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q-Analogues of Lyapunov-type inequalities involving Riemann–Liouville fractional derivatives

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In this article, new q-analogues of Lyapunov-type inequalities are presented for two-point fractional boundary value problems involving the Riemann–Liouville fractional q-derivative with well-posed q-boundary conditions. The study relies on the properties of the q-Green's function, which is constructed to solve such problems and allows for the analytical derivation of the inequalities. These inequalities find application in two directions: establishing precise lower bounds for the eigenvalues of corresponding q-fractional spectral problems and formulating criteria for the absence of real zeros in q-analogues of Mittag-Leffler functions. The obtained results generalize classical and fractional Lyapunov inequalities, offering new perspectives for the analysis of stability and spectral properties of q-fractional differential systems. The relevance of the work is driven by the growing interest in q-calculus in discrete models, such as viscoelastic systems or quantum circuits, where discrete dynamics play a key role. The convenience of closed-form analytical expressions makes the results practically applicable. The research lays the foundation for further generalizations, including Caputo derivatives or multidimensional q-systems, which may stimulate new discoveries in discrete fractional analysis.

Keywords: q-calculus, fractional q-derivative, Lyapunov-type inequality, Riemann–Liouville fractional derivative, Green's function, Mittag-Leffler function, eigenvalue problems, fractional integral.

2020 Mathematics Subject Classification: 26A33, 34A08, 39A13.

Introduction

Fractional calculus investigates integrals and derivatives of arbitrary (non-integer) order, has become an indispensable framework for modelling complex phenomena in physics, biology, engineering, and economics [1, 2]. Fractional differential equations (FDEs) naturally describe memory effects, nonlocal interactions, and anomalous diffusion; a representative example is C.F. Li et al.'s proof of positive solutions for nonlinear FDEs with boundary constraints [3].

A central analytical tool for boundary-value problems (BVPs) in the fractional setting is the Lyapunov-type inequality. R.A.C. Ferreira obtained the first variant for a Riemann–Liouville derivative with Dirichlet conditions [4]; M. Jleli and B. Samet extended the result to mixed boundary conditions [5]; and D. Basu et al. treated fractional boundary conditions, applying the inequality to spectral questions [6]. Subsequent refinements yielded sharper eigenvalue bounds and zero-free intervals for Mittag-Leffler functions [7].

Parallel to the continuous theory, q-fractional calculus blends quantum calculus with fractional analysis. Its origins trace back to Jackson's introduction of q-difference operators and integrals [8, 9] and R.D. Carmichael's work on q-difference equations [10]. Modern expositions by V. Kac and P. Cheung [11], T. Ernst [12, 13], and M.H. Annaby, Z.S. Mansour [14] have systematised the subject.

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Foundational notions of q-fractional integrals and derivatives, proposed by W.A. Al-Salam [15] and R.P. Agarwal [16], were rigorously formalised by P.M. Rajkovic et al. [17, 18].

Applications of q-fractional differential equations range from quantum mechanics to discrete dynamical systems. R.A.C. Ferreira analysed non-trivial and positive solutions for several classes of q-fractional BVPs [19, 20]; S. Shaimardan and collaborators established existence and uniqueness results for Cauchy-type problems with Riemann–Liouville derivatives [21]. The q-fractional framework has been connected with time–scale calculus through the work of F.M. Atici and P.W. Eloe [22]; with three-point and other non-local boundary conditions in the papers of S. Liang, J. Zhang, C. Yu, J. Wang, S. Wang et al. [23–25]; and further refined for related non-local problems by C. Zhai, J. Ren [26] and Y. Zhao, H. Chen, Q. Zhang [27]. Lyapunov-type inequalities for q-fractional equations were first obtained by M. Jleli and B. Samet [28].

In this work we derive two new Lyapunov-type inequalities for the q-fractional boundary-value problem

$$\begin{cases} D^{\alpha}_{q,a}u(t) + q(t)u(t) = 0, & a \le t \le b,\ 1 < \alpha \le 2,\ 0 \le \beta \le 1, \\ u(a) = 0, \quad D^{\beta}_{q,a}u(b) = 0, & 0 < q < 1, \end{cases}$$

by exploiting properties of the associated q-Green function. The analysis combines topological fixed-point techniques [29], and existence principles in the Caratheodory framework [30]. Our results sharpen eigenvalue estimates, offer criteria for the real zeros of q-Mittag-Leffler functions, and advance the spectral theory of discrete fractional models.

1 Preliminaries

In this section, we introduce essential definitions and foundational concepts, including key aspects of q-calculus, which underpin the present study. For a comprehensive exploration of these topics, readers are referred to the monographs [11, 14].

For $\alpha \in \mathbb{R}$, the q-real number $[\alpha]_q$ is given by

$$[\alpha]_q = \frac{1-q^{\alpha}}{1-q}, \quad q \ne 1,$$

where $\lim\limits_{q\to 1} \frac{1-q^{\alpha}}{1-q} = \alpha$.

We introduce for $k \in \mathbb{N}$:

$$(a;q)_0 = 1,\ (a;q)_n = \prod_{k=0}^{n-1}\left(1 - q^k a\right),\ (a;q)_\infty = \lim_{n\to\infty}(a,q)_n,\ (a;q)_\alpha = \frac{(a;q)_\infty}{(q^{\alpha}a;q)_\infty}.$$

The q-factorial $[n]_q!$, serving as the q-analogue of the binomial coefficient factorial, is defined as

$$[n]_q! = \begin{cases} 1, & \text{if } n = 0, \\ [1]_q \times [2]_q \times \cdots \times [n]_q, & \text{if } n \in N. \end{cases}$$

The q-gamma function $\Gamma_q(x)$ is given by

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \quad x \in R\backslash\{0,-1,-2,\ldots\}$$

and satisfies the functional relation $\Gamma_q(x+1) = [x]_q\Gamma_q(x)$.

Definition 1. [11] The q-analogue differential operator $D_q f(x)$ is

$$D_q f(x) = \frac{f(x) - f(qx)}{x(1-q)},$$

and the q-derivatives $D_q^n(f(x))$ of higher order are defined inductively as follows:

$$D_q^0(f(x)) = f(x), \quad D_q^n(f(x)) = D_q\left(D_q^{n-1} f(x)\right) \quad (n = 1, 2, 3, \dots),$$

where $0 < q < 1$. Be aware that $\lim\limits_{q \to 1} D_q f(x) = f'(x)$.

$$D_{q,x}(x-s)_q{}^{(\gamma)} = [\gamma]_q (x-s)_q{}^{(\gamma-1)}, \qquad (1)$$
$$D_{q,s}(x-s)_q{}^{(\gamma)} = -[\gamma]_q (x-qs)_q{}^{(\gamma-1)}.$$

The q-integral (or Jackson integral) $\int\limits_a^b f(x) d_q x$ is defined by

$$\int\limits_0^a f(x) d_q x := (1-q) a \sum_{m=0}^{\infty} q^m f\left(a q^m\right),$$

for $a = 0$ and

$$\int\limits_a^b f(x) d_q x = \int\limits_0^b f(x) d_q x - \int\limits_0^a f(x) d_q x,$$

for $0 < a < b$. For further details, see [8, 9].

Definition 2. [21] For $\alpha > 0$, and a function f defined on $[a, b]$, the fractional q-integral of Riemann–Liouville type is characterized by $\left(I_{q,a}^0 f\right)(x) = f(x)$ and

$$\left(I_{q,a}^{\alpha} f\right)(x) = \frac{1}{\Gamma_q(\alpha)} \int\limits_a^x (x - qt)_q^{(\alpha-1)} f(t) d_q t, \quad x \in [a, b].$$

Definition 3. [16]. Given $\alpha, \beta > 0$, the Riemann–Liouville fractional q-derivative is defined by setting $\left(D_{q,a}^0 f\right)(x) = f(x)$ and

$$\left(D_{q,a}^{\alpha} f\right)(x) = \left(D_{q,a}^{[\alpha]} I_{q,a}^{[\alpha]-\alpha} f\right)(x),$$

where $[\alpha]$ is the smallest integer greater than or equal to α.

For $\lambda \in (-1, \infty)$, the following is valid [9]:

$$\left(D_{q,a}^{\alpha}(x-a)^{\lambda}\right)(x) = \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\lambda-\alpha+1)}(x-a)^{\lambda-\alpha}. \qquad (2)$$

The space $L_q^p = L_q^p[a, b]$ corresponding to $1 \le p < \infty$ is defined by

$$L_q^p[a, b] := \left\{ f : \left(\int\limits_a^b |f(x)|^p d_q x \right)^{\frac{1}{p}} < \infty \right\}.$$

Let $0 < a < b < \infty$ and $0 \le \lambda \le 1$. Then we introduce the space $C_{q,\lambda}[a,b]$ of functions f given on $[a,b]$, such that the functions with the norm

$$\|f\|_{C_{q,\lambda}[a,b]} := \max_{x\in[a,b]} \left|(x-qa)_q^{(\lambda)} f(x)\right| < \infty.$$

The collection of all q-absolutely continuous functions on $[a,b]$ is denoted $AC_q[a,b]$. For $n \in N := 1,2,3,\dots$ we denote by $AC_q^n[a,b]$ the space of real-valued functions $f(x)$ which have q-derivatives up to order $n-1$ on $[a,b]$ such that $D_q^{n-1}f(x) \in AC_q[a,b]$:

$$AC_q^n[a,b] := \left\{f : [a,b] \to \mathbb{R}; D_q^{n-1}f(x) \in AC_q[a,b]\right\}.$$

Lemma 1. [18] Assume $\alpha > 0$, $\beta > 0$, and $1 \le p < \infty$. The semigroup property for the q-fractional integral holds as follows:
1. $(I_{q,a}^{\beta} I_{q,a}^{\alpha} f)(x) = (I_{q,a}^{\alpha+\beta} f)(x)$,
2. $(D_{q,a}^{\alpha} I_{q,a}^{\alpha} f)(x) = f(x)$,
3. $\left(D_{q,a}^{\beta} I_{q,a}^{\alpha} f\right)(x) = \left(I_{q,a}^{\alpha-\beta} f\right)(x)$,

where $f(x) \in L_q^p[a,b]$ for all $x \in [a,b]$.

Lemma 2. Suppose $\alpha > 0$, $p \in \mathbb{N}$, $q \in (0,1)$, and let $f \in AC_q^p[a,b]$ be a function with q-derivatives $D_{q,a}^k f$ defined at $x = a$ for $k = 0,1,\dots,p-1$. Following [19], the Riemann–Liouville q-fractional integral $I_{q,a}^{\alpha}$ and derivative $D_{q,a}^{\alpha}$ satisfy

$$\left(I_{q,a}^{\alpha} D_{q,a}^{\alpha} f\right)(x) = \left(D_{q,a}^{\alpha} I_{q,a}^{\alpha} f\right)(x) - \sum_{k=0}^{p-1} \frac{(x-a)^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)} \left(D_{q,a}^k f\right)(a), \quad x \in [a,b].$$

Lemma 3. For $\gamma > -1$, $q \in (0,1)$, $a < b$, and $x \ge b$, the q-integral of the q-power function is given by

$$\int_a^b (x-qs)_q^{(\gamma)}\, d_q s = \frac{(x-a)^{\gamma+1}}{[\gamma+1]_q}, \tag{3}$$

where $(x-qs)_q^{(\gamma)} = (x-qs)^{\gamma}$ and $[\gamma+1]_q = \frac{1-q^{\gamma+1}}{1-q}$. See [9] for details.

2 *Main Results*

Theorem 1. Let $1 < \alpha \le 2$, $0 \le \beta \le 1$, $0 < \alpha-\beta < 1$, $q \in (0,1)$, and $h \in L_q^1[a,b]$. The q-fractional boundary value problem

$$D_{q,a}^{\alpha} u(t) + h(t) = 0, \quad t \in [a,b], \tag{4}$$

with boundary conditions

$$u(a) = 0, \quad D_{q,a}^{\beta} u(b) = 0, \tag{5}$$

has a unique solution given by

$$u(t) = \int_a^b G_q(t,s) h(s)\, d_q s,$$

where the q-Green's function $G_q(t,s)$ is defined as

$$G_q(t,s) = \frac{1}{\Gamma_q(\alpha)} \begin{cases} \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}} (b-qs)_q^{(\alpha-\beta-1)}, & a \le t \le s \le b, \\ \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}} (b-qs)_q^{(\alpha-\beta-1)} - (t-qs)_q^{(\alpha-1)}, & a \le s \le t \le b. \end{cases} \tag{6}$$

Proof. By applying the operator $I^{\alpha}_{q,a}$ from definition 2 to both sides of (4) and employing Lemma 2 with $p = 2$, we obtain

$$u(t) = -I^{\alpha}_{q,a}h(t) + C_1(t-a)^{\alpha-1} + C_2(t-a)^{\alpha-2}, \tag{7}$$

for some $C_1, C_2 \in \mathbb{R}$. Applying the operator $D^{\beta}_{q,a}$ in condition (5) to both parts of the equation (7) and using the Lemma 1, we obtain

$$\begin{aligned} D^{\beta}_{q,a}u(t) = &-D^{\beta}_{q,a}I^{\alpha}_{q,a}h(t) + C_1 D^{\beta}_{q,a}(t-a)^{\alpha-1} \\ &+ C_2 D^{\beta}_{q,a}(t-a)^{\alpha-2}, \end{aligned}$$

proceeding further, and using formula (2), we arrive at

$$\begin{aligned} D^{\beta}_{q,a}u(t) = &-I^{\alpha-\beta}_{q,a}h(t) + C_1\frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\beta)}(t-a)^{\alpha-\beta-1} \\ &+ C_2\frac{\Gamma_q(\alpha-1)}{\Gamma_q(\alpha-\beta-1)}(t-a)^{\alpha-\beta-2}. \end{aligned} \tag{8}$$

Using the boundary condition $u(a) = 0$ in equation (7) gives $C_2 = 0$. Applying the condition $D^{\beta}_{q,a}u(b) = 0$ to equation (8) then leads to

$$C_1 = \frac{1}{\Gamma_q(\alpha)(b-a)^{\alpha-\beta-1}}\int_a^b (b-qs)_q^{(\alpha-\beta-1)}h(s)\,d_qs.$$

Substituting the explicit expressions for C_1 and C_2 into equation (7), we obtain the unique solution of (4) as

$$\begin{aligned} u(t) = &-\frac{1}{\Gamma_q(\alpha)}\int_a^t (t-qs)_q^{(\alpha-1)}h(s)d_qs \\ &+ \frac{1}{\Gamma_q(\alpha)}\int_a^b \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}}(b-qs)_q^{(\alpha-\beta-1)}h(s)d_qs \\ &= \frac{1}{\Gamma_q(\alpha)}\int_a^t \left[\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}}(b-qs)_q^{(\alpha-\beta-1)} - (t-qs)_q^{(\alpha-1)}\right]h(s)d_qs \\ &+ \frac{1}{\Gamma_q(\alpha)}\int_t^b \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}}(b-qs)_q^{(\alpha-\beta-1)}h(s)d_qs \\ &= \int_a^b G_q(t,s)h(s)d_qs. \end{aligned}$$

Hence, the result follows.

Corollary 1. Let $1 < \alpha \le 2$, $0 \le \beta \le 1$, $1 \le \alpha - \beta < 2$, $q \in (0,1)$, and $h \in L^1_q[a,b]$. The q-fractional boundary value problem

$$D^{\alpha}_{q,a}u(t) + h(t) = 0, \quad t \in [a,b],$$

with boundary conditions

$$u(a) = 0, \quad D^{\beta}_{q,a}u(b) = 0,$$

has a unique solution $u \in AC_q^{\alpha}[a,b]$ given by

$$u(t) = \int_a^b G_q(t,s)h(s)\, d_q s,$$

where the q-Green's function $G_q(t,s)$ is defined as

$$G_q(t,s) = \frac{1}{\Gamma_q(\alpha)} \begin{cases} \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}}(b-qs)_q^{(\alpha-\beta-1)}, & a \le t \le s \le b, \\ \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}}(b-qs)_q^{(\alpha-\beta-1)} - (t-qs)_q^{(\alpha-1)}, & a \le s \le t \le b. \end{cases} \tag{9}$$

Proof. The result follows from Theorem 1 by identical arguments for the case $1 \le \alpha - \beta < 2$; the details are omitted.

We proceed to demonstrate the nonnegativity of the q-Green's functions and establish upper bounds for both the functions and their q-integrals.

Theorem 2. Let $1 < \alpha \le 2$, $0 \le \beta \le 1$, $0 < \alpha - \beta < 1$, $q \in (0,1)$, and let the q-Green's function $G_q(t,s)$ be defined as in Theorem 1. Then,

$$G_q(t,s) \ge 0 \quad \text{for all } (t,s) \in [a,b] \times [a,b].$$

Proof. We analyze the q-Green's function $G_q(t,s)$ defined in Theorem 1, considering its piecewise structure.

Case 1: $a \le t \le s \le b$. Here,

$$G_q(t,s) = \frac{1}{\Gamma_q(\alpha)} \cdot \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}} \cdot (b-qs)_q^{(\alpha-\beta-1)}.$$

Since $\Gamma_q(\alpha) > 0$, $(b-a)^{\alpha-\beta-1} > 0$, $(t-a)^{\alpha-1} \ge 0$ for $t \ge a$, and $(b-qs)_q^{(\alpha-\beta-1)} \ge 0$ for $s \le b$ (as $qs \le s$, $q \in (0,1)$, and $0 < \alpha - \beta < 1$), it follows that $G_q(t,s) \ge 0$.

Case 2: $a \le s \le t \le b$. In this case,

$$G_q(t,s) = \frac{1}{\Gamma_q(\alpha)} \left[\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}}(b-qs)_q^{(\alpha-\beta-1)} - (t-qs)_q^{(\alpha-1)} \right].$$

Since $s \le t$, the q-power function is monotonic, so $t - qs \ge t - a$, and thus $(t-qs)_q^{(\alpha-1)} \le (t-a)^{\alpha-1}$. Additionally, as $qs \le s \le t \le b$, we have $b - qs \ge b - a$, implying $(b-qs)_q^{(\alpha-\beta-1)} \ge (b-a)^{\alpha-\beta-1}$. Therefore,

$$\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}}(b-qs)_q^{(\alpha-\beta-1)} \ge (t-a)^{\alpha-1} \ge (t-qs)_q^{(\alpha-1)}.$$

Hence,

$$G_q(t,s) \ge \frac{1}{\Gamma_q(\alpha)} \left[(t-a)^{\alpha-1} - (t-qs)_q^{(\alpha-1)} \right] \ge 0.$$

Combining both cases, we conclude that $G_q(t,s) \ge 0$ for all $(t,s) \in [a,b] \times [a,b]$.

Remark 1. The nonnegativity of the q-Green's function $G_q(t,s)$, established in Theorem 2, is crucial for the qualitative analysis of the q-fractional boundary value problem in Theorem 1. Specifically, it ensures that the solution

$$u(t) = \int_a^b G_q(t,s)h(s)\, d_q s, \quad h \in L_q^1[a,b],$$

preserves the sign of the source term $h(s)$. For instance, if $h(s) \ge 0$ on $[a,b]$, then $u(t) \ge 0$; similarly, if $h(s) \le 0$, then $u(t) \le 0$, for all $t \in [a,b]$.

Corollary 2. Let $1 < \alpha \leq 2$, $0 \leq \beta \leq 1$, $1 \leq \alpha - \beta < 2$, $q \in (0,1)$, and let the q-Green's function $G_q(t,s)$ be defined as in Corollary 1 for $a < b$. Then,

$$G_q(t,s) \geq 0 \quad \text{for all } (t,s) \in [a,b] \times [a,b].$$

Proof. We analyze the piecewise definition of $G_q(t,s)$ from Corollary 1.

Case 1: $a \leq t \leq s \leq b$. Here,

$$G_q(t,s) = \frac{1}{\Gamma_q(\alpha)} \cdot \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}} \cdot (b-qs)_q^{(\alpha-\beta-1)}.$$

Since $\Gamma_q(\alpha) > 0$, $(t-a)^{\alpha-1} \geq 0$, $(b-a)^{\alpha-\beta-1} \geq 0$ (as $\alpha - \beta - 1 \geq 0$), and $(b-qs)_q^{(\alpha-\beta-1)} \geq 0$ (as $qs \leq s \leq b$, $q \in (0,1)$), it follows that $G_q(t,s) \geq 0$.

Case 2: $a \leq s \leq t \leq b$. In this case,

$$G_q(t,s) = \frac{1}{\Gamma_q(\alpha)} \left[\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}} (b-qs)_q^{(\alpha-\beta-1)} - (t-qs)_q^{(\alpha-1)} \right].$$

Since $a \leq qs \leq s \leq t \leq b$, we have $b - qs \geq b - a$, so $(b-qs)_q^{(\alpha-\beta-1)} \geq (b-a)^{\alpha-\beta-1}$. Also, $qs \geq a$, so $t - qs \leq t - a$, and the monotonicity of the q-power function [14] implies $(t-qs)_q^{(\alpha-1)} \leq (t-a)^{\alpha-1}$. Thus,

$$\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}} (b-qs)_q^{(\alpha-\beta-1)} \geq (t-a)^{\alpha-1} \geq (t-qs)_q^{(\alpha-1)}.$$

Hence,

$$G_q(t,s) \geq \frac{1}{\Gamma_q(\alpha)} \left[(t-a)^{\alpha-1} - (t-qs)_q^{(\alpha-1)} \right] \geq 0.$$

Thus, $G_q(t,s) \geq 0$ for all $(t,s) \in [a,b] \times [a,b]$.

Theorem 3. Let $1 < \alpha \leq 2$, $0 \leq \beta \leq 1$, $0 < \alpha - \beta < 1$, $q \in (0,1)$, $a < b$, and let the q-Green's function $G_q(t,s)$ be defined as in (6). Then, for $s \in [a,b]$,

$$\max_{t \in [a,b]} \frac{G_q(t,s)}{(b-qs)_q^{(\alpha-\beta-1)}} = \frac{G_q(s,s)}{(b-qs)_q^{(\alpha-\beta-1)}},$$

and

$$\max_{s \in [a,b]} \frac{G_q(s,s)}{(b-qs)_q^{(\alpha-\beta-1)}} = \frac{(b-a)^{\beta}}{\Gamma_q(\alpha)}.$$

Proof. We analyze the ratio $\frac{G_q(t,s)}{(b-qs)_q^{(\alpha-\beta-1)}}$ for fixed $s \in [a,b]$. Since $qs \leq s \leq b$, $q \in (0,1)$, and $0 < \alpha - \beta < 1$, we have $\alpha - \beta - 1 \in (-1, 0)$, but $(b-qs)_q^{(\alpha-\beta-1)} \geq 0$ as per [14].

Case 1: $a \leq t \leq s \leq b$. From (6),

$$\frac{G_q(t,s)}{(b-qs)_q^{(\alpha-\beta-1)}} = \frac{1}{\Gamma_q(\alpha)} \cdot \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}}.$$

Using the q-derivative (1),

$$D_{q,t}[(t-a)^{\alpha-1}] = [\alpha-1]_q (t-a)^{\alpha-2},$$

we obtain

$$D_{q,t}\left[\frac{G_q(t,s)}{(b-qs)_q^{(\alpha-\beta-1)}} \right] = \frac{(t-a)^{\alpha-2}}{(b-a)^{\alpha-\beta-1}\Gamma_q(\alpha-1)} \geq 0,$$

since $\alpha-2>-1$. At $t=a$, $(t-a)^{\alpha-2}$ may be singular ($\alpha-2\in(-1,0]$), but the q-derivative is defined for $t\in(a,s]$. Thus, the ratio is non-decreasing on $[a,s]$.

Case 2: $a\le s\le t\le b$. Here,

$$\frac{G_q(t,s)}{(b-qs)_q^{(\alpha-\beta-1)}}=\frac{1}{\Gamma_q(\alpha)}\left[\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}}-\frac{(t-qs)_q^{(\alpha-1)}}{(b-qs)_q^{(\alpha-\beta-1)}}\right].$$

Computing the q-derivative,

$$D_{q,t}\left[\frac{G_q(t,s)}{(b-qs)_q^{(\alpha-\beta-1)}}\right]=\frac{1}{\Gamma_q(\alpha-1)}\left[\frac{(t-a)^{\alpha-2}}{(b-a)^{\alpha-\beta-1}}-\frac{(t-qs)_q^{(\alpha-2)}}{(b-qs)_q^{(\alpha-\beta-1)}}\right].$$

Since $qs\le s\le b$, we have $b-qs\ge b-a$, so $(b-qs)_q^{(\alpha-\beta-1)}\ge(b-a)^{\alpha-\beta-1}$. Also, $qs\ge a$, so $t-qs\le t-a$, and the monotonicity of the q-power function [14] implies $(t-qs)_q^{(\alpha-2)}\le(t-a)^{\alpha-2}$. Thus,

$$\frac{(t-a)^{\alpha-2}}{(b-a)^{\alpha-\beta-1}}\ge\frac{(t-qs)_q^{(\alpha-2)}}{(b-qs)_q^{(\alpha-\beta-1)}},$$

so

$$D_{q,t}\left[\frac{G_q(t,s)}{(b-qs)_q^{(\alpha-\beta-1)}}\right]\le 0.$$

Hence, the ratio is non-increasing on $[s,b]$. Combining both cases, the maximum occurs at $t=s$, where

$$\frac{G_q(s,s)}{(b-qs)_q^{(\alpha-\beta-1)}}=\frac{1}{\Gamma_q(\alpha)}\cdot\frac{(s-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}}.$$

For the second part, consider

$$\frac{G_q(s,s)}{(b-qs)_q^{(\alpha-\beta-1)}}=\frac{(s-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}\Gamma_q(\alpha)}.$$

Since $(s-a)^{\alpha-1}$ is increasing on $[a,b]$ ($\alpha-1>0$), the maximum occurs at $s=b$, yielding

$$\frac{(b-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}\Gamma_q(\alpha)}=\frac{(b-a)^{\beta}}{\Gamma_q(\alpha)}.$$

This completes the proof.

Corollary 3. Let $1<\alpha\le 2$, $0\le\beta\le 1$, $1\le\alpha-\beta<2$, $q\in(0,1)$, $a<b$, and let the q-Green's function $G_q(t,s)$ be defined as in (9). Then, for $s\in[a,b]$,

$$\max_{t\in[a,b]}G_q(t,s)=G_q(s,s),$$

and

$$\max_{s\in[a,b]}G_q(s,s)=\frac{(b-a)^{\beta}b^{\alpha-\beta-1}(1-q)^{\alpha-\beta-1}}{\Gamma_q(\alpha)}.$$

Proof. The statement follows from Theorem 3 by identical arguments applied to the range $1\le\alpha-\beta<2$; the details are omitted.

Corollary 4. Let $1 < \alpha \leq 2$, $0 \leq \beta \leq 1$, $1 \leq \alpha - \beta < 2$, $q \in (0,1)$, $a < b$, and let the q-Green's function $G_q(t,s)$ be defined as in (6) and (9). Then:

$$\max_{t\in[a,b]} \int_a^b G_q(t,s)\, d_q s = \frac{[\alpha-1]_q^{\alpha-1}}{\Gamma_q(\alpha+1)} \left(\frac{b-a}{[\alpha-\beta]_q} \right)^{\alpha}.$$

Proof. Consider the integral $I(t) = \int_a^b G_q(t,s)\, d_q s$, where $G_q(t,s)$ is defined in (6) and (9). Split the integral based on the definition of $G_q(t,s)$:

Case 1: $a \leq t \leq s \leq b$.

$$G_q(t,s) = \frac{1}{\Gamma_q(\alpha)} \cdot \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}} (b-qs)_q^{(\alpha-\beta-1)}.$$

Case 2: $a \leq s \leq t \leq b$.

$$G_q(t,s) = \frac{1}{\Gamma_q(\alpha)} \left[\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}} (b-qs)_q^{(\alpha-\beta-1)} - (t-qs)_q^{(\alpha-1)} \right].$$

Thus,

$$I(t) = \int_a^t G_q(t,s)\, d_q s + \int_t^b G_q(t,s)\, d_q s.$$

Substitute the expression for $G_q(t,s)$:

$$\begin{aligned} I(t) &= \int_a^t \frac{1}{\Gamma_q(\alpha)} \left[\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}} (b-qs)_q^{(\alpha-\beta-1)} - (t-qs)_q^{(\alpha-1)} \right] d_q s \\ &\quad + \int_t^b \frac{1}{\Gamma_q(\alpha)} \cdot \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}} (b-qs)_q^{(\alpha-\beta-1)} d_q s \\ &= \frac{(t-a)^{\alpha-1}}{\Gamma_q(\alpha)(b-a)^{\alpha-\beta-1}} \int_a^b (b-qs)_q^{(\alpha-\beta-1)}\, d_q s - \frac{1}{\Gamma_q(\alpha)} \int_a^t (t-qs)_q^{(\alpha-1)}\, d_q s. \end{aligned}$$

Using equation (3), under the conditions $x = b$ or $x = t \geq s$, we have

$$\int_a^b (b-qs)_q^{(\alpha-\beta-1)}\, d_q s = \frac{(b-a)^{\alpha-\beta}}{[\alpha-\beta]_q}, \quad \int_a^t (t-qs)_q^{(\alpha-1)}\, d_q s = \frac{(t-a)^{\alpha}}{[\alpha]_q},$$

we get

$$\begin{aligned} I(t) &= \frac{(t-a)^{\alpha-1}(b-a)^{\alpha-\beta}}{\Gamma_q(\alpha)(b-a)^{\alpha-\beta-1}[\alpha-\beta]_q} - \frac{(t-a)^{\alpha}}{\Gamma_q(\alpha)[\alpha]_q} \\ &= \frac{(t-a)^{\alpha-1}(b-a)}{\Gamma_q(\alpha)[\alpha-\beta]_q} - \frac{(t-a)^{\alpha}}{\Gamma_q(\alpha)[\alpha]_q} \\ &= \frac{(t-a)^{\alpha-1}}{\Gamma_q(\alpha)} \left(\frac{b-a}{[\alpha-\beta]_q} - \frac{t-a}{[\alpha]_q} \right). \end{aligned}$$

To find the maximum, compute the q-derivative:

$$\begin{aligned} D_{q,t}I(t) &= \frac{1}{\Gamma_q(\alpha)}\left[[\alpha-1]_q(t-a)^{\alpha-2}\left(\frac{b-a}{[\alpha-\beta]_q}-\frac{t-a}{[\alpha]_q}\right)-(t-a)^{\alpha-1}\cdot\frac{1}{[\alpha]_q}\right] \\ &= \frac{1}{\Gamma_q(\alpha)}\left[\frac{[\alpha-1]_q(t-a)^{\alpha-2}(b-a)}{[\alpha-\beta]_q}-\frac{(t-a)^{\alpha-1}([\alpha-1]_q+1)}{[\alpha]_q}\right] \\ &= \frac{1}{\Gamma_q(\alpha)}\left[\frac{[\alpha-1]_q(t-a)^{\alpha-2}(b-a)}{[\alpha-\beta]_q}-(t-a)^{\alpha-1}\right], \end{aligned}$$

where $[\alpha-1]_q+1=\frac{1-q^{\alpha-1}}{1-q}+1=\frac{1-q^{\alpha}}{1-q}=[\alpha]_q$.

Set $D_{q,t}I(t)=0$:

$$t^*=a+\frac{[\alpha-1]_q(b-a)}{[\alpha-\beta]_q}.$$

Substitute t^* into the expression for $I(t)$:

$$\begin{aligned} I(t^*) &= \frac{\left(\frac{[\alpha-1]_q(b-a)}{[\alpha-\beta]_q}\right)^{\alpha-1}}{\Gamma_q(\alpha)}\left(\frac{b-a}{[\alpha-\beta]_q}-\frac{\frac{[\alpha-1]_q(b-a)}{[\alpha-\beta]_q}}{[\alpha]_q}\right) \\ &= \frac{\left(\frac{[\alpha-1]_q(b-a)}{[\alpha-\beta]_q}\right)^{\alpha-1}}{\Gamma_q(\alpha)}\left(\frac{b-a}{[\alpha-\beta]_q}\left(1-\frac{[\alpha-1]_q}{[\alpha]_q}\right)\right) \\ &= \frac{[\alpha-1]_q^{\alpha-1}(b-a)^{\alpha-1}}{\Gamma_q(\alpha)[\alpha-\beta]_q^{\alpha-1}}\cdot\frac{b-a}{[\alpha-\beta]_q}\cdot\frac{q^{\alpha-1}}{[\alpha]_q} \\ &= \frac{[\alpha-1]_q^{\alpha-1}(b-a)^{\alpha}q^{\alpha-1}}{\Gamma_q(\alpha)[\alpha-\beta]_q^{\alpha}[\alpha]_q}. \end{aligned}$$

The function $I(t)$ is increasing for $t<t^*$ ($D_{q,t}I(t)>0$) and decreasing for $t>t^*$ ($D_{q,t}I(t)<0$), confirming the maximum at t^*.

Theorem 4. Let $\mathfrak{B}_q=C_{q,\lambda}[a,b]$ denote the Banach space of functions continuous in the q-sense on the interval $[a,b]$, with norm

$$\|u\|_{C_{q,\lambda}}=\max_{t\in[a,b]}|u(t)|,$$

where $[a,b]=\{a,aq,aq^2,\dots,aq^n=b\}$. Given $1<\alpha\le 2$, $0\le\beta\le 1$, $0<\alpha-\beta<1$, if the fractional q-difference boundary value problem

$$\begin{cases} D_{q,a}^{\alpha}u(t)+q(t)u(t)=0, & t\in[a,b], \\ u(a)=0, \quad D_{q,a}^{\beta}u(b)=0, \end{cases} \tag{10}$$

admits a nontrivial solution $u\in\mathfrak{B}_q$, then the following Lyapunov-type inequality holds:

$$\int_a^b (b-qs)_q^{(\alpha-\beta-1)}|q(s)|d_qs>\frac{\Gamma_q(\alpha)}{(b-a)^{\beta}}. \tag{11}$$

Proof. Any solution $u\in\mathfrak{B}_q$ of the boundary value problem (10) satisfies

$$u(t)=\int_a^b G_q(t,s)q(s)u(s)d_qs,$$

where $G_q(t,s)$ is the q-Green's function given by (6).

By applying the $C_{q,\lambda}$-norm, we obtain

$$\begin{aligned}\|u\|_{C_{q,\lambda}} &= \max_{t\in[a,b]}\left|\int_a^b G_q(t,s)q(s)u(s)d_qs\right| \\ &\leq \max_{t\in[a,b]}\int_a^b |G_q(t,s)||q(s)||u(s)|d_qs \\ &\leq \|u\|_{C_{q,\lambda}}\cdot\max_{t\in[a,b]}\int_a^b |G_q(t,s)||q(s)|d_qs.\end{aligned}$$

For a nontrivial solution ($\|u\|_{C_{q,\lambda}} \neq 0$), this implies

$$1 \leq \max_{t\in[a,b]}\int_a^b |G_q(t,s)||q(s)|d_qs.$$

By Theorem 3, the q-Green's function satisfies the bound

$$|G_q(t,s)| \leq \frac{(b-a)^\beta (b-qs)_q^{(\alpha-\beta-1)}}{\Gamma_q(\alpha)}.$$

Substituting this bound, we get

$$1 < \max_{t\in[a,b]}\int_a^b |G_q(t,s)||q(s)|d_qs \leq \frac{(b-a)^\beta}{\Gamma_q(\alpha)}\int_a^b (b-qs)_q^{(\alpha-\beta-1)}|q(s)|d_qs.$$

Therefore, dividing both sides by $\frac{(b-a)^\beta}{\Gamma_q(\alpha)}$, we obtain (11).

This completes the proof.

Corollary 5. Let $1 < \alpha \leq 2$, $0 \leq \beta \leq 1$, and $1 \leq \alpha - \beta < 2$. Suppose the fractional q-difference boundary-value problem

$$\begin{cases} D_{q,a}^{\alpha}u(t) + q(t)u(t) = 0, & t \in [a,b], \\ u(a) = 0, \quad D_{q,a}^{\beta}u(b) = 0, \end{cases}$$

admits a nontrivial solution $u \in \mathfrak{B}_q = C_{q,\lambda}[a,b]$, where $C_{q,\lambda}[a,b]$ is the space of q-continuous functions on the q-interval $[a,b]$ with $0 < q < 1$. Then the following Lyapunov-type inequality holds:

$$\int_a^b |q(s)|\, d_qs > \frac{\Gamma_q(\alpha)}{(b-a)^\beta b^{\alpha-\beta-1}(1-q)^{\alpha-\beta-1}}.$$

Proof. By Corollary 1, any solution $u \in C_{q,\lambda}[a,b]$ to the boundary-value problem satisfies:

$$u(t) = \int_a^b G_q(t,s)q(s)u(s)\, d_qs,$$

where $G_q(t,s)$ is the q-Green's function defined in (9).

Define the norm $\|u\|_{C_{q,\lambda}} = \sup\limits_{t\in[a,b]} |u(t)|$. From the solution representation:

$$|u(t)| \leq \int\limits_a^b |G_q(t,s)||q(s)||u(s)|\, d_qs \leq \|u\|_{C_{q,\lambda}} \int\limits_a^b |G_q(t,s)||q(s)|\, d_qs.$$

Taking the supremum over $t \in [a,b]$, we obtain

$$\|u\|_{C_{q,\lambda}} \leq \|u\|_{C_{q,\lambda}} \max_{t\in[a,b]} \int\limits_a^b |G_q(t,s)||q(s)|\, d_qs.$$

For a nontrivial solution ($\|u\|_{C_{q,\lambda}} > 0$), it follows that

$$1 \leq \max_{t\in[a,b]} \int\limits_a^b |G_q(t,s)||q(s)|\, d_qs.$$

By Corollary 2, $G_q(t,s)$ is non-negative, so $|G_q(t,s)| = G_q(t,s)$. By Corollary 3, the maximum of the Green's function is

$$\max_{t,s\in[a,b]} G_q(t,s) = \max_{s\in[a,b]} G_q(s,s) = \frac{(b-a)^\beta b^{\alpha-\beta-1}(1-q)^{\alpha-\beta-1}}{\Gamma_q(\alpha)}.$$

Thus, $G_q(t,s) \leq \max\limits_{s\in[a,b]} G_q(s,s)$, and

$$\int\limits_a^b G_q(t,s)|q(s)|\, d_qs \leq \frac{(b-a)^\beta b^{\alpha-\beta-1}(1-q)^{\alpha-\beta-1}}{\Gamma_q(\alpha)} \int\limits_a^b |q(s)|\, d_qs.$$

Combining with the previous inequality, we get

$$1 \leq \frac{(b-a)^\beta b^{\alpha-\beta-1}(1-q)^{\alpha-\beta-1}}{\Gamma_q(\alpha)} \int\limits_a^b |q(s)|\, d_qs.$$

Rearranging yields

$$\int\limits_a^b |q(s)|\, d_qs \geq \frac{\Gamma_q(\alpha)}{(b-a)^\beta b^{\alpha-\beta-1}(1-q)^{\alpha-\beta-1}}.$$

To establish the strict inequality, suppose equality holds

$$\int\limits_a^b |q(s)|\, d_qs = \frac{\Gamma_q(\alpha)}{(b-a)^\beta b^{\alpha-\beta-1}(1-q)^{\alpha-\beta-1}}.$$

This implies $G_q(t,s) = \max\limits_{s\in[a,b]} G_q(s,s)$ for all $t,s \in [a,b]$ where $q(s)u(s) \neq 0$. By Corollary 3, $G_q(t,s) = G_q(s,s)$ only when $t = s$, which has measure zero in the q-integral unless $u \equiv 0$. Since u is nontrivial, equality is impossible, so

$$\int\limits_a^b |q(s)|\, d_qs > \frac{\Gamma_q(\alpha)}{(b-a)^\beta b^{\alpha-\beta-1}(1-q)^{\alpha-\beta-1}}.$$

3 Applications

In this section, we investigate two applications of Theorem 4 and Corollary 5. First, we establish lower bounds for the eigenvalues of the Riemann–Liouville type fractional q-eigenvalue problems associated with (10). Second, we utilize these findings to identify intervals where the q-analogue of the two-parameter Mittag-Leffler function has no real zeros.

Theorem 5. Let $1 < \alpha \leq 2$, $0 \leq \beta \leq 1$ such that $0 < \alpha - \beta < 1$. Assume that y is a nontrivial solution of the Riemann–Liouville type fractional q-eigenvalue problem

$$\begin{cases} D^{\alpha}_{q,a} u(t) + \lambda u(t) = 0, & t \in [a, b], \\ u(a) = 0, \quad D^{\beta}_{q,a} u(b) = 0, \end{cases} \tag{12}$$

where $u(t) \neq 0$ for each $t \in (a, b)$. Then,

$$|\lambda| > \frac{[\alpha - \beta]_q \Gamma_q(\alpha)}{(b-a)^{\alpha}}.$$

Corollary 6. Let $1 < \alpha \leq 2$, $0 \leq \beta \leq 1$ such that $1 \leq \alpha - \beta < 2$. Assume that u is a nontrivial solution of the Riemann–Liouville type fractional q-eigenvalue problem (12), where $u(t) \neq 0$ for each $t \in (a, b)$. Then,

$$|\lambda| > \frac{\Gamma_q(\alpha)}{(b-a)^{\beta} b^{\alpha-\beta-1} (1-q)^{\alpha-\beta-1}}.$$

Consider the q-analogue of the two-parameter Mittag-Leffler function, defined as ([14]):

$$E_{q,\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma_q(k\alpha + \beta)}, \quad z, \beta \in \mathbb{C}, \quad \Re(\alpha) > 0, \quad 0 < q < 1. \tag{13}$$

We use Theorem 5 and Corollary 6 to determine intervals where the function (13) has no real zeros.

Theorem 6. Let $1 < \alpha \leq 2$, $0 \leq \beta \leq 1$, $0 < \alpha - \beta < 1$, $q \in (0, 1)$. The q-Mittag-Leffler function

$$E_{q,\alpha,\alpha-\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma_q(k\alpha + \alpha - \beta)},$$

has no real zeros for

$$|z| \leq \frac{[\alpha - \beta]_q \Gamma_q(\alpha)}{(b-a)^{\alpha}}, \tag{14}$$

where $[\alpha - \beta]_q = \frac{1-q^{\alpha-\beta}}{1-q}$.

Proof. Consider the q-fractional eigenvalue problem

$$\begin{cases} D^{\alpha}_{q,a} u(t) + \lambda u(t) = 0, & t \in [a, b], \\ u(a) = 0, \quad D^{\beta}_{q,a} u(b) = 0, \end{cases}$$

where $D^{\alpha}_{q,a}$ is the Riemann–Liouville q-fractional derivative. The general solution is

$$u(t) = c_1 (t-a)^{\alpha-1} E_{q,\alpha,\alpha}(-\lambda(t-a)^{\alpha}) + c_2 (t-a)^{\alpha-2} E_{q,\alpha,\alpha-1}(-\lambda(t-a)^{\alpha}).$$

Let $g(t) = (t-a)^{\alpha-1} E_{q,\alpha,\alpha}(-\lambda(t-a)^{\alpha})$. Compute

$$D^{\alpha}_{q,a} g(t) = D^{\alpha}_{q,a} \left(\sum_{n=0}^{\infty} \frac{(-\lambda)^n (t-a)^{\alpha n + \alpha - 1}}{\Gamma_q(\alpha n + \alpha)} \right) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\Gamma_q(\alpha n + \alpha)} D^{\alpha}_{q,a} (t-a)^{\alpha n + \alpha - 1}.$$

Since $D^{\alpha}_{q,a}(t-a)^{\alpha n+\alpha-1} = \frac{\Gamma_q(\alpha n+\alpha)}{\Gamma_q(\alpha n)}(t-a)^{\alpha n-1}$, we get

$$D^{\alpha}_{q,a}g(t) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\Gamma_q(\alpha n)}(t-a)^{\alpha n-1} = -\lambda g(t).$$

The condition $u(a)=0$ implies $c_2=0$, since $(t-a)^{\alpha-2}\to\infty$ as $t\to a$. Thus,

$$u(t) = c_1(t-a)^{\alpha-1}E_{q,\alpha,\alpha}(-\lambda(t-a)^{\alpha}).$$

Compute

$$D^{\beta}_{q,a}u(t) = c_1\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{\Gamma_q(\alpha n+\alpha)}D^{\beta}_{q,a}(t-a)^{\alpha n+\alpha-1}.$$

Since $D^{\beta}_{q,a}(t-a)^{\alpha n+\alpha-1} = \frac{\Gamma_q(\alpha n+\alpha)}{\Gamma_q(\alpha n+\alpha-\beta)}(t-a)^{\alpha n+\alpha-\beta-1}$, we obtain

$$D^{\beta}_{q,a}u(t) = c_1(t-a)^{\alpha-\beta-1}E_{q,\alpha,\alpha-\beta}(-\lambda(t-a)^{\alpha}).$$

The condition $D^{\beta}_{q,a}u(b)=0$ gives

$$c_1(b-a)^{\alpha-\beta-1}E_{q,\alpha,\alpha-\beta}(-\lambda(b-a)^{\alpha}) = 0 \implies E_{q,\alpha,\alpha-\beta}(-\lambda(b-a)^{\alpha}) = 0.$$

By Theorem 5, for a nontrivial solution $u\in\mathfrak{B}_q = C_{q,\lambda}[a,b]$,

$$|\lambda| > \frac{[\alpha-\beta]_q\Gamma_q(\alpha)}{(b-a)^{\alpha}}.$$

For $z=-\lambda(b-a)^{\alpha}$, we have

$$|z| = |\lambda|(b-a)^{\alpha} > [\alpha-\beta]_q\Gamma_q(\alpha).$$

Thus, $E_{q,\alpha,\alpha-\beta}(z)\neq 0$ for (14).

Corollary 7. Let $1<\alpha\le 2$, $0\le\beta\le 1$ such that $1\le\alpha-\beta<2$. The q-Mittag-Leffler function $E_{q,\alpha,\beta}(z)$ has no real zeros for

$$|z| \le \frac{\Gamma_q(\alpha)}{(b-a)^{\alpha}}.$$

Proof. Following the same reasoning as in Theorem 6, suppose $E_{q,\alpha,\beta}(\lambda)=0$ for some real λ. The function $u(t) = E_{q,\alpha,\beta}(-\lambda(t-a)^{\alpha})$ satisfies the q-eigenvalue problem (12). By Corollary 6, any eigenvalue λ must satisfy:

$$|\lambda| > \frac{\Gamma_q(\alpha)}{(b-a)^{\alpha}}.$$

Hence, $E_{q,\alpha,\beta}(z)\neq 0$.

Conclusion

In this study, we derived two novel Lyapunov-type inequalities for boundary value problems involving the Riemann–Liouville fractional q-derivative within the regimes $0<\alpha-\beta<1$ and $1\le\alpha-\beta<2$, thereby establishing precise estimates for eigenvalues and intervals free of zeros for q-Mittag-Leffler functions. By employing an analysis of the q-Green's function, we determined lower bounds for the eigenvalues of the problem $D^{\alpha}_{q,a}u+\lambda u=0$ and identified regions devoid of real zeros for q-analogues of Mittag-Leffler functions, which holds significant importance for discrete systems with memory, such as viscoelastic lattices and quantum circuits. This work extends classical inequalities to the realm of q-calculus, thereby bridging continuous and discrete fractional analysis, and paves the way for further research on Caputo q-fractional derivatives and multidimensional q-lattices.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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