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## Compactness of Commutators for Riesz Potential on Local Morrey-type spaces

The paper considers Morrey-type local spaces from  $LM_{p\theta}^w$ . The main work is the proof of the commutator compactness theorem for the Riesz potential  $[b, I_\alpha]$  in local Morrey-type spaces from  $LM_{p\theta}^{w_1}$  to  $LM_{q\theta}^{w_2}$ . We also give new sufficient conditions for the commutator to be bounded for the Riesz potential  $[b, I_\alpha]$  in local Morrey-type spaces from  $LM_{p\theta}^{w_1}$  to  $LM_{q\theta}^{w_2}$ . In the proof of the commutator compactness theorem for the Riesz potential, we essentially use the boundedness condition for the commutator for the Riesz potential  $[b, I_\alpha]$  in local Morrey-type spaces  $LM_{p\theta}^w$ , and use the sufficient conditions from the theorem of precompactness of sets in local spaces of Morrey type  $LM_{p\theta}^w$ . In the course of proving the commutator compactness theorem for the Riesz potential, we prove lemmas for the commutator ball for the Riesz potential  $[b, I_\alpha]$ . Similar results were obtained for global Morrey-type spaces  $GM_{p\theta}^w$  and for generalized Morrey spaces  $M_p^w$ .

*Keywords:* Compactness, Commutators, Riesz Potential, Local Morrey-type spaces.

### Introduction

First we give some definitions.

By  $\mathfrak{M}(I)$  we denote the set of all measurable functions on  $I$ . The symbol  $\mathfrak{M}^+(I)$  stands for the collection of all  $f \in \mathfrak{M}(I)$  which are non-negative on  $I$ , while  $\mathfrak{M}^+(I; \downarrow)$  and  $\mathfrak{M}^+(I; \uparrow)$  are used to denote the subset of those functions which are non-increasing and non-decreasing on  $I$ , respectively. When  $I = (0, \infty)$ , we write simply  $\mathfrak{M}^+$ ,  $\mathfrak{M}^\downarrow$  and  $\mathfrak{M}^\uparrow$  instead of  $\mathfrak{M}^+(I)$ ,  $\mathfrak{M}^+(I; \downarrow)$  and  $\mathfrak{M}^+(I; \uparrow)$ , accordingly. The family of all weight functions (also called just weights) on  $I$ , that is, locally integrable non-negative functions on  $(0, \infty)$ , is given by  $\mathcal{W}(I)$ .

For  $p \in (0, \infty)$  and  $w \in \mathfrak{M}^+(I)$ , we define the functional  $\|\cdot\|_{p,w,I}$  on  $\mathfrak{M}(I)$ , by

$$\|f\|_{p,w,I} := \begin{cases} (\int_I |f(x)|^p w(x) dx)^{\frac{1}{p}}, & \text{if } p < \infty; \\ \text{ess sup}_I |f(x)|w(x), & \text{if } p = \infty. \end{cases}$$

If, in addition,  $w \in \mathcal{W}(I)$ , then the weighted Lebesgue space  $L^p(w, I)$  is given by

$$L^p(w, I) = \{f \in \mathfrak{M}(I) : \|f\|_{p,w,I} < \infty\},$$

and it is equipped with the quasi-norm  $\|\cdot\|_{p,w,I}$ . When  $w \equiv 1$  on  $I$ , we write simply  $L^p(I)$  and  $\|\cdot\|_{p,I}$  instead of  $L^p(w, I)$  and  $\|\cdot\|_{p,w,I}$ , respectively.

Let  $1 \leq p, \theta \leq \infty$ ,  $w$  be a measurable non-negative function on  $(0, \infty)$ . The Local Morrey-type space  $LM_{p\theta}^w \equiv LM_{p\theta}^w(\mathbb{R}^n)$  is defined as the set of all functions  $f \in L_p^{loc}(\mathbb{R}^n)$  with finite quasi-norm

$$\|f\|_{LM_{p\theta}^w} \equiv \left\| w(r) \|f\|_{L_p(B(0,r))} \right\|_{L_\theta(0,\infty)},$$

where  $B(t, r)$  the ball with center at the point  $t$  and of radius  $r$ .

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The space  $LM_{p\theta}^w$  coincides with the known Morrey space  $M_p^\lambda$  at  $w(r) = r^{-\lambda}, \theta = \infty$ , where  $0 \leq \lambda \leq \frac{n}{p}$ , which, in turn, for  $\lambda = 0$  coincides with the space  $L_p(\mathbb{R}^n)$ .

Following the notation of [1, 2], we denote by  $\Omega_\theta$  the set of all functions which are non-negative, measurable on  $(0, \infty)$ , not equivalent to 0 and such that for some  $t > 0$

$$\|w(r)\|_{L_\theta(t, \infty)} < \infty.$$

Note that the space  $LM_{p\theta}^w$  is non-trivial, that is consists not only of functions equivalent to 0 on  $\mathbb{R}^n$ , if and only if  $w \in \Omega_\theta$ .

In this paper we consider the Riesz Potential in the following form

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}} dy.$$

The Riesz Potential  $I_\alpha$  plays an important role in the harmonic analysis and theory of operators.

For a function  $b \in L_{loc}(\mathbb{R}^n)$  by  $M_b$  denote multiplier operator  $M_b f = bf$ , where  $f$  is measurable function. Then the commutator between  $I_\alpha$  and  $M_b$  is defined by

$$[b, I_\alpha] = M_b I_\alpha - I_\alpha M_b = \int_{\mathbb{R}^n} \frac{[b(x) - b(y)]f(y)}{|x - y|^{n-\alpha}} dy.$$

The commutators for Riesz Potential were investigated [3–9].

It is said that the function  $b(x) \in L_\infty(\mathbb{R}^n)$  belongs to the space  $BMO(\mathbb{R}^n)$ , if

$$\|b\|_* = \sup_{Q \subset \mathbb{R}^n} \frac{1}{|Q|} \int_Q |b(x) - b_Q| dx = \sup_{Q \in \mathbb{R}^n} M(b, Q) < \infty,$$

where  $Q$  - cube  $\mathbb{R}^n$  and  $b_Q = \frac{1}{|Q|} \int_{\mathbb{R}^n} f(y) dy$ .

By  $VMO(\mathbb{R}^n)$  we denote the  $BMO$ -closure  $C_0^\infty(\mathbb{R}^n)$ , where  $C_0^\infty(\mathbb{R}^n)$  the set of all functions from  $C^\infty(\mathbb{R}^n)$  with compact support. Through the  $\chi(A)$  denotes the characteristic function of the set  $B \subset \mathbb{R}^n$ , and  ${}^c A$  denotes the complement of  $A$ .

The main purpose of this work is to find sufficient conditions for the compactness of commutators operators  $[b, I_\alpha]$  on the Local Morrey-type space  $LM_{p\theta}^w(\mathbb{R}^n)$ .

We note that in the case of the Morrey space this question was investigated in [4]. The following well-known theorem gives necessary and sufficient conditions for the boundedness and compactness for  $[b, I_\alpha]$  on the Local Morrey-type spaces  $LM_{p\theta}^w(\mathbb{R}^n)$ .

### 1 Formulas and theorems

To formulate the following theorem on the boundedness of the Hardy operator in weighted Lebesgue spaces, we introduce the notation.

Denote by

$$H^* g(t) := \int_t^\infty g(s) ds, \quad g \in \mathfrak{M}^+,$$

the Hardy operator.

$$W(t) := \int_0^t w(t) dw,$$

$$U_*(t) := \int_t^\infty u(t)du,$$

$$V_*(t) := \int_t^\infty v(t)dv.$$

*Theorem 1.* Let  $0 < q, p \leq \infty$ . Assume that  $u, v, w \in \mathcal{W}(0, \infty)$ . Then inequality

$$\|H_u^*(f)\|_{q,w,(0;\infty)} \leq c \|f_u^*\|_{p,w,(0;\infty)}, f \in \mathfrak{M}^\dagger$$

with the best constant  $c$  holds if and only if the following holds:

$$A_0^* := \sup_{t>0} \left( \int_t^\infty U_*^q(\tau)w(\tau)d\tau \right)^{\frac{1}{q}} V_*^{-\frac{1}{p}}(t),$$

$$A_1^* := \sup_{t>0} W^{\frac{1}{q}}(t) \left( \int_t^\infty \left( \frac{U_*(\tau)}{V_*(\tau)} \right)^{p'} v(\tau)d\tau \right)^{\frac{1}{p'}},$$

and in this case  $c \approx A_0^* + A_1^*$ .

*Theorem 2.* (see. [2]) Let  $1 < p < q < \infty$ ,  $0 < \alpha = n(\frac{1}{p} - \frac{1}{q})$ ,  $0 < \theta < \infty$ ,  $(w_1, w_2)$  satisfy the following condition

$$\left\| w_2(r) \left( \frac{r}{t+r} \right)^{\frac{n}{p}} \right\|_{L_\theta(0,\infty)} \leq \|w_1(r)\|_{L_\theta(t,\infty)}. \quad (1)$$

Then the operator  $I_\alpha$  is bounded from  $LM_{p\theta}^{w_1}(\mathbb{R}^n)$  to  $LM_{q\theta}^{w_2}(\mathbb{R}^n)$ .

It is well known that the boundedness of such operators on Morrey space  $LM_{p\theta}^\lambda(\mathbb{R}^n)$  was considered in [1, 2].

The following theorem on sufficient conditions for the precompactness of sets on Local Morrey-type and other spaces was proved in [10–14].

*Theorem 3.* (see. [13]) Suppose that  $1 \leq p \leq \theta \leq \infty$  and  $w \in \Omega_{p\theta}$ . Suppose that a subset  $S$  of  $LM_{p\theta}^w$  satisfies the following conditions:

$$\sup_{f \in S} \|f\|_{LM_{p\theta}^w} < \infty, \quad (2)$$

$$\limsup_{u \rightarrow 0} \sup_{f \in S} \|f(\cdot + u) - f(\cdot)\|_{LM_{p\theta}^w} = 0, \quad (3)$$

$$\limsup_{r \rightarrow \infty} \sup_{f \in S} \|f \chi_{cB(0,r)}\|_{LM_{p\theta}^w} = 0. \quad (4)$$

Then  $S$  is a pre-compact set in  $LM_{p\theta}^w(\mathbb{R}^n)$ .

*Theorem 4.* Let  $1 < p \leq q < \infty$ ,  $0 < \alpha < n$  and  $b \in BMO(\mathbb{R}^n)$ .  $1 < p < \frac{n}{\alpha} \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ ,  $w_1, w_2 \in \Omega_\theta$ . Then the condition

$$A_0^* := \sup_{t>0} \left( \int_t^\infty \int_\tau^\infty (1 + \ln \frac{\tau}{r}) dr w(\tau) d\tau \right)^{\frac{1}{q}} \left[ \int_t^\infty v(t)dv \right]^{-\frac{1}{p}}, \quad (5)$$

$$A_1^* := \sup_{t>0} W^{\frac{1}{q}}(t) \left( \int_t^\infty \left( \frac{U_*(\tau)}{V_*(\tau)} \right)^{p'} v(\tau)d\tau \right)^{\frac{1}{p'}} < \infty. \quad (6)$$

Then the commutator  $[b, I_\alpha]$  is the boundedness operator from  $LM_{p\theta}^{w_1}$  to  $LM_{q\theta}^{w_2}$ .

Note that for the case of Morrey space  $LM_{p\theta}^\lambda (0 < \lambda < 1)$  (i.e., if  $w(r) = r^{-\lambda}$ ) this assertion was proved earlier in [4], and in the case of  $\lambda = 0$  is - known Frechet-Kolmogorov theorem [15]. We note that the pre-compactness some sets in Banach function spaces were investigated in [16]. Theorem 4 is proved using theorem 5.4 from [17] and theorem 3.4 from [5].

Now we give theorem about the compactness of the operators  $[b, I_\alpha]$  on Local Morrey-type space  $LM_{p\theta}^w(\mathbb{R}^n)$ .

*Theorem 5.* Let  $1 < p \leq q < \infty$ ,  $0 < \alpha < n$  and  $b \in VMO(\mathbb{R}^n)$ .  $1 < p < \frac{n}{\alpha}$ ,  $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ ,  $w_1, w_2 \in \Omega_\theta$  satisfy the conditions (1), (5), (6). Then the commutator  $[b, I_\alpha]$  is a compact operator from  $LM_{p\theta}^{w_1}$  to  $LM_{p\theta}^{w_2}$ .

To prove this theorem we need the following auxiliary assertions.

*Lemma 1.* Let  $n \in \mathbb{N}$ ,  $1 \leq p \leq \infty$ ,  $1 \leq q \leq \infty$ ,  $0 < \alpha < n \left(1 - \frac{1}{q}\right)$ ,  $\beta > 0$ . Then there exists  $C > 0$ , depending only on  $n, p, q, \alpha$ , such that for some  $f \in L_p(B(0, \beta))$  satisfying the condition  $supp f \subset \overline{B(0, \beta)}$ , and for some  $\gamma \geq 2\beta$ ,  $t \in \mathbb{R}^n$ ,  $r > 0$

$$\|(I_\alpha f) \chi_{B(0, \gamma)}\|_{L_q(B(t, r))} \leq C \gamma^{\alpha-n} (\min\{\gamma, r\})^{\frac{n}{q}} \|f\|_{L_p(B(0, \beta))}. \tag{7}$$

*Proof.* From the definition of the operator  $I_\alpha$ , we have

$$\begin{aligned} I &= \left\| (I_\alpha f) \chi_{B(0, \gamma)} \right\|_{L_q(B(t, r))} = \\ &= \left( \int_{B(t, r) \cap^c B(0, \gamma)} \left| \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy \right|^q dx \right)^{\frac{1}{q}} \leq \\ &\leq \left( \int_{B(t, r) \cap^c B(0, \gamma)} \left| \int_{B(0, \beta)} \frac{f(y)}{|x-y|^{n-\alpha}} dy \right|^q dx \right)^{\frac{1}{q}}. \end{aligned}$$

It is clear that  $\beta \leq \frac{\gamma}{2}$  for  $x \in^c B(0, \gamma)$ ,  $y \in B(0, \beta)$  we have

$$|x-y| \geq |x| - |y| \geq |x| - \beta = \frac{|x|}{2} + \frac{|x|}{2} - \beta \geq \frac{|x|}{2}. \tag{8}$$

From this it follows that

$$\begin{aligned} I &\leq 2^{n-\alpha} \left( \int_{B(0, \gamma)} \frac{dx}{|x|^{(n-\alpha)q}} \right)^{\frac{1}{q}} \int_{B(0, \beta)} |f(y)| dy \leq \\ &\leq 2^{n-\alpha} \left( \delta_n \int_\gamma^\infty \rho^{(n-\alpha)q+n-1} d\rho \right)^{\frac{1}{q}} (v_n \beta^n)^{1-\frac{1}{p}} \|f\|_{L_p(B(0, \beta))} = \\ &= 2^{n-\alpha} \left( \frac{\delta_n}{(n-\alpha)q-n} \right)^{\frac{1}{q}} v_n^{1-\frac{1}{p}} \beta^{n(1-\frac{1}{p})} \gamma^{\alpha-n(1-\frac{1}{p})} \|f\|_{L_p(B(0, \beta))} \equiv \\ &\equiv C_1 \gamma^{\alpha-n(1-\frac{1}{p})} \|f\|_{L_p(B(0, \beta))}. \end{aligned} \tag{9}$$

$\beta = \frac{\gamma}{2}$  for  $x \in^c B(0, \gamma)$ ,  $y \in B(0, \beta)$ , then using (8) we get  $|x-y| \geq \frac{|x|}{2}$ .

Next, we consider

$$\begin{aligned}
 I &\leq 2^{n-\alpha}\gamma^{\alpha-n} \left( \int_{B(t,r)} dx \right)^{\frac{1}{q}} \int_{B(0,\beta)} |f(y)| dy \leq \\
 &\leq 2^{n-\alpha}\gamma^{\alpha-n} (v_n r^n)^{\frac{1}{q}} (v_n \beta^n)^{1-\frac{1}{p}} \|f\|_{L_p(B(0,\beta))} = \\
 &= C_2 \gamma^{\alpha-n} r^{\frac{n}{q}} \|f\|_{L_p(B(0,\beta))}. \tag{10}
 \end{aligned}$$

From inequality (9) and (10) it follows (7), where  $C = \max\{C_1, C_2\}$ .

*Lemma 2.* Let  $n \in \mathbb{N}$ ,  $1 \leq p \leq \infty$ ,  $1 \leq q \leq \infty$ ,  $0 < \alpha < n(1 - \frac{1}{q})$ ,  $\beta > 0$ . Then there exists  $C > 0$ , depending only on  $n, p, q, \alpha$  such that for some  $f \in L_p(B(0, \beta))$ ,  $b \in L_\infty(\mathbb{R}^n)$ , satisfying the condition  $\text{supp } b \subset \overline{B(0, \beta)}$ , and for some  $\gamma \geq 2\beta$ ,  $t \in \mathbb{R}^n$ ,  $r > 0$

$$\left\| ([b, I_\alpha]f) \chi_{B(0,\gamma)} \right\|_{L_q(B(t,r))} \leq C \gamma^{\alpha-n} (\min\{\gamma, r\})^{\frac{n}{q}} \|b\|_{L_\infty(\mathbb{R}^n)} \|f\|_{L_p(B(0,\beta))}. \tag{11}$$

*Proof.* Let  $\gamma > \beta, \text{supp } b \subset B(0, \beta)$ , for  $x \in^c B(0, \gamma), b(x) = 0$ . Then

$$\begin{aligned}
 &\left\| [b, I_\alpha] f \chi_{B(0,\gamma)} \right\|_{L_q(B(t,r))} = \\
 &= \left( \int_{B(t,r) \cap^c B(0,\gamma)} \left| \int_{\mathbb{R}^n} \frac{(b(x) - b(y))f(y)}{|x - y|^{n-\alpha}} dy \right|^q dx \right)^{\frac{1}{q}} \leq \\
 &\leq \left( \int_{B(t,r) \cap^c B(0,\gamma)} \left| \int_{\mathbb{R}^n} \frac{b(y)f(y)}{|x - y|^{n-\alpha}} dy \right|^q dx \right)^{\frac{1}{q}} \leq \left( \int_{B(t,r) \cap^c B(0,\gamma)} \left| \int_{B(0,\beta)} \frac{|b(y)||f(y)|}{|x - y|^{n-\alpha}} dy \right|^q dx \right)^{\frac{1}{q}} \leq \\
 &\leq \left( \int_{B(t,r) \cap^c B(0,\gamma)} \left| \int_{B(0,\beta)} \frac{|f(y)|}{|x - y|^{n-\alpha}} dy \right|^q dx \right)^{\frac{1}{q}} \|b\|_{L_\theta(\mathbb{R}^n)} \leq \\
 &\leq \left( \int_{B(t,r) \cap^c B(0,\gamma)} \left| \int_{\mathbb{R}^n} \frac{|f(y)|}{|x - y|^{n-\alpha}} dy \right|^q dx \right)^{\frac{1}{q}} \|b\|_{L_\theta(\mathbb{R}^n)} = \left\| (I_\alpha f) \chi_{B(0,\gamma)} \right\|_{L_q(B(t,r))} \|b\|_{L_\theta(\mathbb{R}^n)}.
 \end{aligned}$$

From this and from Lemma 1 we obtain the inequality (11).

*Proof of Theorem 5.* To the proof of Theorem 5 it is sufficient to show that the conditions (2)–(4) of Theorem 3 are hold.

Let  $F$  be an arbitrary bounded subset of  $LM_{p\theta}^{w_1}$ . Since  $C_c^\infty(\mathbb{R}^n)$  is dense in  $VMO(\mathbb{R}^n)$  we only need to prove that the set  $G = \{[b, I_\alpha]f : f \in F, b \in C_c^\infty\}$  is pre-compact in the  $GM_p^{w_2}$ . By Theorem 3, we only need to verify the conditions (2), (3) and (4) hold uniformly  $F$  for  $b \in C_c^\infty$ .

Suppose that

$$\|f\|_{LM_{p\theta}^{w_1}} \leq D.$$

Applying condition (1), we have

$$\|[b, I_\alpha]f\|_{LM_{p\theta}^{w_2}} \leq C \cdot \|b\|_* \sup_{f \in F} \|f\|_{M_p^{w_1}} \leq C \cdot D \|b\|_* < \infty.$$

This implies that the condition (2) of Theorem 3 is hold.

Now we prove that condition (4) of Theorem 3 also is hold, i. e.

$$\lim_{\gamma \rightarrow \infty} \left\| ([b, I_\alpha] f) \chi_{B(0,\gamma)}^c \right\|_{LM_{p\theta}^{w_2}} = 0.$$

It follows from Lemma 2. Indeed

$$\begin{aligned} & \left\| ([b, I_\alpha] f) \chi_{B(0,\gamma)}^c \right\|_{LM_{p\theta}^{w_2}} = \\ & = \left\| w(r) \left\| ([b, I_\alpha] f) \chi_{B(0,\gamma)}^c \right\|_{L_p(B(0,r))} \right\|_{L_\theta(0,\infty)} \leq \\ & \leq C \gamma^{-n} \|b\|_{L_\theta(\mathbb{R}^n)} \|f\|_{L_p B(0,\beta)} \sup_{\substack{r>0, \\ x \in \mathbb{R}^n}} \left\| w_2(r) (\min\{\gamma, r\})^{\frac{n}{p}} \right\|_{L_\theta(0,\infty)}. \end{aligned}$$

When  $r < l < \gamma$  we have  $(\min\{\gamma, r\})^{\frac{n}{p}} = r^{\frac{n}{p}}$ . By condition  $\left\| w_2(r) r^{\frac{n}{p}} \right\|_{L_\theta(l,\infty)} < \infty$ .

When  $\gamma < t < r$  we have  $(\min\{\gamma, r\})^{\frac{n}{p}} = \gamma^{\frac{n}{p}}$ . By condition  $\|w_2(r)\|_{L_\theta(0,t)} < \infty$ .

Therefore

$$\lim_{\gamma \rightarrow \infty} \left\| ([b, I_\alpha] f) \chi_{B(0,\gamma)}^c \right\|_{LM_{p\theta}^{w_2}} = 0.$$

This implies the required condition (4).

Now we prove that condition (3) of Theorem 3 for the set  $[b, I_\alpha](f)$ ,  $f \in F$ , is hold i.e. we show that for any  $0 < \varepsilon < \frac{1}{2}$  and if  $|z|$  is sufficiently small depending only on  $\varepsilon$ , then for every  $f \in F$ .

$$\|([b, I_\alpha f)(\cdot + z)] - [b, I_\alpha] f(\cdot)\|_{LM_{p\theta}^{w_2}} \leq C \cdot \varepsilon.$$

Let  $\varepsilon$  arbitrary number such that  $0 < \varepsilon < \frac{1}{2}$ . For  $|z| \in \mathbb{R}^n$  we have, that

$$\begin{aligned} [f, I_\alpha] f(x+z) - [b, I_\alpha] f(x) &= \int_{\mathbb{R}^n} \frac{[b(x+z) - b(y)]f(y)}{|x+z-y|^{n-\alpha}} dy - \int_{\mathbb{R}^n} \frac{[b(x) - b(y)]f(y)}{|x-y|^{n-\alpha}} dy = \\ &= \int_{\mathbb{R}^n} \frac{[b(x+z) - b(y)]f(y)}{|x+z-y|^{n-\alpha}} dy - \int_{\mathbb{R}^n} \frac{[b(x) + b(x+z) - b(x+z) - b(y)]f(y)}{|x-y|^{n-\alpha}} dy = \\ &= \int_{\mathbb{R}^n} \frac{[b(x+z) - b(y)]f(y)}{|x+z-y|^{n-\alpha}} dy + \int_{\mathbb{R}^n} \frac{[b(y) - b(x+z)]f(y)}{|x-y|^{n-\alpha}} dy + \\ & \quad + \int_{\mathbb{R}^n} \frac{[b(x+z) - b(x)]f(y)}{|x-y|^{n-\alpha}} dy = \\ &= \int_{\mathbb{R}^n} [b(y) - b(x+z)] \left( \frac{f(y)}{|x-y|^{n-\alpha}} - \frac{f(y)}{|x+z-y|^{n-\alpha}} \right) dy + \int_{\mathbb{R}^n} \frac{[b(x+z) - b(x)]f(y)}{|x-y|^{n-\alpha}} dy = \\ &= \int_{|x-y| > |z|e^{\frac{1}{\varepsilon}}} \frac{[b(x+z) - b(x)]f(y)}{|x-y|^{n-\alpha}} dy + \\ & \quad + \int_{|x-y| > |z|e^{\frac{1}{\varepsilon}}} \left( \frac{f(y)}{|x-y|^{n-\alpha}} - \frac{f(y)}{|x+z-y|^{n-\alpha}} \right) [b(x+z) - b(y)] dy + \end{aligned}$$

$$\begin{aligned}
 & + \int_{|x-y| \leq |z|e^{\frac{1}{\varepsilon}}} \frac{[b(y) - b(x)]f(y)}{|x-y|^{n-\alpha}} dy - \int_{|x-y| \leq |z|e^{\frac{1}{\varepsilon}}} \frac{[b(y) - b(x+z)]f(y)}{|x+z-y|^{n-\alpha}} dy = \\
 & = J_1 + J_2 + J_3 - J_4.
 \end{aligned} \tag{12}$$

Since  $b \in C_0^n(\mathbb{R}^n)$ , we have

$$|b(x) - b(x+z)| \leq |\nabla f(x)| \cdot |z| \leq C|z|.$$

Then

$$|J_1| \leq C|z|I_\alpha(|f|)(x).$$

By Theorem 5

$$\|J_1\|_{LM_{p\theta}^{w_2}} \leq C|z| \|I_\alpha(f)\|_{LM_{p\theta}^{w_2}} \leq C|z| \|f\|_{LM_{p\theta}^w} \leq CD|z|. \tag{13}$$

For  $J_2$  we have that

$$(b(x+z) - b(y)) \leq 2 \|b\|_\infty \leq C.$$

Therefore

$$|J_2| \leq C|z| \int_{|x-y| > |z|e^{\frac{1}{\varepsilon}}} \frac{f(y)}{|x-y|^n} dy \leq C_\varepsilon I_\alpha(|f|)(x).$$

Again by the of Theorem 1 we get

$$\|J_2\|_{LM_{p\theta}^{w_2}} \leq c\varepsilon \|I_\alpha(f)\|_{LM_{p\theta}^{w_1}} \leq c\varepsilon \|C \cdot D \cdot \varepsilon\|.$$

Consequently,

$$|J_3| \leq C \int_{|x-y| \leq |z|e^{\frac{1}{\varepsilon}}} \frac{f(y)}{|x-y|^{n-\alpha}} dy \leq C\varepsilon^{-1}|z| \int_{|x-y| \leq |z|e^{\frac{1}{\varepsilon}}} \frac{f(y)}{|x-y|^{n-\alpha}} dy \leq C \cdot \varepsilon^{-1}|z|I_\alpha(|f|)(x).$$

Thus, we have

$$\|J_3\|_{LM_{p\theta}^{w_2}} \leq C \cdot \varepsilon^{-1}|z| \|I_\alpha(f)\|_{LM_{p\theta}^{w_2}} \leq C \cdot \varepsilon^{-1}|z| \|f\|_{LM_{p\theta}^{w_1}} \leq \varepsilon^{-1}|z|. \tag{14}$$

Similarly, using the estimate finally by

$$|b(x+z) - b(y)| \leq C|x+z-y|,$$

we have

$$|J_4| \leq C \int_{|x-y| \leq e^{\varepsilon^{-1}}(y)} |x+z-y|^{-n+1+\alpha} |b(y)| dy \leq C(\varepsilon^{-1}|z| + |z|)I_\alpha(|f|)(x+z).$$

Therefore

$$\|J_4\|_{LM_{p\theta}^{w_2}} \leq C \cdot (\varepsilon^{-1}|z| + |z|) \|f\|_{LM_{p\theta}^{w_1}} \leq C \cdot (\varepsilon^{-1}|z| + |z|). \tag{15}$$

Here  $C$  does not depend on  $z$  and  $\varepsilon$ . Finally from (12)–(15) taking a  $|z|$  small enough we have

$$\|[b, I_\alpha(f)](\cdot + z) - [b, I_\alpha]f(\cdot)\|_{LM_{p\theta}^{w_2}} \leq \|J_1\|_{LM_{p\theta}^{w_2}} + \|J_2\|_{LM_{p\theta}^{w_2}} + \|J_3\|_{LM_{p\theta}^{w_2}} + \|J_4\|_{LM_{p\theta}^{w_2}} \leq C \cdot D \cdot \varepsilon$$

i.e. the set  $[b, I_\alpha](f)$ ,  $f \in F$  satisfies the condition (3) of Theorem 3. Then by Theorem 3, the set  $[b, I_\alpha](f)$ ,  $f \in F$  is precompact in the  $LM_{p\theta}^{w_2}$ . Which completes the proof of the theorem.

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## Локальді Морри типтес кеңістігінде Рисс потенциал коммутаторының компактылығы

Мақалада  $LM_{p\theta}^w$  локальді Морри типті кеңістіктері қарастырылған. Негізгі жұмыс  $LM_{p\theta}^{w_1}$ -дан  $LM_{q\theta}^{w_2}$ -ге дейінгі локальді Морри типті кеңістіктердегі  $[b, I_\alpha]$  Рисс потенциалы үшін коммутатордың компактылық теоремасын дәлелдеу. Соңдай-ақ, Рисс потенциалы үшін коммутатордың  $[b, I_\alpha]$  локальді Морри типті кеңістіктердегі  $LM_{p\theta}^{w_1}$ -дан  $LM_{q\theta}^{w_2}$  шенелгендігі үшін жаңа жеткілікті шарттар берілген. Рисс потенциалы үшін коммутатордың компактылық теоремасын дәлелдеуде негізінен,  $[b, I_\alpha]$  Рисс потенциалы үшін коммутатордың  $LM_{p\theta}^w$  локальді Морри типті кеңістіктерінде шектелген шарты, сонымен қатар  $LM_{p\theta}^w$  локальді Морри типті кеңістіктеріндегі жиындардың компактылық теоремасының жеткілікті шарттары пайдаланылған. Рисс потенциалы үшін коммутатордың жинақылық теоремасын дәлелдеу барысында  $[b, I_\alpha]$  Рисс потенциалы үшін коммутатор шарының шенелген леммалары анықталған. Осыған ұқсас нәтижелер  $GM_{p\theta}^w$  глобалды Морри типті кеңістіктер үшін және  $M_p^w$  жалпыланған Морри кеңістігі үшін де алынған.

*Кілт сөздер:* компактылы, коммутаторлар, Рисс потенциалы, локальді Морри типті кеңістіктер.

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## Компактность коммутаторов для потенциала Рисса в локальных пространствах типа Морри

В статье рассмотрены локальные пространства типа Морри из  $LM_{p\theta}^w$ . Основной работой является доказательство теоремы компактности коммутатора для потенциала Рисса  $[b, I_\alpha]$  в локальных пространствах типа Морри из  $LM_{p\theta}^{w_1}$  в  $LM_{q\theta}^{w_2}$ . Приведены также новые достаточные условия ограниченности коммутатора для потенциала Рисса  $[b, I_\alpha]$  в локальных пространствах типа Морри из  $LM_{p\theta}^{w_1}$  в  $LM_{q\theta}^{w_2}$ . В доказательстве теоремы компактности коммутатора для потенциала Рисса существенно использованы условия ограниченности коммутатора для потенциала Рисса  $[b, I_\alpha]$  в локальных пространствах типа Морри  $LM_{p\theta}^w$ , а также достаточные условия из теоремы предкомпактности множеств в локальных пространствах типа Морри  $LM_{p\theta}^w$ . В ходе доказательства теоремы компактности коммутатора для потенциала Рисса подтверждены леммы оценки по шару коммутатора для потенциала Рисса  $[b, I_\alpha]$ . Аналогичные результаты были получены для глобальных пространств типа Морри  $GM_{p\theta}^w$  и для обобщенных пространств Морри  $M_p^w$ .

*Ключевые слова:* компактность, коммутаторы, потенциал Рисса, локальные пространства типа Морри.

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