}(I^m)$  and the quantity

$$\sigma(f) \equiv \left\{ \sum_{s_m=1}^{\infty} 2^{s_m \theta_m \left(\frac{1}{p_m} - \frac{1}{q_m}\right)} \left[ \cdots \left[ \sum_{s_1=1}^{\infty} 2^{s_1 \theta_1 \left(\frac{1}{p_1} - \frac{1}{q_1}\right)} \|\delta_{\bar{s}}(f)\|_{\bar{p}}^{\theta_1} \right]^{\frac{\theta_2}{\theta_1}} \cdots \right]^{\frac{\theta_m}{\theta_{m-1}}} \right]^{\frac{1}{\theta_m}}$$

is finite, then the function  $f$  belongs to  $L_{\bar{q},\bar{\theta}}(I^m)$  and the following inequality holds:

$$\|f\|_{\bar{q},\bar{\theta}} \leq C(\bar{p}, \bar{q}, \bar{\theta}) \cdot \sigma(f).$$

**Proof.** Blozinski [1, p. 161] proved the relation

$$\|f\|_{\bar{q},\bar{\theta}} \asymp \|\bar{f}\|_{\bar{q},\bar{\theta}}, \quad 1 \leq q_j < +\infty, \quad 1 \leq \theta_j < +\infty, \quad j = 1, \dots, m.$$

Using this relation, we find

$$\|f\|_{\bar{q},\bar{\theta}} \leq C\|\bar{f}\|_{\bar{q},\bar{\theta}} = C \left\{ \sum_{n_m=0}^{\infty} \int_{2^{-n_m-1}}^{2^{-n_m}} t_m^{\frac{\theta_m}{q_m}-1} \left[ \dots \left[ \sum_{n_1=0}^{\infty} \int_{2^{-n_1-1}}^{2^{-n_1}} t_1^{\frac{\theta_1}{q_1}-1} \right. \right. \right. \\ \left. \left. \left. \times (\bar{f}(t_1, \dots, t_m))^{\theta_1} dt_1 \right]^{\frac{\theta_2}{\theta_1}} \dots \right]^{\frac{\theta_m}{\theta_{m-1}}} dt_m \right\}^{\frac{1}{\theta_m}}. \quad (10)$$

Further, applying Theorem 1 and taking into account the relation

$$\int_{2^{-n-1}}^{2^{-n}} t^{\beta-1} dt \asymp 2^{-n\beta}, \quad (11)$$

from (10) we obtain

$$\|f\|_{\bar{q},\bar{\theta}} \leq C \left\{ \sum_{n_m=0}^{\infty} 2^{n_m \theta_m \left(\frac{1}{p_m} - \frac{1}{q_m}\right)} \left[ \dots \left[ \sum_{n_1=0}^{\infty} 2^{n_1 \theta_1 \left(\frac{1}{p_1} - \frac{1}{q_1}\right)} \|f - U_{\bar{n}}(f)\|_{\bar{p}}^{\theta_1} \right]^{\frac{\theta_2}{\theta_1}} \dots \right]^{\frac{\theta_m}{\theta_{m-1}}} \right\}^{\frac{1}{\theta_m}} \\ + \left\{ \sum_{n_m=0}^{\infty} \int_{2^{-n_m-1}}^{2^{-n_m}} t_m^{\frac{\theta_m}{q_m}-1} \left[ \dots \left[ \sum_{n_1=0}^{\infty} \int_{2^{-n_1-1}}^{2^{-n_1}} t_1^{\frac{\theta_1}{q_1}-1} \left( \sum_{e \subset \{1, \dots, m\}, j \notin e} \prod t_j^{-\frac{1}{p_j}} \right. \right. \right. \\ \left. \left. \left. \times \sum_{\bar{s} \in G_e(\bar{n})} \prod_{j \in e} 2^{s_j} \|\delta_{\bar{s}}(f)\|_{\bar{p}} \right)^{\theta_1} \right]^{\frac{\theta_2}{\theta_1}} \dots \right]^{\frac{\theta_m}{\theta_{m-1}}} dt_m \right\}^{\frac{1}{\theta_m}}. \quad (12)$$

Suppose that  $e = \{1, \dots, i\}$ ,  $i \leq m$ . Then, using relation (11) and successively applying the triangle inequality, we obtain

$$\left\{ \sum_{n_m=0}^{\infty} \int_{2^{-n_m-1}}^{2^{-n_m}} t_m^{\frac{\theta_m}{q_m}-1} \left[ \dots \left[ \sum_{n_1=0}^{\infty} \int_{2^{-n_1-1}}^{2^{-n_1}} t_1^{\frac{\theta_1}{q_1}-1} \left( \prod_{j=i+1}^m t_j^{-\frac{1}{p_j}} \sum_{s_m=n_m+1}^{\infty} \dots \sum_{s_{i+1}=n_{i+1}+1}^{\infty} \right. \right. \right. \\ \left. \left. \left. \times \sum_{s_i=1}^{n_i} \dots \sum_{s_1=1}^{n_1} \prod_{j=1}^i 2^{s_j} \|\delta_{\bar{s}}(f)\|_{\bar{p}} \right)^{\theta_1} \right]^{\frac{\theta_2}{\theta_1}} \dots \right]^{\frac{\theta_m}{\theta_{m-1}}} dt_m \right\}^{\frac{1}{\theta_m}} \\ \leq C \left\{ \sum_{n_m=0}^{\infty} 2^{n_m \theta_m \left(\frac{1}{p_m} - \frac{1}{q_m}\right)} \left[ \dots \sum_{n_{i+1}=0}^{\infty} 2^{n_{i+1} \theta_{i+1} \left(\frac{1}{p_{i+1}} - \frac{1}{q_{i+1}}\right)} \right. \right. \\ \left. \left. \times \left[ \sum_{n_i=0}^{\infty} 2^{-n_i \frac{\theta_i}{q_i}} \left[ \dots \left[ \sum_{n_1=0}^{\infty} 2^{-n_1 \frac{\theta_1}{q_1}} \left( \sum_{s_m=n_m+1}^{\infty} \dots \sum_{s_{i+1}=n_{i+1}+1}^{\infty} \sum_{s_i=1}^{n_i} \dots \sum_{s_1=1}^{n_1} \right. \right. \right. \right. \right. \\ \left. \left. \left. \times \prod_{j=1}^i 2^{s_j} \|\delta_{\bar{s}}(f)\|_{\bar{p}} \right)^{\theta_1} \right]^{\frac{\theta_2}{\theta_1}} \dots \right]^{\frac{\theta_{i+1}}{\theta_i}} \right]^{\frac{\theta_i}{\theta_{i-1}}} \dots \right]^{\frac{\theta_m}{\theta_{m-1}}} \right\}^{\frac{1}{\theta_m}} \\ \leq C \left\{ \sum_{n_m=0}^{\infty} 2^{n_m \theta_m \left(\frac{1}{p_m} - \frac{1}{q_m}\right)} \left( \sum_{s_m=n_m+1}^{\infty} \left[ \sum_{n_{m-1}=0}^{\infty} 2^{n_{m-1} \theta_{m-1} \left(\frac{1}{p_{m-1}} - \frac{1}{q_{m-1}}\right)} \right. \right. \right.$$

$$\begin{aligned}
& \times \left( \sum_{s_{m-1}=n_{m-1}+1}^{\infty} \left[ \dots \left[ \sum_{n_{i+1}=0}^{\infty} 2^{n_{i+1}\theta_{i+1}\left(\frac{1}{p_{i+1}}-\frac{1}{q_{i+1}}\right)} \left( \sum_{s_{i+1}=n_{i+1}+1}^{\infty} \left[ \sum_{n_i=0}^{\infty} 2^{-n_i\frac{\theta_i}{q_i}} \right. \right. \right. \right. \right. \\
& \times \left( \sum_{s_i=1}^{n_i} 2^{\frac{s_i}{p_i}} \left[ \sum_{n_{i-1}=0}^{\infty} 2^{-n_{i-1}\frac{\theta_{i-1}}{q_{i-1}}} \left( \sum_{s_{i-1}=1}^{n_{i-1}} 2^{\frac{s_{i-1}}{p_{i-1}}} \dots \left[ \sum_{n_1=0}^{\infty} 2^{-n_1\frac{\theta_1}{q_1}} \right. \right. \right. \right. \\
& \times \left. \left. \left. \left. \left( \sum_{s_1=1}^{n_1} 2^{\frac{s_1}{p_1}} \|\delta_{\bar{s}}(f)\|_{\bar{p}} \right)^{\theta_1} \right]^{\frac{\theta_2}{\theta_1}} \dots \right)^{\theta_{i-1}} \right]^{\frac{\theta_i}{\theta_{i-1}}} \right]^{\theta_i} \right]^{\frac{\theta_{i+1}}{\theta_i}} \right]^{\theta_{i+1}} \dots \right]^{\frac{\theta_m}{\theta_{m-1}}} \right)^{\theta_m} \left. \right\}^{\frac{1}{\theta_m}}
\end{aligned}$$

(we apply part a) of Lemma A to the sum over the indices  $n_j$ ,  $j = 1, \dots, i$ , and part b) to the sum over  $n_j$ ,  $j = i + 1, \dots, m$ )

$$\begin{aligned}
& \leq C \left\{ \sum_{n_m=0}^{\infty} 2^{n_m\theta_m\left(\frac{1}{p_m}-\frac{1}{q_m}\right)} \left[ \dots \left[ \sum_{n_{i+1}=0}^{\infty} 2^{n_{i+1}\theta_{i+1}\left(\frac{1}{p_{i+1}}-\frac{1}{q_{i+1}}\right)} \left[ \sum_{n_i=0}^{\infty} 2^{-n_i\frac{\theta_i}{q_i}} \left( \sum_{s_i=1}^{n_i} 2^{\frac{s_i}{p_i}} \right. \right. \right. \right. \right. \\
& \times \left. \left. \left. \left. \left[ \sum_{n_{i-1}=0}^{\infty} 2^{-n_{i-1}\frac{\theta_{i-1}}{q_{i-1}}} \left( \sum_{s_{i-1}=1}^{n_{i-1}} 2^{\frac{s_{i-1}}{p_{i-1}}} \dots \left[ \sum_{n_1=0}^{\infty} 2^{-n_1\frac{\theta_1}{q_1}} \left( \sum_{s_1=1}^{n_1} 2^{\frac{s_1}{p_1}} \|\delta_{\bar{s}}(f)\|_{\bar{p}} \right)^{\theta_1} \right]^{\frac{\theta_2}{\theta_1}} \dots \right. \right. \right. \right. \\
& \leq C \left\{ \sum_{n_m=0}^{\infty} 2^{n_m\theta_m\left(\frac{1}{p_m}-\frac{1}{q_m}\right)} \left[ \sum_{n_{m-1}=0}^{\infty} 2^{n_{m-1}\theta_{m-1}\left(\frac{1}{p_{m-1}}-\frac{1}{q_{m-1}}\right)} \dots \right. \right. \\
& \times \left. \left. \left[ \sum_{n_1=0}^{\infty} 2^{n_1\theta_1\left(\frac{1}{p_1}-\frac{1}{q_1}\right)} \|\delta_{\bar{s}}(f)\|_{\bar{p}} \right]^{\frac{\theta_2}{\theta_1}} \dots \right]^{\frac{\theta_m}{\theta_{m-1}}} \right\}^{\frac{1}{\theta_m}}. \tag{13}
\end{aligned}$$

Inequalities (12) and (13) imply

$$\begin{aligned}
\|f\|_{\bar{q},\bar{\theta}} & \leq C \left[ \left\{ \sum_{n_m=0}^{\infty} 2^{n_m\theta_m\left(\frac{1}{p_m}-\frac{1}{q_m}\right)} \left[ \dots \left[ \sum_{n_1=0}^{\infty} 2^{n_1\theta_1\left(\frac{1}{p_1}-\frac{1}{q_1}\right)} \right. \right. \right. \right. \\
& \times \left. \left. \left. \left. \left[ \sum_{n_{i+1}=0}^{\infty} 2^{-n_{i+1}\frac{\theta_{i+1}}{q_{i+1}}} \left( \sum_{s_{i+1}=1}^{n_{i+1}} 2^{\frac{s_{i+1}}{p_{i+1}}} \dots \left[ \sum_{n_i=0}^{\infty} 2^{-n_i\frac{\theta_i}{q_i}} \left( \sum_{s_i=1}^{n_i} 2^{\frac{s_i}{p_i}} \|\delta_{\bar{s}}(f)\|_{\bar{p}} \right)^{\theta_i} \right]^{\frac{\theta_{i+1}}{\theta_i}} \dots \right. \right. \right. \right. \\
& + \left. \left. \left. \left. \left[ \sum_{n_m=0}^{\infty} 2^{n_m\theta_m\left(\frac{1}{p_m}-\frac{1}{q_m}\right)} \left[ \sum_{n_{m-1}=0}^{\infty} 2^{n_{m-1}\theta_{m-1}\left(\frac{1}{p_{m-1}}-\frac{1}{q_{m-1}}\right)} \dots \right. \right. \right. \right. \\
& \times \left. \left. \left. \left. \left[ \sum_{n_1=0}^{\infty} 2^{n_1\theta_1\left(\frac{1}{p_1}-\frac{1}{q_1}\right)} \|\delta_{\bar{s}}(f)\|_{\bar{p}} \right]^{\frac{\theta_2}{\theta_1}} \dots \right]^{\frac{\theta_m}{\theta_{m-1}}} \right\} \right]. \tag{14}
\end{aligned}$$

Since

$$f(\bar{x}) - U_{\bar{n}}(f, \bar{x}) = \sum_{s_m=n_m+1}^{\infty} \dots \sum_{s_1=n_1+1}^{\infty} \delta_{\bar{s}}(f, \bar{x}),$$

using the property of the norm and Lemma A, we obtain

$$\begin{aligned}
& \left\{ \sum_{n_m=0}^{\infty} 2^{n_m\theta_m\left(\frac{1}{p_m}-\frac{1}{q_m}\right)} \left[ \dots \left[ \sum_{n_1=0}^{\infty} 2^{n_1\theta_1\left(\frac{1}{p_1}-\frac{1}{q_1}\right)} \|f - U_{\bar{n}}(f)\|_{\bar{p}} \right]^{\frac{\theta_2}{\theta_1}} \dots \right]^{\frac{\theta_m}{\theta_{m-1}}} \right\}^{\frac{1}{\theta_m}} \\
& \leq \left\{ \sum_{n_m=0}^{\infty} 2^{n_m\theta_m\left(\frac{1}{p_m}-\frac{1}{q_m}\right)} \left[ \dots \left[ \sum_{n_1=0}^{\infty} 2^{n_1\theta_1\left(\frac{1}{p_1}-\frac{1}{q_1}\right)} \|\delta_{\bar{n}}(f)\|_{\bar{p}} \right]^{\frac{\theta_2}{\theta_1}} \dots \right]^{\frac{\theta_m}{\theta_{m-1}}} \right\}^{\frac{1}{\theta_m}}. \tag{15}
\end{aligned}$$

Inequalities (14) and (15) imply the assertion of the theorem.  $\square$

**Remark 2.** It is well known that, in the case  $1 < q_j < +\infty$ ,  $1 < \theta_j < +\infty$ ,  $j = 1, \dots, m$ , the space  $L_{\bar{q}, \bar{\theta}}(I^m)$  has absolutely continuous norms. Therefore, the adjoint space  $L_{\bar{q}, \bar{\theta}}^*(I^m)$  coincides with the dual space  $L_{\bar{q}', \bar{\theta}'}(I^m)$ , where

$$\bar{q}' = (q'_1, \dots, q'_m), \quad \bar{\theta}' = (\theta'_1, \dots, \theta'_m), \quad q'_j = \frac{q_j}{q_j - 1}, \quad \theta'_j = \frac{\theta_j}{\theta_j - 1}, \quad j = 1, \dots, m$$

(see [1, p. 161]).

Now, using the duality principle, just as in [7], and Theorem 2, we can verify the validity of the following assertion.

**Theorem 3.** Suppose that  $1 < q_j < \tau_j < +\infty$ ,  $1 < \theta_j < +\infty$ ,  $j = 1, \dots, m$ . If  $f \in L_{\bar{q}, \bar{\theta}}(I^m)$ ,

$$f(\bar{x}) \sim \sum_{\bar{s} \in \mathbb{Z}_+^m} b_{\bar{s}} \sum_{\bar{k} \in \rho(\bar{s})} e^{i\langle \bar{k}, \bar{x} \rangle},$$

then the following inequality holds:

$$\|f\|_{\bar{q}, \bar{\theta}} \geq C \cdot \left\{ \sum_{s_m=1}^{\infty} 2^{s_m \theta_m \left(\frac{1}{\tau_m} - \frac{1}{q_m}\right)} \left[ \dots \left[ \sum_{s_1=1}^{\infty} 2^{s_1 \theta_1 \left(\frac{1}{\tau_1} - \frac{1}{q_1}\right)} \|\delta_{\bar{s}}(f)\|_{\bar{\tau}}^{\frac{\theta_1}{\theta_1 - 1}} \right]^{\frac{\theta_2}{\theta_2 - 1}} \dots \right]^{\frac{\theta_m}{\theta_m - 1}} \right\}^{\frac{1}{\theta_m}}.$$

Further, let us consider the order of approximation of the functional class  $H_{\bar{p}}^{\bar{\tau}}$ .

First, note that the partial sum operator

$$S_{Q_n^{\bar{\tau}}}(f, \bar{x}) = \sum_{\bar{k} \in Q_n^{\bar{\tau}}} a_{\bar{k}}(f) \cdot e^{i\langle \bar{k}, \bar{x} \rangle}$$

is bounded in the space  $L_{\bar{q}, \bar{\theta}}(I^m)$ ,  $1 < q_j < +\infty$ ,  $1 < \theta_j < +\infty$ ,  $j = 1, \dots, m$ , i.e.,

$$\|S_{Q_n^{\bar{\tau}}}(f)\|_{\bar{q}, \bar{\theta}} \leq C \cdot \|f\|_{\bar{q}, \bar{\theta}}, \quad f \in L_{\bar{q}, \bar{\theta}}(I^m). \tag{16}$$

This assertion follows from the Marcinkiewicz theorem on multipliers (see [17, p. 239] and [2, Corollary 1, p. 106]).

From inequality (16), we obtain

$$\|f - S_{Q_n^{\bar{\tau}}}(f)\|_{\bar{q}, \bar{\theta}} \leq C(q, \theta) \cdot E_n^{(\bar{\tau})}(f)_{\bar{q}, \bar{\theta}}, \quad f \in L_{\bar{q}, \bar{\theta}}(I^m). \tag{17}$$

**Theorem 4.** Suppose that

$$1 \leq p_j < q_j < +\infty, \quad 1 \leq \theta_j < +\infty, \quad \frac{1}{p_j} - \frac{1}{q_j} < r_j, \quad j = 1, \dots, m,$$

$$r_1 = \dots = r_\nu < r_{\nu+1} \leq \dots \leq r_m, \quad \gamma_j = \frac{r_j}{r_1}, \quad j = 1, \dots, m,$$

$$\frac{1}{p_j} - \frac{1}{q_j} = \frac{1}{p_1} - \frac{1}{q_1}, \quad j = 1, \dots, \nu, \quad r_1 \left(\frac{1}{p_j} - \frac{1}{q_j}\right) < r_j \left(\frac{1}{p_1} - \frac{1}{q_1}\right), \quad j = \nu + 1, \dots, m.$$

Then the following relation holds:

$$\sup_{f \in H_{\bar{p}}^{\bar{\tau}}} E_n^{(\bar{\tau})}(f)_{\bar{q}, \bar{\theta}} \asymp 2^{-n \left(r_1 + \frac{1}{q_1} - \frac{1}{p_1}\right)} \cdot n^{\sum_{j=2}^{\nu} \frac{1}{\theta_j}}.$$

**Proof.** Let us prove that, for any  $f \in H_{\bar{p}}^{\bar{\gamma}}$ , we have the following inequality:

$$\|f - S_{Q_n^{\bar{\gamma}}}(f)\|_{\bar{q}, \bar{\theta}} \leq C \cdot 2^{-n(r_1 + \frac{1}{q_1} - \frac{1}{p_1})} \cdot n^{\sum_{j=2}^{\nu} \frac{1}{\theta_j}}. \quad (18)$$

Applying Theorem 2 to the function  $f - S_{Q_n^{\bar{\gamma}}}(f) \in L_{\bar{q}, \bar{\theta}}(I^m)$ , we see that

$$\begin{aligned} \|f - S_{Q_n^{\bar{\gamma}}}(f)\|_{\bar{q}, \bar{\theta}} &\leq C \cdot \left\{ \sum_{s_m=0}^{\infty} 2^{s_m(\frac{1}{p_m} - \frac{1}{q_m})\theta_m} \left[ \dots \left[ \sum_{s_1=0}^{\infty} 2^{s_1(\frac{1}{p_1} - \frac{1}{q_1})\theta_1} \right. \right. \right. \\ &\quad \left. \left. \left. \times \|\delta_{\bar{s}}(f - S_{Q_n^{\bar{\gamma}}}(f))\|_{\bar{p}} \right]^{\frac{\theta_2}{\theta_1}} \dots \right]^{\frac{\theta_m}{\theta_{m-1}}} \right\}^{\frac{1}{\theta_m}}. \end{aligned} \quad (19)$$

Since

$$\delta_{\bar{s}}(f - S_{Q_n^{\bar{\gamma}}}(f)) = \begin{cases} 0 & \text{for any } \bar{s} \in Y^m(\bar{\gamma}, n), \\ \delta_{\bar{s}}(f) & \text{for any } \bar{s} \notin Y^m(\bar{\gamma}, n), \end{cases}$$

from inequality (19) we obtain

$$\begin{aligned} \|f - S_{Q_n^{\bar{\gamma}}}(f)\|_{\bar{q}, \bar{\theta}} &\leq C \cdot \left\| \left\{ \prod_{j=1}^m 2^{s_j(\frac{1}{p_j} - \frac{1}{q_j})} \|\delta_{\bar{s}}(f)\|_{\bar{p}} \right\}_{\bar{s} \in Y^m(\bar{\gamma}, n)} \right\|_{\ell_{\bar{\theta}}(\mathbb{Z}_+^m)} \\ &\leq C \cdot \left\| \left\{ \prod_{j=1}^m 2^{-s_j(r_j + \frac{1}{q_j} - \frac{1}{p_j})} \right\}_{\bar{s} \in Y^m(\bar{\gamma}, n)} \right\|_{\ell_{\bar{\theta}}(\mathbb{Z}_+^m)}. \end{aligned} \quad (20)$$

In Lemma 2, set

$$\alpha = r_1 + \frac{1}{q_1} - \frac{1}{p_1} > 0, \quad \gamma_j = \frac{r_j + 1/q_j - 1/p_j}{r_1 + 1/q_1 - 1/p_1}, \quad \gamma'_j = \frac{r_j}{r_1}, \quad j = 1, \dots, m.$$

Then the condition  $r_1 = \dots = r_\nu$  implies  $\gamma'_j = 1, j = 1, \dots, \nu$ , and

$$\begin{aligned} \frac{1}{p_1} - \frac{1}{q_1} = \dots = \frac{1}{p_\nu} - \frac{1}{q_\nu} &\implies \gamma_j = 1, \quad j = 1, \dots, \nu, \\ r_1 \left( \frac{1}{p_j} - \frac{1}{q_j} \right) < r_j \left( \frac{1}{p_1} - \frac{1}{q_1} \right) &\implies \gamma'_j < \gamma_j, \quad j = \nu + 1, \dots, m. \end{aligned}$$

In view of Lemma 2, it follows from (20) that

$$\|f - S_{Q_n^{\bar{\gamma}}}(f)\|_{\bar{q}, \bar{\theta}} \leq C \cdot 2^{-n(r_1 + \frac{1}{q_1} - \frac{1}{p_1})} \cdot n^{\sum_{j=2}^{\nu} \frac{1}{\theta_j}}$$

for any function  $f \in H_{\bar{p}}^{\bar{\gamma}}$ . Estimate (18) is proved.

Let us prove the lower bound. To do this, consider the function

$$f_0(\bar{x}) = \sum_{\bar{s} \in \mathbb{Z}_+^m} \prod_{j=1}^m 2^{-s_j(r_j + 1 - \frac{1}{p_j})} \sum_{\bar{k} \in \rho(\bar{s})} e^{i\langle \bar{k}, \bar{x} \rangle}.$$

It is well known that  $(1/C_0)f_0 \in H_{\bar{p}}^{\bar{\gamma}}, 1 < p_j < +\infty, j = 1, \dots, m$ , where  $C_0$  is a positive constant.

Let us choose a number  $\tau \in (1, p_j)$  for all  $j = 1, \dots, m$  and apply Theorem 3 to the function  $f_0 - S_{Q_n^{\bar{\gamma}}}(f_0)$ . Then

$$\begin{aligned} &\|f_0 - S_{Q_n^{\bar{\gamma}}}(f_0)\|_{\bar{q}, \bar{\theta}} \\ &\geq C \cdot \left\{ \sum_{s_m=0}^{\infty} 2^{s_m \theta_m (\frac{1}{\tau} - \frac{1}{q_m})} \left[ \dots \left[ \sum_{s_1=0}^{\infty} 2^{s_1 \theta_1 (\frac{1}{\tau} - \frac{1}{q_1})} \|\delta_{\bar{s}}(f_0 - S_{Q_n^{\bar{\gamma}}}(f_0))\|_{\tau} \right]^{\frac{\theta_2}{\theta_1}} \dots \right]^{\frac{\theta_m}{\theta_{m-1}}} \right\}^{\frac{1}{\theta_m}} \end{aligned}$$

$$\geq C \cdot \left\{ \sum_{s_m \leq \frac{n}{\gamma_m}} 2^{s_m \theta_m \left(\frac{1}{\tau} - \frac{1}{q_m}\right)} \left[ \sum_{s_{m-1} \leq \frac{1}{\gamma_{m-1}} (n - s_m \gamma_m)} 2^{s_{m-1} \theta_{m-1} \left(\frac{1}{\tau} - \frac{1}{q_{m-1}}\right)} \dots \right. \right. \\ \left. \left. \times \left[ \sum_{s_1 \leq (n - \sum_{j=2}^m s_j \gamma_j)} 2^{s_1 \theta_1 \left(\frac{1}{\tau} - \frac{1}{q_1}\right)} \|\delta_{\bar{s}}(f_0)\|_{\tau}^{\theta_1} \right]^{\frac{\theta_2}{\theta_1}} \dots \right]^{\frac{\theta_m}{\theta_{m-1}}} \right\}^{\frac{1}{\theta_m}}. \tag{21}$$

Taking into account the relation

$$\|\delta_{\bar{s}}(f_0)\|_{\tau} \asymp \prod_{j=1}^m 2^{s_j \left(\frac{1}{p_j} - \frac{1}{\tau} - r_j\right)},$$

from (21) and (17) we obtain

$$E_n^{(\bar{\gamma})}(f_0)_{\bar{q}, \bar{\theta}} \geq C \cdot \left\{ \sum_{s_m \leq \frac{n}{\gamma_m}} 2^{-s_m \theta_m \left(r_m + \frac{1}{q_m} - \frac{1}{p_m}\right)} \right. \\ \times \left[ \sum_{s_{m-1} \leq \frac{1}{\gamma_{m-1}} (n - s_m \gamma_m)} 2^{-s_{m-1} \theta_{m-1} \left(r_{m-1} + \frac{1}{q_{m-1}} - \frac{1}{p_{m-1}}\right)} \dots \right. \\ \left. \times \left[ \sum_{s_1 \leq n - \sum_{j=2}^m s_j \gamma_j} 2^{-s_1 \theta_1 \left(r_1 + \frac{1}{q_1} - \frac{1}{p_1}\right)} \right]^{\frac{\theta_2}{\theta_1}} \dots \right]^{\frac{\theta_m}{\theta_{m-1}}} \right\}^{\frac{1}{\theta_m}} \\ \geq C \cdot 2^{-n \left(r_1 + \frac{1}{q_1} - \frac{1}{p_1}\right)} \cdot n^{\sum_{j=2}^m \frac{1}{\theta_j}}.$$

The theorem is proved. □

**Theorem 5.** *Suppose that*

$$1 \leq p_j < q_j < +\infty, \quad j = 1, \dots, m, \quad \bar{\gamma} = \frac{\bar{r}}{r_1}, \quad r_1 = \dots = r_\nu < r_{\nu+1} \leq \dots \leq r_m, \\ \frac{1}{p_1} - \frac{1}{q_1} = \dots = \frac{1}{p_\nu} - \frac{1}{q_\nu}, \quad r_1 \left(\frac{1}{p_j} - \frac{1}{q_j}\right) < r_j \left(\frac{1}{p_1} - \frac{1}{q_1}\right), \quad j = \nu + 1, \dots, m.$$

Then the following estimate holds:

$$\sup_{f \in H_{\bar{p}}^{\bar{\gamma}}} E_n^{\bar{\gamma}}(f)_{\bar{q}, \infty} \leq C(p, r, q) \cdot 2^{-n \left(r_1 + \frac{1}{q_1} - \frac{1}{p_1}\right)} \cdot n^{\nu-1}.$$

**Proof.** Using the definition of the Marcinkiewicz space, Theorem 1 and taking the condition  $1/p_j - 1/q_j > 0, j = 1, \dots, m$ , into account, we obtain

$$\|f\|_{\bar{q}, \infty} \leq C(p, q) \cdot \sum_{\bar{s} \in \mathbb{Z}_+^m} \prod_{j=1}^m 2^{s_j \left(\frac{1}{p_j} - \frac{1}{q_j}\right)} \|\delta_{\bar{s}}(f)\|_{\bar{p}}.$$

This inequality, which holds for any function  $f \in H_{\bar{p}}^{\bar{\gamma}}$ , yields

$$\|f - S_{Q_n^{\bar{\gamma}}}(f)\|_{\bar{q}, \infty} \leq C(p, q) \cdot \sum_{\langle \bar{s}, \bar{\gamma} \rangle > n} \prod_{j=1}^m 2^{-s_j \left(r_j + \frac{1}{q_j} - \frac{1}{p_j}\right)}. \tag{22}$$

Set

$$\gamma'_j = \frac{r_j + 1/q_j - 1/p_j}{r_1 + 1/q_1 - 1/p_1}.$$

Then  $\gamma_j < \gamma'_j$  for all  $j = \nu + 1, \dots, m$ , because, by the assumption of the theorem,

$$r_1 \left( \frac{1}{p_j} - \frac{1}{q_j} \right) < r_j \left( \frac{1}{p_1} - \frac{1}{q_1} \right), \quad j = \nu + 1, \dots, m.$$

Now, using the estimate (see [7, Lemma C])

$$\sum_{(\bar{s}, \bar{\gamma}) > n} 2^{-\alpha(\bar{s}, \bar{\gamma})} \leq C(\gamma) \cdot 2^{-n\alpha} \cdot n^{\nu-1},$$

from inequality (22) we obtain

$$\|f - S_{Q_n}(f)\|_{\bar{q}, \infty} \leq C(p, q) \cdot 2^{-n \left( r_1 + \frac{1}{q_1} - \frac{1}{p_1} \right)} n^{\nu-1}.$$

The theorem is proved.  $\square$

**Remark 3.** Note that, in the case  $q_1 = \dots = q_m$ ,  $\theta_1 = \dots = \theta_m$ ,  $p_1 = \dots = p_m$ , Theorems 2–4 imply certain results due to Temlyakov [7].

**Remark 4.** Estimates of the value of a nonincreasing rearrangement of a function of one variable via its best approximation by trigonometric polynomials were established by É. A. Storozhenko [18].

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