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Asymptotic estimations of the solution for a singularly perturbed equation with unlimited boundary conditions

The paper studies a two-point boundary value problem with unlimited boundary conditions for a linear singularly perturbed differential equation. Asymptotic estimates are given for a linearly independent system of solutions of a homogeneous perturbed equation. Auxiliary, so-called boundary functions, the Cauchy function are defined. For sufficiently small values of the parameter, estimates for the Cauchy function and boundary functions are found. An algorithm for constructing the desired solution of the boundary value problem has been developed. A theorem on the solvability of a solution to a boundary value problem is proved. For sufficiently small values of the parameter, an asymptotic estimate for the solution of the inhomogeneous boundary value problem is established. The initial conditions for the degenerate equation are determined. The formula is determined; the phenomena of the initial jump are studied.

Keywords: two-point boundary value problem, initial jumps, degenerate problem, small parameter, initial function, boundary functions.

Introduction

Researchers [1–14] have developed efficient asymptotic methods for singularly perturbed problems. For sufficiently small values of the parameter, these methods make it possible to construct uniform asymptotic approximations. However, for some singularly perturbed two-point boundary value problems with initial jumps, the choice of an appropriate method for constructing asymptotic approximations without a preliminary study turns out to be almost impossible. The first studies devoted to the phenomena of initial jumps were the works of Vishik, Lyusternik [15] and Kasymov [16]. In [17–19], these studies were summarized and continued. The jump phenomenon in many real problems of practice is a significant component, which is taken into account when building a model of these processes. In this case, the value of the jump is the condition for the perturbed problem to be replaced by a degenerate problem. For example, a new justification for the Painlev paradox, the existence of contrast structures, and the jump phenomenon were established by Neimark and Smirnova [20]. Asymptotic behavior, jump phenomena of the solution of a general two-point perturbed boundary value problem with finite boundary conditions were considered in [21–23]. In these papers, using the formula for solving a two-point boundary value problem for sufficiently small values of the parameter, asymptotic estimates are established, a theorem on the solvability of a solution to a two-point boundary value problem is formulated and proved, and the phenomena of initial and boundary jumps are revealed.

1 *Statement of the problem*

The next natural continuation in this direction is the study of the asymptotic behavior of solutions to perturbed two-point boundary value problems with unlimited boundary conditions. This work is devoted to the consideration of such problems.

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Consider the following inhomogeneous differential equation:

$$L_\varepsilon y \equiv \varepsilon y^{'''} + A(t)y^{''} + B(t)y^{'} + C(t)y = F(t) \tag{1}$$

with unlimited boundary conditions of the form:

$$y(0,\varepsilon) = a_1, \quad y^{'}(0,\varepsilon) = \frac{a_2}{\varepsilon}, \quad y(1,\varepsilon) = a_3 \tag{2}$$

where $\varepsilon > 0$ is a small positive parameter, $a_2 \neq 0$, a_i, $i = 1,2,3$ are known constants, $A(t)$, $B(t)$, $C(t)$, $F(t)$ are functions defined on the interval $[0,1]$.

In this paper, based on the analytical representation of the solution to problem (1), (2), the existence and uniqueness of the sought solution is proved.

Assume that the following conditions hold:

C1) $A(t), B(t), C(t), F(t)$ *are sufficiently smooth functions defined on the interval [0,1];*

C2) $A(t) \geq \gamma = const > 0,\ 0 \leq t \leq 1$;

C3) $\overline{J} = \begin{vmatrix} y_{10}(0) & y_{20}(0) \\ y_{10}(1) & y_{20}(1) \end{vmatrix} \neq 0$;

C4) Let $a_1 + \frac{a_2}{A(0)} \neq 0$.

2 *The fundamental set of solutions to the homogeneous perturbed equation*

Consider the following homogeneous equation associated with (1)

$$L_\varepsilon y(t,\varepsilon) \equiv \varepsilon y^{'''} + A(t)y^{''} + B(t)y^{'} + C(t)y = 0, \tag{3}$$

corresponding to the inhomogeneous equation (1). For the fundamental system of solutions to equation (3), the following lemma [1] is valid.

Lemma 1. If conditions (C1) and (C2) are satisfied, then the fundamental set of solutions $y_i(t,\varepsilon)$, $i = 1,2,3$ of (3) in the interval $0 \leq t \leq 1$ has the following asymptotic representation as $\varepsilon \to 0$:

$$\begin{cases} y_i^{(j)}(t,\varepsilon) = y_{i0}^{(j)}(t) + O(\varepsilon), \quad i=1,2, \quad j = 0,1,2, \\ y_3^{(j)}(t,\varepsilon) = \dfrac{1}{\varepsilon^j} exp\left(\dfrac{1}{\varepsilon}\displaystyle\int_0^t \mu(x)dx\right)\mu^j(t)y_{30}(t)[1 + O(\varepsilon)], \quad j=0,1,2, \end{cases} \tag{4}$$

where $\mu(t) = -A(t) < 0,\ y_{i0}(t),\ i = 1,2$, are solutions of the problem

$$L_0 y_{i0}(t) \equiv A(t)y_{i0}^{''} + B(t)y_{i0}^{'} + C(t)y_{i0} = 0, \quad i = 1,2, \tag{5}$$

with initial conditions:

$$y_{10}(0) = 1, \quad y_{10}^{'}(0) = 0, \quad y_{20}(0) = 0, \quad y_{20}^{'}(0) = 1,$$

Functions $y_{30}(t)$ has the form

$$y_{30}(t) = (A(0)/A(t))^2 exp\left(\int_0^t (B(x)/A(x))dx\right) \neq 0. \tag{6}$$

By applying asymptotic representation (4), for the $W[y_1(t,\varepsilon), y_2(t,\varepsilon), y_3(t,\varepsilon)]$ with sufficiently small ε we get

$$W(t,\varepsilon) = \frac{1}{\varepsilon^2} exp\left(\frac{1}{\varepsilon}\int\limits_0^t \mu(x)dx\right) y_{30}(t)\mu^2(t)\overline{W}(t)[1+O(\varepsilon)] \neq 0, \tag{7}$$

where $\overline{W}(t)$ is the Wronsky determinant of the fundamental system of solutions $y_{i0}(t)$, i=$1,2,3$ of equations (5),(6), and $\overline{W}(t) \neq 0$.

3 Constructing the initial and boundary functions

Also as in previous works [17, 18], we introduce the initial function

$$K(t,s,\varepsilon) = \frac{W(t,s,\varepsilon)}{W(s,\varepsilon)}, \tag{8}$$

determined from the next problem:

$$L_\varepsilon K(t,s,\varepsilon) = 0, \quad K(s,s,\varepsilon) = 0, \quad K_t'(s,s,\varepsilon) = 0, \quad K_t''(s,s,\varepsilon) = 1, \tag{9}$$

where $W(t,s,\varepsilon)$ is the determinant obtained from the Wronskian $W(s,\varepsilon)$ by replacing the third row with the fundamental set of solutions $y_1(t,\varepsilon), y_2(t,\varepsilon), y_3(t,\varepsilon)$.

Obviously, the initial function $K(t,s,\varepsilon)$ satisfies equation (3) and initial conditions (9), where the function $K(t,s,\varepsilon)$ does not depend on the choice of the solution fundamental set for equation (3). Therefore, the initial function for equation (3) exists, it can be expressed by formula (8) and it is determined by a unique form.

Let us consider the determinant

$$J(\varepsilon) = \begin{vmatrix} y_1(0,\varepsilon) & y_2(0,\varepsilon) & y_3(0,\varepsilon) \\ y_1'(0,\varepsilon) & y_2'(0,\varepsilon) & y_3'(0,\varepsilon) \\ y_1(1,\varepsilon) & y_2(1,\varepsilon) & y_3(1,\varepsilon) \end{vmatrix}. \tag{10}$$

Due to asymptotic estimation (4), elements of determinant (10) has next form $\varepsilon \to 0$:

$$y_i^j(0,\varepsilon) = y_{io}^j(0) + O(\varepsilon), \; i = 1,2, \; j = 0,1, \; y_i(1,\varepsilon) = y_{i0}(1) + O(\varepsilon), \; i = 1,2, \tag{11}$$

$$y_3(0,\varepsilon) = y_{30}[1+O(\varepsilon)], \; y_3'(0,\varepsilon) = \frac{1}{\varepsilon} y_{30}(0)\mu(0)[1+O(\varepsilon)],$$

$$y_3(1,\varepsilon) = exp\left(\frac{1}{\varepsilon}\int\limits_0^1 \mu(x)dx\right)[y_{30}(1) + O(\varepsilon)].$$

Then the determinant $J(\varepsilon)$ taking into account (11) has the following representation as $\varepsilon \to 0$

$$J(\varepsilon) = -\frac{1}{\varepsilon}\mu(0)\overline{J}(1+O(\varepsilon)). \tag{12}$$

Definition 1. The functions $\Phi_i(t,\varepsilon)$, $i = 1,2,3$, are called boundary functions for boundary value problem (1) and (2), if they satisfy homogeneous equation (3) and boundary conditions

$$\Phi_i^{(j)}(0,\varepsilon) = \begin{cases} 1, & j = i-1, \; i{=}1,2, \\ 0, & j \neq i-1, \; i = 1,2,3, \; j = 0,1, \end{cases} \tag{13}$$

$$\Phi_i(1,\varepsilon)=\begin{cases}1, & i=3,\\ 0, & i=1,2.\end{cases}$$

The following theorem is valid.

Theorem 1. If conditions (C1)–(C3) are satisfied, then the boundary functions $\Phi_i(t,\varepsilon)$, $i=1,2,3$, on the interval [0,1] exist, unique and can be expressed by formula:

$$\Phi_i(t,\varepsilon)=\frac{J_i(t,\varepsilon)}{J(\varepsilon)},\quad i=1,2,3, \tag{14}$$

where $J_i(t,\varepsilon)$, $i=1,2,3$ is the determinant obtained from $J(\varepsilon)$ by replacing the i-th row with the fundamental set of solutions $y_1(t,\varepsilon)$, $y_2(t,\varepsilon)$, $y_3(t,\varepsilon)$.

Proof. We seek the boundary functions $\Phi_i(t,\varepsilon)$, $i=1,2,3$ in the next form which satisfy the condition (13)

$$\Phi_i(t,\varepsilon)=c_1^i y_1(t,\varepsilon)+c_2^i y_2(t,\varepsilon)+c_3^i y_3(t,\varepsilon),\ i=1,2,3. \tag{15}$$

where c_1^i, c_2^i, c_3^i are unknown constants which are defined from the function (15), that function satisfies boundary condition (13). Obviously, the function (15) depending on one variable t satisfies the homogeneous equation (3). By substituting (15) into (13), we obtain

$$\Phi_i^k(0,\varepsilon)=\begin{cases}1, & k=i-1,\ i=1,2,\\ 0, & k\neq i-1,\ i=1,2,3,\ k=0,1,\end{cases} \tag{16}$$

$$\Phi_i(1,\varepsilon)=c_1^i y_1(1,\varepsilon)+c_2^i y_2(1,\varepsilon)+c_3^i y_3(1,\varepsilon)=\begin{cases}1, & i=3,\\ 0, & i=1,2.\end{cases}$$

With a fixed value i system (16) has a linear algebraic system of equations for determining c_1^i, c_2^i, c_3^i, which determinant is $J(\varepsilon)$. Then, by means of (12) for a sufficiently small ε the system (16) is uniquely solvable. Solving (16), we have

$$c_k^i=\frac{J_{ik}}{J(\varepsilon)},\quad i=1,2,3, \tag{17}$$

where J_{ik} is the algebraic complement of the determinant element $J(\varepsilon)$, at the intersection of the $i-th$ row and $k-th$ column. Substituting (17) into (16) and comparing the decomposition obtained with the determinant decomposition $J_i(t,\varepsilon)$ by elements $i-th$ row, we get formula (14). Hereby, the functions $\Phi_i(t,\varepsilon)$, $i=1,2,3$, defined by the formula (14) satisfy the equation (3) and boundary condition (13). Consequently, functions $\Phi_i(t,\varepsilon)$, $i=1,2,3$, are defined on the interval $0\le t\le 1$, are boundary functions of the perturbed problem (1), (2). The theorem is proved.

Lemma 2. The initial function $K(t,s,\varepsilon)$ and its derivatives by variable t to the second order are defined on the interval [0,1] at $s\le t$ have following asymptotic representation as $\varepsilon\to 0$:

$$K^{(j)}(t,s,\varepsilon)=\frac{\varepsilon}{\mu(s)\overline{W}(s)}\left[-\overline{W}^j(t,s)+\varepsilon^{1-j}\frac{y_{30}(t)\mu^j(t)}{y_{30}(s)\mu(s)}exp\left(\frac{1}{\varepsilon}\int_s^t\mu(x)dx\right)\cdot\right.$$

$$\left.\cdot\,\overline{W}(s)+O\left(\varepsilon+\varepsilon^{2-j}exp\left(\frac{1}{\varepsilon}\int_s^t\mu(x)dx\right)\right)\right],\quad j{=}0,1,2, \tag{18}$$

where $\overline{W}(t,s)=\begin{vmatrix}y_{10}(s) & y_{20}(s)\\ y_{10}(t) & y_{20}(t)\end{vmatrix}$.

Proof . We expand $W(t,s,\varepsilon)$ by the elements of the third column:

$$W^{(j)}(t,s,\varepsilon) = y_3(s,\varepsilon)W_{13}(t,s) - y_3'(s,\varepsilon)W_{23}(t,s) + y_3^{(j)}(t,s)W_{33}(s,\varepsilon), \tag{19}$$

where the minors $W_{13}(t,s)$ $(i = 1,2,3)$ by virtue of (11) as $\varepsilon \to 0$ can be represented in the form

$$W_{13}(t,s,\varepsilon) = y_{10}'(s)y_{20}^{(j)}(t) - y_{20}'(s)y_{10}^{(j)}(t) + O(\varepsilon), \tag{20}$$

$$W_{23}(t,\varepsilon) = \overline{W}^{(j)}(t,s) + O(\varepsilon),$$

$$W_{33}(s,\varepsilon) = \overline{W}(s) + O(\varepsilon)$$

where $\overline{W}^j(t,s)$ is determinant obtained from the $\overline{W}t,s$ after deleting second row to $y_{10}^{(j)}(t)$, $y_{20}^{(j)}(t)$. Then, taking into account estimates (20) and (4), from (19) for the function $W^{(j)}(t,s,\varepsilon)$ we have following representation as $\varepsilon \to 0$

$$W^{(j)}(t,s,\varepsilon) = \frac{1}{\varepsilon} exp\left(\frac{1}{\varepsilon}\int_0^s \mu(x)dx\right) y_{30}(s)\mu(s)\Bigg[-\overline{W}^{(j)}(t,s) +$$
$$+\varepsilon^{1-j} exp\left(\frac{1}{\varepsilon}\int_s^t \mu(x)dx\right)\cdot\frac{y_{30}(t)\mu^{(j)}(t)}{y_{30}(s)\mu^{(j)}(s)}\overline{W}(s) +$$
$$+O\left(\varepsilon + \varepsilon^{2-j} exp\left(\frac{1}{\varepsilon}\int_s^t \mu(x)dx\right)\right)\Bigg]. \tag{21}$$

Use (21) and (7) in (8) we obtain estimate (18). Lemma is proved.

Lemma 3. Under conditions (C1)–(C3), on the interval for the boundary functions $\Phi_i(t,\varepsilon)$, $i = 1,2,3$, the following asymptotic representation holds as $\varepsilon \to 0$

$$\Phi_1^{(j)}(t,\varepsilon) = \frac{\overline{J}_1^{(j)}(t)}{\overline{J}} - \frac{\varepsilon}{\varepsilon^j} exp\left(\frac{1}{\varepsilon}\int_0^t \mu(x)dx\right)\frac{y_{30}(t)\mu^j(t)}{\overline{J}y_{30}(0)\mu(0)}\frac{\overline{J}_1^{(2)}(0)}{\overline{J}} +$$
$$+O\left(\varepsilon + \frac{\varepsilon^2}{\varepsilon^j} exp\left(\frac{1}{\varepsilon}\int_0^t \mu(x)dx\right)\right), \tag{22}$$

$$\Phi_2^{(j)}(t,\varepsilon) = -\varepsilon\frac{\overline{J}_1^{(j)}(t)}{\mu(0)\overline{J}} + \frac{\varepsilon}{\varepsilon^j} exp\left(\frac{1}{\varepsilon}\int_0^t \mu(x)dx\right)\frac{y_{30}(t)\mu^j(t)}{y_{30}(0)\mu(0)} +$$
$$+O\left(\varepsilon^2 + \frac{\varepsilon^2}{\varepsilon^j} exp\left(\frac{1}{\varepsilon}\int_0^t \mu(x)dx\right)\right),$$

$$\Phi_3^{(j)}(t,\varepsilon) = \frac{\overline{J}_2^{(j)}(t)}{\overline{J}} - \frac{\varepsilon}{\varepsilon^j} exp\left(\frac{1}{\varepsilon}\int_0^t \mu(x)dx\right)\frac{y_{30}(t)\mu^j(t)}{y_{30}(0)\mu(0)}\frac{\overline{J}_2^{(2)}(0)}{\overline{J}} +$$

$$+O\left(\varepsilon+\frac{\varepsilon^2}{\varepsilon^j}exp\left(\frac{1}{\varepsilon}\int\limits_0^t\mu(x)dx\right)\right),\quad j=0,1,2,$$

where $\overline{J}_i^{(j)}(t)$ is the determinant obtained from $\overline{J}$ by replacing the $i-th$ row with the fundamental set of solutions $y_{10}^{(j)}(t)$, $y_{20}^{(j)}(t)$.

Proof. By spreading $J_i^{(j)}(t,\varepsilon)$ the element of the third column and taking into account the estimation (4), we have

$$J_1^{(j)}(t,\varepsilon)=-\frac{1}{\varepsilon}y_{30}(0)\mu(0)\left[\overline{J}_1^{(j)}(t)+\varepsilon^{1-j}\frac{y_{30}(t)\mu^j(t)}{y_{30}(0)\mu(0)}exp\left(\frac{1}{s}\int\limits_0^t\mu(x)dx\right)\overline{J}_1^{(2)}(0)+\right.$$
$$\left.+O\left(\varepsilon+\varepsilon^{2-j}exp\left(\frac{1}{\varepsilon}\int\limits_0^t\mu(x)dx\right)\right)\right],$$
$$J_2^{(j)}(t,\varepsilon)=y_{30}(0)\left[\overline{J}_1^{(j)}(t)-\varepsilon^{-j}\frac{y_{30}(t)\mu^j(t)}{y_{30}(0)}exp\left(\frac{1}{\varepsilon}\int\limits_0^t\mu(x)dx\right)\overline{J}+\right.$$
$$\left.+O\left(\varepsilon+\varepsilon^{1-j}exp\left(\frac{1}{\varepsilon}\int\limits_0^t\mu(x)dx\right)\right)\right],\tag{23}$$
$$J_3^{(j)}(t,\varepsilon)=-\frac{1}{\varepsilon}y_{30}(0)\mu(0)\left[\overline{J}_2^{(j)}(t)-\varepsilon^{1-j}\frac{y_{30}(t)\mu^j(t)}{y_{30}(0)\mu(0)}exp\left(\frac{1}{\varepsilon}\int\limits_0^t\mu(x)dx\right)\cdot\right.$$
$$\left.\cdot\,\overline{J}_2^{(2)}(0)+O\left(\varepsilon+\varepsilon^{2-j}exp\left(\frac{1}{\varepsilon}\int\limits_0^t\mu(x)dx\right)\right)\right].$$

Then, using (12) and (23) in (14), we get (22). Lemma is proved.

4 *Constructing the solution of the boundary value problem*

Theorem 2. If conditions (C1)–(C3) are satisfied then for sufficiently small $\varepsilon>0$ boundary value problem (1), (2) on the interval [0,1] has a unique solution $y(t,\varepsilon)$, which can be presented in the following form

$$y(t,\varepsilon)=a_1\Phi_1(t,\varepsilon)+\frac{a_2}{\varepsilon}\Phi_2(t,\varepsilon)+a_3\Phi_3(t,\varepsilon)-$$
$$-\Phi_3(t,\varepsilon)\frac{1}{\varepsilon}\int\limits_0^1K(1,s,\varepsilon)F(s)ds+\int\limits_0^tK(t,s,\varepsilon)F(s)ds.\tag{24}$$

Proof. We seek the solution $y(t,\varepsilon)$ of BVP (1), (2) in the form:

$$y(t,\varepsilon)=C_1\Phi_1(t,\varepsilon)+C_2\Phi_2(t,\varepsilon)+C_3\Phi_3(t,\varepsilon)+\frac{1}{\varepsilon}\int\limits_0^tK(t,s,\varepsilon)F(s)ds,\tag{25}$$

where $C_i,\ \ i=1,2,3$, are unknown constants. By directly substituting (25) in (1) we make sure that the function $y(t,\varepsilon)$ is defined by formula (25) is a solution of equation (1). For determination $C_i,\ i=1,2,3$, we use (25) in (2). Then we will have:

$$C_1=a_1,\ \ C_2=\frac{a}{\varepsilon},\ \ C_3=a_3-\frac{1}{\varepsilon}\int\limits_0^1 K(1,s,\varepsilon)F(s)ds. \tag{26}$$

Substituting found values (26) into (25), we obtain (24). From here and from that, the boundary function $\Phi_i(t,\varepsilon)$ does not depend on the choice of the fundamental solution system of equation (3) it follows that solutions of the boundary value problem (1), (2) exist, are unique and are expressed by formula (25). The theorem is proved.

Theorem 3. Under conditions (C1)–(C3), for the solution $y(t,\varepsilon)$ of the boundary value problem (1) and (2) in the interval [0,1] the following asymptotic estimations hold uniformly by variable t and as $\varepsilon\to 0$

$$|y(t,\varepsilon)|\le\Big[|a_1-\frac{a_2}{\mu(0)}||\overline{\Phi}_1(t)|+|a_3||\overline{\Phi}_2(t)|+$$
$$+|a_2|\left(\exp\left(-\gamma\frac{t}{\varepsilon}\right)\right)+\max_{0\le t\le 1}|F(t)|\Big], \tag{27}$$

where $C\ge 0$ are constants independent of t and ε, functions $\overline{\Phi}_1(t)=\frac{\overline{J}_1(t)}{\overline{J}}$, $\overline{\Phi}_2(t)=\frac{\overline{J}_2(t)}{\overline{J}}$ satisfy the degenerate homogeneous equation

$$A(t)\overline{y}^{''}+B(t)\overline{y}^{'}+C(t)\overline{y}=0 \tag{28}$$

and boundary conditions

$$\overline{\Phi}_1(0)=1,\quad \overline{\Phi}_2(0)=0, \tag{29}$$
$$\overline{\Phi}_1(1)=0,\quad \overline{\Phi}_2(1)=1.$$

Proof . In (24) the expression $\frac{1}{\varepsilon}\int\limits_0^1 K(1,s,\varepsilon)F(s)ds$ can be expressed in the next form

$$\frac{1}{\varepsilon}\int\limits_0^1 K(1,s,\varepsilon)F(s)ds=-\int\limits_0^1\overline{K}(1,s)\frac{F(s)}{\mu(s)}ds+O(\varepsilon)=$$
$$=\int\limits_0^1\overline{K}(1,s)\frac{F(s)}{A(s)}ds+O(\varepsilon), \tag{30}$$

where the function

$$\overline{K}(t,s)=\frac{\overline{W}(t,s)}{\overline{W}(s)}, \tag{31}$$

is an initial function of equation (28), by means of (31) and by variable t satisfies equation (28) and initial conditions

$$\overline{K}(s,s)=0,\overline{K'}(s,s)=1$$

and, the function $\overline{K}(t,s)$ does not depend on the choice of the fundamental set of solution $y_{10}(t)$, $y_{20}(t)$ to equation (28). The function

$$\overline{\Phi}_k(t) = \frac{\overline{J}_k(t)}{\overline{J}}, \quad k = 1, 2, \tag{32}$$

satisfies the degenerate homogeneous equation (28) and the boundary conditions (29).

Consequently, functions (32) are boundary functions of the unknown degenerate problem. The function $\overline{\Phi}_k(t)$ does not depend on the choice of the fundamental set of solution $y_{10}(t)$, $y_{20}(t)$ of equation (28) too. From (24), and by means of (30),(18), (22), we have

$$y(t,\varepsilon) = a_1 \frac{\overline{J}_1(t)}{\overline{J}} + a_2 \left[-\frac{\overline{J}_1(t)}{\mu(0)\overline{J}} + exp\left(\frac{1}{\varepsilon} \int\limits_0^t \mu(x)dx \right) \frac{y_{30}(t)}{y_{30}(t)\mu(0)} \right] - \tag{33}$$

$$-a_3 \frac{\overline{J}_2(t)}{\overline{J}} - \frac{\overline{J}_2(t)}{\overline{J}} \int\limits_0^1 \overline{K}(1,s)\frac{F(s)}{A(s)}ds + \int\limits_0^t \overline{K}(t,s)\frac{F(s)}{A(s)}ds + O(\varepsilon).$$

Hence, we obtain the desired estimate (27). The theorem is proved.

5 *Constructing a solution of the unperturbed problem*

Now we formulate a degenerate problem. Let's consider the degenerate equation

$$L_0\overline{y} \equiv A(t)\overline{y}^{''} + B(t)\overline{y}^{'} + C(t)\overline{y} = F(t). \tag{34}$$

In order to select the boundary conditions of the degenerate problem, we turn to the estimate (27). In the first approximation, in these estimates, for the boundary functions of the degenerate problem, there are constants $a_1 - \frac{a_2}{\mu(0)}$, a_3 Taking this into account consideration, we construct a solution to equation (34) under the conditions:

$$\overline{y}(0) = a_1 - \frac{a_2}{\mu(0)}, \quad \overline{y}(1) = a_3. \tag{35}$$

Theorem 4. Under the conditions (C1)–(C4), the solution $\overline{y}(t)$ of the boundary value problem (34), (35) on the interval [0,1] is unique and can be presented in the following form

$$\overline{y}(t) = \left(a_1 - \frac{a_2}{\mu(0)} \right) \overline{\Phi}_1(t) + a_3\overline{\Phi}_2(t) -$$

$$-\overline{\Phi}_2(t) \int\limits_0^1 \overline{K}(1,s)\frac{F(s)}{A(s)}ds + \int\limits_0^t \overline{K}(1,s)\frac{F(s)}{A(s)}ds, \tag{36}$$

where $\overline{\Phi}_k(t)$, $k = 1, 2$, $\overline{K}(t,s)$ are functions defined in (31), (32).

The proof of Theorem 3 is carried out similarly to the proof of Theorem 2.

6 *About limit transition and initial jump*

Theorem 5. Under the conditions (C1)–(C4), for a sufficiently small $\varepsilon \geq 0$ the difference between solution $y(t,\varepsilon)$ of BVP (1), (2) and solution $\overline{y}(t)$ of problem (34), (35) on the interval [0,1] satisfies the following inequality:

$$|y(t,\varepsilon) - \overline{y}(t)| \leq C\left(\varepsilon + exp\left(\frac{1}{\varepsilon} \int\limits_0^t \mu(x)dx \right) \right). \tag{37}$$

Proof . We introduce a function $u(t,\varepsilon) = y(t,\varepsilon) - \overline{y}(t)$. The problem (1), (2) have the next form:

$$L_\varepsilon u(t,\varepsilon) = -\varepsilon \overline{y}'''(t),\ u(0,\varepsilon) = \frac{a_2}{\mu(0)},\ u'(0,\varepsilon) = \frac{a_2}{\varepsilon} - \overline{y}'(0),\ u(1,\varepsilon) = 0. \tag{38}$$

By applying Theorem 3 to boundary value problem (38), taking into account (37), we obtain the following inequality:

$$|u(t,\varepsilon)| \leq C\left(\varepsilon + exp\left(\frac{1}{\varepsilon}\int\limits_0^t \mu(x)dx\right)\right),$$

which proves the estimation (38).

Thus, from Theorem 5 it follows that

$$\lim_{\varepsilon\to 0} y(t,\varepsilon) = \overline{y}(t),\ 0 < t \leq 1. \tag{39}$$

Now we determine the magnitude of the initial jump. For this, we turn to the estimate (33). From (33) taking into account (36), we obtain

$$\lim_{\varepsilon\to 0} y(0,\varepsilon) - \overline{y}(0) = \frac{a_2}{\mu(0)},\quad y'(0,\varepsilon) = \frac{a_2}{\varepsilon}. \tag{40}$$

Based on (39) and (40), we conclude, that the solution $y(t,\varepsilon)$ of a singularly perturbed equation (1) with unlimited boundary conditions (2) has the zero order of initial jump at point $t = 0$, which is one of the features of the studied problem.

Thus, we conclude that the established algorithm for studying the solution of a boundary value problem with unlimited boundary conditions allows us to investigate the asymptotic behavior of the solution of a general boundary value problem with unlimited boundary conditions for higher order linear equations. However, the proposed algorithm does not allow to construct the asymptotic solution of substantially nonlinear boundary value problems with unlimited boundary conditions that possess the phenomena of initial jumps. A natural direction for further research is the construction linear asymptotic solution, and nonlinear singularly perturbed boundary problems with unlimited boundary conditions possessing initial jumps. Therefore, the study of the asymptotic behavior and the construction asymptotic solution of the singularly perturbed boundary problems with unlimited boundary conditions that possess the phenomena of initial jumps are still relevant, of particular theoretical interest and important in applications. The obtained results provide opportunities for further research and development of the theory of boundary value problems for ordinary differential equations with a small parameter at the highest derivatives. The constructed initial approximations can be used when considering various problems of chemical kinetics.

7 Conclusion

Thus, the initial and boundary functions for perturbed and unperturbed problems are introduced and constructed, and their asymptotic estimations are found. Using these functions, we constructed an analytical representation of the solution to a singularly perturbed boundary value problem (1), (2) with unlimited boundary conditions. The unperturbed boundary value problem is formulated. The difference between the solutions of the degenerate and initial boundary value problems is estimated for sufficiently small $\varepsilon > 0$, and thus it is proved that the solution of the perturbed problem tends to solve the degenerate problem as the small parameter tends to zero. The growth of the derivative with respect to a small parameter is established. The class of boundary problems with unlimited boundary conditions with the phenomenon of initial jumps is distinguished.

The obtained results give the opportunity for further research in the theory of singularly perturbed boundary value problems, to reduce the boundary value problem (1), (2) to the Cauchy problem with unbounded initial conditions, which in turn can be considered as the basis for constructing the asymptotic expansions of some singularly perturbed boundary value problems with unlimited boundary conditions.

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Сингулярлы ауытқыған дифференциалдық теңдеу үшін шектелмеген шекаралық шарттары бар есептің асимптотикалық бағалаулары

Мақалада сызықты сингулярлы ауытқыған дифференциалдық теңдеу үшін шектелмеген шекаралық шарттары бар екі нүктелі шекаралық есеп зерттелген. Ауытқыған біртекті теңдеу шешімдерінің сызықты тәуелсіз жүйесі үшін асимптотикалық бағалаулар берілген. Шекаралық функциялар, Коши функциясы деп аталатын көмекші функциялар анықталған. Параметрдің жеткілікті аз мәндері үшін

Коши функциясы мен шекаралық функциялардың бағалаулары табылған. Зерттелетін шекаралық есептің қажетті шешімін құру алгоритмі құрастырылды. Шеттік есептің шешімінің шешілетіндігі туралы теорема дәлелденді. Параметрдің жеткілікті аз мәндері үшін қарастырылып отырған біртекті емес шекаралық есептің шешімі үшін асимптотикалық бағалау берілді. Өзгертілген теңдеудің бастапқы шарттары анықталған. Формула анықталды, бастапқы секіріс құбылысы зерттелген.

Кілт сөздер: екі нүктелі шекаралық есеп, бастапқы секіріс, ауытқыған есеп, кіші параметр, бастапқы функция, шекаралық функциялар.

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Асимптотические оценки решения сингулярно возмущенной краевой задачи с неограниченными граничными условиями

В статье исследована двухточечная краевая задача с неограниченными краевыми условиями для линейного сингулярно возмущенного дифференциального уравнения. Даны асимптотические оценки для линейно независимой системы решений однородного возмущенного уравнения. Определены вспомогательные, так называемые граничные функции, функция Коши. При достаточно малых значениях параметра найдены оценки для функции Коши и граничных функций. Разработан алгоритм построения искомого решения исследуемой краевой задачи. Доказана теорема о разрешимости решения краевой задачи. При достаточно малых значениях параметра установлена асимптотическая оценка решения рассматриваемой неоднородной краевой задачи. Определены начальные условия для вырождающегося уравнения. Формула определена, изучены явления начального скачка.

Ключевые слова: двухточечная краевая задача, начальные скачки, вырожденная задача, малый параметр, начальная функция, граничные функции.

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