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A.Kh. Attaev

Institute of Applied Mathematics and Automation of Kabardin-Balkar Scientific Centre of RAS, Nalchik, Russia
(E-mail: attaev.anatoly@yandex.ru)

On Problem of Internal Boundary Control for String Vibration Equation

The article deals with the vibration control problem described by one dimensional wave equation with integral type boundary condition. As usual, the initial and final moments of time for arbitrary displacements and velocities of the wave are specified by points on a string (Cauchy data). It is shown that the minimum time for the realizable control is uniquely determined by the condition of correct solvability to the Cauchy problem involving data lying on disconnected manifold. This suggests that the internal boundary conditions does not affect the minimum time value. Necessary and sufficient conditions for the existence of the desired internal-boundary controls that move the process from the state initially specified to a predetermined final one are obtained and written out. The controls are presented in explicit analytical form. Moreover, it is shown that for the inner-boundary controls expressions, one should use not the representation of the solution to the Cauchy problem in the sought-for domain, but the formula for the general solution of the string oscillation equation (d'Alembert's formula).

Keywords: string vibration equation, boundary control, Cauchy problem, trunk and branched pipeline networks, nonlocal mixed problem.

Introduction

One of the main parts of systems with distributed parameters in the Control Theory is one-dimensional distributed-parameter system for objects with the motion described by hyperbolic partial differential equations. As a rule, these objects control requires considering oscillation and wave propagation. We refer to objects include such technical facilities as a compressor and pumping stations providing distributing water via trunk and branched pipeline networks at a given flow rate and pressure. The problem of pressure pulsation dampening in pipelines is considered to be a classical one. A detailed account on engineering aspects and mathematical formulations for this problem and also various solution techniques can be found in [1], as well as in [2, 3]. There are many works devoted to boundary control problems for hyperbolic equations, including loaded ones, and to optimization problems in terms of an arbitrary sufficiently large time interval, and others, we here mention only some of them [4–14]. These papers investigated boundary control problems for hyperbolic equations with both local and nonlocal multi-point boundary conditions. The boundary control problem is formulated as follows: Define the control action for moving the internal state of a system from any initial state to any other final state in a finite time interval. If the controls are determined on the boundary such problem is called the boundary control problem. If controls are defined inside the domain as well as on the boundary such a problem is logically called the problem of the internal boundary control. In mathematical terms, the unique solvability of the control problem is equivalent to the well-posed solvability of the Cauchy problem with data lying on a disconnected manifold. It is this fact that makes it possible to determine the minimum time during which unambiguous control is carried out.

The following results were obtained:

1. Necessary and sufficient conditions $\varphi_0(x)$, $\varphi_1(x)$, $\psi_0(x)$, $\psi_1(x)$, $k_1(x,t)$, $k_2(x,t)$, are established ensuring the existence of the boundary controls $\mu(t)$ and $\nu(t)$, in the form of

$$\varphi_0^{'}(0) - \psi_0(0) - \varphi_1^{'}(l) + \psi_1(l) = 0, \tag{1}$$

$$\varphi_0^{'}(l) + \psi_0(l) - \varphi_1^{'}(0) - \psi_1(0) = 0, \tag{2}$$

$$\varphi_0^{''}(0) - \psi_0^{'}(0) - \varphi_1^{''}(l) + \psi_1^{'}(l) = 0, \tag{3}$$

$$\varphi_0^{''}(l) + \psi_0^{'}(l) - \varphi_1^{''}(0) - \psi_1^{'}(0) = 0, \tag{4}$$

$$\varphi_0(0) + \psi_0(l) + \int_0^l \psi_0(\xi)d\xi - \varphi_1(0) - \varphi_1(l) + \int_0^l \psi_1(\xi)d\xi = 0, \tag{5}$$

$$\int_0^l k_i(\xi,0)\varphi_0(\xi)d\xi = 0, \ \int_0^l k_i(\xi,l)\varphi_1(\xi)d\xi = 0, \ i = 1,2, \tag{6}$$

$$\int_0^l k_i^{'}(\xi,0)\varphi_0(\xi)d\xi + \int_0^l k_i(\xi,0)\psi_0(\xi)d\xi = 0, \ i = 1,2, \tag{7}$$

$$\int_0^l k_i^{'}(\xi,l)\varphi_0(\xi)d\xi + \int_0^l k_i(\xi,l)\psi_1(\xi)d\xi = 0, \ i = 1,2. \tag{8}$$

2. Under conditions (1)–(4) , an explicit analytical form of the sought controls is found

$$\mu(t) = \frac{1}{2}\varphi_0(t) + \frac{1}{2}\int_0^t \psi_0(\xi)d\xi + \frac{1}{2}\varphi_1(l-t) + \frac{1}{2}\int_{l-t}^l \psi_1(\xi)d\xi + \frac{1}{2}\varphi_0(0) - \frac{1}{2}\varphi_1(l) + U_{k_1}(t), \tag{9}$$

$$\nu(t) = \frac{1}{2}\varphi_0(l-t) + \frac{1}{2}\int_{l-t}^l \psi_0(\xi)d\xi + \frac{1}{2}\varphi_1(t) + \frac{1}{2}\int_0^t \psi_1(\xi)d\xi + \frac{1}{2}\varphi_0(l) - \frac{1}{2}\varphi_1(0) + U_{k_2}(t), \tag{10}$$

where

$$U_{k_i}(t) = \frac{1}{2}\int_0^{l-t} k_i(t+\xi,t)\varphi_0(\xi)d\xi + \frac{1}{2}\int_t^l k_i(\xi-t,t)\varphi_0(\xi)d\xi +$$
$$\frac{1}{2}\int_t^l k_i(\xi-t,t)d\xi \int_0^t \psi_0(\xi)d\xi - \frac{1}{2}\int_0^{l-t}\left(\int_\xi^{l-t} k_i(t+\eta,t)d\eta\right)\psi_0(\xi)d\xi +$$
$$\frac{1}{2}\int_t^l\left(\int_\xi^l k_i(\eta-t,t)d\eta\right)\psi_0(\xi)d\xi + \frac{1}{2}\int_0^t k_i(\xi+l-t,t)\varphi_1(\xi)d\xi +$$
$$\frac{1}{2}\int_0^t k_i(\xi+l-t,t)d\xi\int_t^l \psi_1(\xi)d\xi + \frac{1}{2}\int_l^{l-t}\left(\int_{l-\xi}^t k_i(t-\eta,t)d\eta\right)\psi_1(\xi)d\xi +$$
$$\frac{1}{2}\int_0^t\left(\int_0^\xi k_i(\eta+l-t,t)d\eta\right)\psi_1(\xi)d\xi + \frac{1}{2}\left[\varphi_0(0) - \varphi_1(l)\right]\left[\int_0^t k_i(z,t)dz - \int_{l-t}^l k_i(z,t)dz\right], \ i = 1,2.$$

Below the problem of internal-boundary-value control for the string oscillation equation is formulated and the possibility for obtaining unique solution in minimum time interval using the control and the Cauchy problems is discussed.

Main part

In the domain $\Omega = \{(x,t) : 0 < x < l, 0 < t < T\}$ the one-dimensional equation of string vibration is considered

$$u_{xx} - u_{tt} = 0, \tag{11}$$

describing, for example, the string vibration with the ends fixed at the points $x = 0$ and $x = l$. Assume that arbitrary displacements and arbitrary speeds are set at the initial time $t = 0$ and the final time $t = T$:

$$u(x,0) = \varphi_0(x),\ u_t(x,0) = \psi_0(x),\ 0 \leq x \leq l, \tag{12}$$

$$u(x,T) = \varphi_1(x),\ u_t(x,T) = \psi_1(x),\ 0 \leq x \leq l. \tag{13}$$

Internal boundary conditions are specified by the following relations:

$$u(0,t) + \int_0^l k_1(\xi,t)u(\xi,t)d\xi = \mu(t), \tag{14}$$

$$u(l,t) + \int_0^2 k_2(\xi,t)u(\xi,t)d\xi = \nu(t), \tag{15}$$

where $k_1(x,t)$, $k_2(x,t)$ — are the given functions, and $k_i(x,t)$, $\frac{\partial}{\partial t}k_i(x,t) \in C(\bar{\Omega})$, $i = 1, 2$.

Further in this paper the function $u(x,t) \in C^2(\Omega) \cap C^1(\bar{\Omega})$ — is understood as the solution to equation (11). The problem is to find such values of $\mu(t)$ and $\nu(t)$ for unambiguous moving the system from state (14) to state (15) in the minimum time interval.

From the problem formulated above it immediately follows that $\mu(t)$ and $\nu(t)$ are uniquely determined in the domain Ω if and only if the Cauchy problem (12), (13) for equation (11) in the domain Ω is well-posed.

Indeed, in case when the Cauchy problem (12), (13) for equation (11) occurs underdetermined, the infinite set of $(\mu(t), \nu(t))$ can perform the desired control. This implies the non-uniqueness of the solution to the Cauchy problem (12), (13) for equation (11) in the domain Ω. If the Cauchy problem (12), (13) for equation (11) in the domain Ω is underdetermined, then control is possible only for linearly depend (12) and (13). These items are discussed in [4] and in more detail in [14]. There you can also find out that the Cauchy problem (12), (13) for equation (11) in the domain Ω is well-posed if and only if the value of T is equal to l. If $T > l$, the Cauchy problem is underdetermined. IF $T < l$, the Cauchy problem is redefined. This suggests that the desired control is unambiguously feasible if and only if $T = l$.

Consider $T = l$. It is well known that any regular solution to equation (11) can be represented as follows:

$$u(x,t) = f(x-t) + g(x+t), \tag{16}$$

where $f(x)$ and $g(x)$ - g(x) — are arbitrary twice-continuously differentiable functions. Further, for conveniences for the functions $f(x)$ and $g(x)$ from formula (16), introduce the following notation:

$$f(x) = f_0(x),\ g(x) = g_0(x), x \in [0,l],$$

$$f(x) = f_1(x), x \in [-l,0],$$

$$g(x) = g_1(x), x \in [l,2l].$$

Satisfying conditions (12) for (16) and using the notation, obtain

$$f_0(x) = \frac{\varphi_0(x)}{2} - \frac{1}{2}\int_0^x \psi_0(\xi)d\xi + C_1, \tag{17}$$

$$g_0(x) = \frac{\varphi_0(x)}{2} + \frac{1}{2}\int_0^x \psi_0(\xi)d\xi - C_1. \tag{18}$$

Now satisfying conditions (13) for (16), as $T = l$, we obtain

$$f(x-l) + g(x+l) = \varphi_1(x),$$

$$-f'(x-l) + g'(x+l) = \psi_1(x).$$

Hence, it is obvious that

$$f'(x-l) = \frac{\varphi_1'(x)}{2} - \frac{\psi(x)}{2}.$$

Using the last identity integration from l to $l-x$, get

$$f(-x) = \frac{\varphi_1'(l-x)}{2} - \frac{\varphi_1'(l)}{2} - \frac{1}{2}\int_l^{l-x} \psi_1(\xi)d\xi + f(0).$$

Taking into account that $f_0(0) = \frac{\varphi_0(0)}{2} + C_1$ by (17) and using our notation, obtain

$$f_1(-x) = \frac{\varphi_1(l-x)}{2} + \frac{1}{2}\int_{l-x}^{l} \psi_1(\xi)d\xi + \frac{\varphi_0(0)}{2} + \frac{\varphi_1(l)}{2} + C_1,\ x\in[0,l], \tag{19}$$

and by (13) get

$$g_1(l+x) = \frac{\varphi_1(x)}{2} - \frac{1}{2}\int_{x}^{l} \psi_1(\xi)d\xi - \frac{\varphi_0(0)}{2} + \frac{\varphi_1(l)}{2} - C_1,\ x\in[0,l]. \tag{20}$$

Now let us find the control actions $\mu(t)$ and $\nu(t)$. Generally speaking, the solution to the Cauchy problem in the domain Ω as $T=l$ could be found and the resulting expression could be substituted into (14) and (15) respectively. We find another way that does not require solving the Cauchy problem.

Substituting (16) sequentially (13) and (14), we obtain

$$\mu(t) = V_0(t) + V_{k_1}(t), \tag{21}$$

$$\nu(t) = V_l(t) + V_{k_2}(t), \tag{22}$$

$$V_{k_i}(t) = \int_0^l k_i(\xi,t)f(\xi-t)d\xi + \int_0^l k_i(\xi,t)g(\xi+t)d\xi, \tag{23}$$

$$V_0(t) = f_1(-t) + g_0(t),\ V_l(t) = f_0(l-t) + g_1(l+t),\ t\in[0,l].$$

Replacing $t-\xi=z$ in the first integral of (23), and $t+\xi=l+z$ in the second, we obtain

$$V_{k_i}(t) = \int_{t-l}^{t} k_i(t-z,t)f(z)dz + \int_{t-l}^{t} k_i(z+l-t,t)g(l+z)dz =$$

$$= \int_0^t k_i(t-z,t)f(z)dz + \int_0^t k_i(z+l-t,t)g(l+z)dz +$$

$$+ \int_{t-l}^{0} k_i(t-z,t)f(z)dz + \int_{t-l}^{0} k_i(z+l-t,t)g(l+z)dz.$$

Substituting $-z=z$ and $l+z=z$ in the last two integrals, respectively, and using our notation for the functions $f(x)$ and $g(x)$, finally obtain the expression for $V_{k_i}(t)$.

$$V_{k_i}(t) = \int_0^t k_i(t-z,t)f_1(-z)dz + \int_0^t k_i(z+l-t,t)g_1(l+z)dz +$$

$$+ \int_0^{l-t} k_i(t+z,t)f_0(z)dz + \int_t^l k_i(z-t,t)g_0(z)dz,\ i=1,2.$$

Replace f_0, g_0, f_1, g_1 with expressions from (17), (18) and (19), (20), substitute then the expressions into (21) and (22) for $V_{k_1}(t)$ and $V_{k_2}(t)$, respectively, and making some transformations, obtain (9), (10).

Since the functions $u(x,t)$ belong to the class $C^2(\Omega)\cap C^1(\bar{\Omega})$, the continuity conditions

$$f_0(0) = f_1(0),\ f_0'(0) = f_1'(0),\ f_0''(0) = f_1''(0), \tag{24}$$

$$g_0(l) = g_1(l),\ g_0'(l) = g_1'(l),\ g_0''(l) = g_1''(l) \tag{25}$$

and consistency conditions

$$\mu(0) = \varphi_0(0),\, \mu(l) = \varphi_1(0),\, \mu'(0) = \psi_0(0),\, \mu'(l) = \psi_1(0), \tag{26}$$

$$\nu(0) = \varphi_0(l),\, \nu(l) = \varphi_1(l),\, \nu'(0) = \psi_0(l),\, \nu'(l) = \psi_1(l). \tag{27}$$

Conditions (24), (25) are necessary and sufficient for the existence of a solution to the mixed problem (12), (13) for equation (11) in the domain Ω as $T = l$, in the form of

$$u(x,t) = f_i(x-t) + g_j(x+t),\ -il < x-t < (1-i)l,\ jl < x+t < (1+j)l,\ i,j = 0,1.$$

Conditions (24)—(27) are necessary and sufficient for the functions $\varphi_0(x)$, $\varphi_1(x) \in C^2(0,l) \cap C^1[0,l]$, $\psi_0(x)$, $\psi_1(x) \in C^1[0,l]$, there were internal boundary controls $\mu(t)$, $\nu(t) \in C^2(0,l) \cap C^1[0,l]$ satisfying conditions (13) solutions $u(x,t) \in C^2(\Omega) \cap C^1(\bar{\Omega})$ to the mixed problem (12) for equation (11).

The proof of the uniqueness of the solution to the mixed problem (12), (13) and nonlocal mixed problems (12), (14), (15) and (13), (14), (15) for equation (11) is carried out the same way as in [4]. Substituting (17) and (18) into (24) and (25), respectively, and (14) and (15), into (26) and (27), we obtain (1)—(8).

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А.Х. Аттаев

Ішек тербелісінің теңдеуі үшін ішкі-шеттік басқарудың бір есебі жайында

Мақалада интегралды типті ішкі-шеттік шарттары бар бір өлшемді ішектің тербеліс теңдеуімен сипатталатын процесті басқару есебі қарастырылған. Әдеттегідей, уақыттың бастапқы және соңғы сәттерінде еркін ығысулар мен ішек нүктелерінің жылдамдығы (Коши мәліметтері) беріледі. Ізделінді басқарудың жалғыз мүмкін болатын ең аз уақыты шекаралық үзілісті көпбейнеліктерімен Коши есебінің корректілі шешілу жағдайынан нақты анықталатындығы көрсетілген. Бұл ішкі-шеттік шарттардың түрі минималды уақыт мәніне әсер етпейтінін көрсетеді. Тербелмелі жүйені бастапқы берілген күйден алдын-ала берілген соңғы күйге аударатын ішкі-шеттік басқармалардың қажетті және жеткілікті жағдайлары алынып жазылды. Басқармалардың өздері нақты аналитикалық түрде жазылған. Сонымен қатар, ішкі-шеттік басқару үшін өрнектерді алуда ізделінді облыстағы Коши есебін шешімімен емес, ішек тербеліс теңдеуінің жалпы шешімінің формуласын (Даламбер формуласы) қолдану керек екендігі көрсетілген.

Кілт сөздер: ішектің тербеліс теңдеуі, шекаралық басқару есебі, Коши есебі, магистральдық және тармақталған құбыр желілері, бейлокалды аралас есеп.

А.Х. Аттаев

Об одной задаче внутренне-краевого управления для уравнения колебания струны

В статье рассмотрена задача управления процессом, который описывается уравнением колебания одномерной струны с внутренне-краевыми условиями интегрального типа. Как обычно, в начальный и финальный моменты времени задаются произвольные смещения и скорости точек струны (данные Коши). Показано, что минимальное время, в течение которого единственным образом осуществимо искомое управление однозначным образом, определяется из условия корректной разрешимости задачи Коши с данными на граничном разрывном многообразии. Это свидетельствует о том, что сам вид внутренне-краевых условий на значение минимального времени не влияет. Получены и выписаны необходимые и достаточные условия существования искомых внутренне-краевых управлений, переводящих колебательную систему из начально заданного состояния в наперед заданное финальное состояние. Сами управления выписаны в явном аналитическом виде. При этом показано, что для получения выражений для внутренне-краевых управлений нужно воспользоваться не самим представлением решения задачи Коши в искомой области, а формулой общего решения уравнения колебания струны (формулой Даламбера).

Ключевые слова: уравнение колебания струны, задача граничного управления, задача Коши, магистральные и разветвленные трубопроводные сети, нелокальная смешанная задача.

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