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Particular solutions of the multidimensional singular ultrahyperbolic equation generalizing the telegraph and Helmholtz equations

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This article deals with the construction of particular solutions for a second-order multidimensional singular partial differential equation, which generalizes the famous telegraph and Helmholtz equations. The constructed particular solutions are expressed in terms of the multiple confluent hypergeometric function, which is analogous to the multiple Lauricella function and the famous Bessel function. A limit correlation theorem for the multiple confluent hypergeometric function is proved, and a system of partial differential equations associated with the confluent function is derived. Thanks to the proven properties of the multiple confluent hypergeometric function. The particular solutions of the multidimensional partial differential equation with the singular coefficients are written in explicit forms and it is determined that these solutions have a singularity at the vertex of a multidimensional cone.

Keywords: particular solution, Lauricella function, multiple confluent hypergeometric function, a limit correlation theorem, a system of the partial differential equations.

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Introduction

It is well known that particular solutions play an essential role in the study of partial differential equations. The set of particular solutions includes fundamental solutions that satisfy certain additional conditions. In case of the singular elliptic equations, the role of particular solutions is played by fundamental solutions. Formulation and solving of many local and non-local boundary value problems are based on these solutions. The explicit form of particular solutions gives a possibility to study the considered equation in detail.

In the case of PDE with singular coefficients, particular solutions, including fundamental ones, are expressed through hypergeometric functions, the number of variables of which is directly related to the number of singular coefficients. For instance, in the paper [1], particular solutions of the generalized Euler-Poisson-Darboux equation with three singular coefficients

$$u_{xx} + u_{yy} + \frac{2\alpha}{x}u_x + \frac{2\beta}{y}u_y = u_{tt} + \frac{2\gamma}{t}u_t,\ x > 0,\ y > 0,\ t > 0,\ \ 0 < 2\alpha,\, 2\beta,\, 2\gamma < 1 \tag{1}$$

are written by a hypergeometric function $F_A^{(3)}$ in three variables introduced by Lauricella [2]. In addition, self-similar solutions of some model degenerate partial differential equations of the higher order are expressed by the higher order hypergeometric functions [3–6].

It is well known [7] that all linearly independent fundamental solutions at the origin of singular elliptic equation

$$\sum_{j=1}^{m} \frac{\partial^2 u}{\partial x_j^2} + \sum_{j=1}^{n} \frac{2\alpha_j}{x_j}\frac{\partial u}{\partial x_j} = 0, \quad m \geq 2, \quad n \leq m \tag{2}$$

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in the first hyperoctant $x_1 > 0, \ldots, x_n > 0$ are expressed explicitly by the Lauricella function $F_A^{(n)}$ in n variables. Various applications of the fundamental solutions of equation (2) to the solution of boundary value problems for this equation can be found in the works [8–11].

In a recent work [12], particular solutions of the equation

$$\sum_{j=1}^{p} \frac{\partial^2 u}{\partial x_j^2} + \sum_{j=1}^{p} \frac{2\alpha_j}{x_j} \frac{\partial u}{\partial x_j} = \sum_{j=p+1}^{n} \frac{\partial^2 u}{\partial x_j^2} + \sum_{j=p+1}^{n} \frac{2\alpha_j}{x_j} \frac{\partial u}{\partial x_j}, \ \ p = \overline{1, n} \tag{3}$$

are also expressed through the Lauricella function $F_A^{(n)}$, the variables of which differ from the variables of the Lauricella function included in the fundamental solutions of the equation (2) only by signs depending on the equation under consideration.

All fundamental solutions of the multidimensional Helmholtz equation with n singular coefficients

$$\sum_{j=1}^{m} \frac{\partial^2 u}{\partial x_j^2} + \sum_{j=1}^{n} \frac{2\alpha_j}{x_j} \frac{\partial u}{\partial x_j} + \lambda u = 0, \ m \geq 2, \ n \leq m, \ -\infty < \lambda < +\infty \tag{4}$$

are presented by the confluent hypergeometric function in $n+1$ variables, the first n variables of which coincide with the variables of the fundamental solutions of equation (2). In this case, the last variable in the confluent hypergeometric function appears due to the presence of the parameter λ (for details, see [13]).

The following so-called multidimensional singular ultrahyperbolic equation

$$\sum_{j=1}^{p} \frac{\partial^2 u}{\partial x_j^2} + \sum_{j=1}^{p} \frac{2\alpha_j}{x_j} \frac{\partial u}{\partial x_j} = \sum_{j=p+1}^{m} \frac{\partial^2 u}{\partial x_j^2} + \sum_{j=p+1}^{n} \frac{2\alpha_j}{x_j} \frac{\partial u}{\partial x_j} + \lambda u, \ \ p \leq n \leq m \tag{5}$$

contains all four equations (1)–(4) considered above. Note that equation (5) generalizes also the well-known Helmholtz $u_{xx} + u_{yy} + cu = 0$ and telegraph $u_{xx} - u_{yy} + cu = 0$ equations.

In this paper we construct particular solutions of equation (5) in some multidimensional cone when $0 < p < n = m$ and prove that these solutions are simultaneously fundamental solutions of the considered equation near the origin. Note, if $p = 0$ or $p = n$, then the equation (5) becomes an equation of the singular elliptic type (4), particular (fundamental) solutions of which are found in [13].

The plan of this paper is as follows. In Section 1, we briefly give some preliminary information, which will be used later, and investigate new properties of the multiple confluent hypergeometric function $\mathrm{H}_A^{(n,1)}$. In Section 2 we compose a system corresponding to the function $\mathrm{H}_A^{(n,1)}$ and find all particular solutions of this system. In Section 3 we study an ultrahyperbolic equation with singular coefficients, all particular solutions of which are written out explicitly through a multiple confluent hypergeometric function $\mathrm{H}_A^{(n,1)}$. In Section 4, the properties of the constructed particular solutions are studied and the order of singularity of these solutions in the neighborhood of the origin is determined.

1 Hypergeometric functions of several variables

The great success of the theory of hypergeometric functions in one variable has stimulated the development of a corresponding theory in two and more variables. Horn [14] gave the general definition of the hypergeometric functions of two variables. He has investigated the convergence of hypergeometric functions in two variables and established the systems of partial differential equations which they satisfy (for details, see [15; Section 5.7]).

Following Horn we define a hypergeometric function of several variables.

Let a multiple power series be given

$$\sum_{|\mathbf{k}|=0}^{\infty} A\left(\mathbf{k}\right) \prod_{j=1}^{n} x_j^{k_j}, \tag{6}$$

where the summation is carried out over a multi-index $\mathbf{k} := (k_1, \ldots, k_n)$ with non-negative integer components $k_j \geq 0$, $j = 1, \ldots, n$, for which, as usual, $|\mathbf{k}| := k_1 + \ldots + k_n$.

A multiple power series (6) is a hypergeometric series if the following n relations

$$\frac{A\left(\mathbf{k}_j + \mathbf{e}_j\right)}{A\left(\mathbf{k}\right)} = f_j\left(\mathbf{k}\right) \tag{7}$$

are rational functions of $\mathbf{k}$, where $\mathbf{e}_j := (0, \ldots, 0, 1, 0, \ldots, 0)$ denotes a vector whose $j-$th component is equal to one, and the rest are equal to zero $(j = 1, \ldots, n)$.

Let's suppose

$$f_j\left(\mathbf{k}\right) = \frac{P_j\left(\mathbf{k}\right)}{Q_j\left(\mathbf{k}\right)}, \tag{8}$$

where P_j and Q_j are polynomials of $\mathbf{k}$ having degrees p_j and q_j respectively. It is assumed that Q_j has a multiplier of $k_j + 1$; P_j and Q_j have no common multipliers, with the possible exception of $k_j + 1$ $(j = 1, \ldots, n)$.

The largest of the numbers $p_1, \ldots, p_n, q_1, \ldots, q_n$ is called *order* of the hypergeometric series (6).

The hypergeometric series (6) is called *complete*, if all the numbers $p_1, \ldots, p_n, q_1, \ldots, q_n$ are the same, i.e. $p_1 = \ldots = p_n = q_1 = \ldots = q_n$, otherwise *confluent*.

A symbol $(\kappa)_\nu$ denotes the general Pochhammer symbol or the shifted factorial, since $(1)_l = l!$ $\left(l \in \mathbb{N} \bigcup \{0\}; \mathbb{N} := \{1, 2, \ldots\}\right)$, which is defined (for $\kappa, \nu \in \mathbb{C}$), in terms of the familiar Gamma function, by

$$(\kappa)_\nu := \frac{\Gamma\left(\kappa + \nu\right)}{\Gamma\left(\kappa\right)} = \begin{cases} 1 & \left(\nu = 0;\ \kappa \in \mathbb{C} \backslash \{0\}\right), \\ \kappa\left(\kappa + 1\right) \ldots \left(\kappa + l - 1\right) & \left(\nu = l \in \mathbb{N};\ \kappa \in \mathbb{C}\right), \end{cases}$$

it is being understood conventionally that $(0)_0 := 1$ assumed tacitly that the $\Gamma-$quotient exists.

A Lauricella function $F_A^{(n)}$ in $n \in N$ real variables $\mathbf{x} := (x_1, \ldots, x_n)$ [2] (see, also [16])

$$F_A^{(n)}\left[\begin{matrix} a, \mathbf{b}; \\ \mathbf{c}; \end{matrix}\ \mathbf{x}\right] = \sum_{|\mathbf{k}|=0}^{\infty} (a)_{|\mathbf{k}|} \prod_{j=1}^{n} \frac{(b_j)_{k_j}}{(c_j)_{k_j}} \frac{x_j^{k_j}}{k_j!}, \ \sum_{j=1}^{n} |x_j| < 1 \tag{9}$$

is also a complete hypergeometric function of the order 2. Hereinafter $\mathbf{b} := (b_1, \ldots, b_n)$, $\mathbf{c} := (c_1, \ldots, c_n)$. In definition (9), as usual, the denominator parameters $c_1, \ldots, c_n$ are neither zero nor a negative integer.

Let a, b_k, c_k be real numbers, where $c_k \neq 0, -1, -2, \ldots$ and $a > |\mathbf{b}| > 0$ and $c_k > b_k$. Then for $n = 1, 2, \ldots$, the following limit correlation is true [17]

$$\lim_{\varepsilon \to 0}\left\{\varepsilon^{-|\mathbf{b}|} F_A^{(n)}\left[\begin{matrix} a, \mathbf{b}; \\ \mathbf{c}; \end{matrix}\ 1 - \frac{z_1(\varepsilon)}{\varepsilon}, \ldots, 1 - \frac{z_n(\varepsilon)}{\varepsilon}\right]\right\} = \frac{\Gamma\left(a - |\mathbf{b}|\right)}{\Gamma(a)} \prod_{k=1}^{n} \frac{|z_k(0)|^{-b_k}\, \Gamma\left(c_k\right)}{\Gamma\left(c_k - b_k\right)}, \tag{10}$$

where $|\mathbf{b}| := b_1 + \ldots + b_n$; $z_k(\varepsilon)$ are arbitrary functions, and $z_k(0) \neq 0$.

Note that the limit correlation formula (10) is applied in the theory of boundary value problems for the multidimensional singular elliptic equation (2), for instance, see [18].

Consider the following confluent hypergeometric function in $n + 1$ variables

$$\mathrm{H}_A^{(n,1)}\left[\begin{matrix} a, \mathbf{b}; \\ \mathbf{c}; \end{matrix}\ \mathbf{x}, y\right] = \sum_{|\mathbf{k}|+l=0}^{\infty} (a)_{|\mathbf{k}|-l} \prod_{j=1}^{n} \frac{(b_j)_{k_j}}{(c_j)_{k_j}} \frac{x_j^{k_j}}{k_j!} \cdot \frac{y^l}{l!}, \quad \sum_{j=1}^{n} |x_j| < 1, \tag{11}$$

where $\mathbf{x}$ and y are real variables, and $l = 0, 1, 2, \dots$

Note, this confluent hypergeometric function $\mathrm{H}_A^{(n,1)}$ was first introduced and studied in a more general form in [13] and its particular cases ($n = 1, 2, 3$) were known in [15,19,20].

The confluent hypergeometric function $\mathrm{H}_A^{(n,1)}$ has the following formula of derivation:

$$\frac{\partial^{|\mathbf{k}|+l}}{\partial x_1^{k_1}\dots\partial x_n^{k_n}\partial y^l}\mathrm{H}_A^{(n,1)}\left[\begin{array}{r}a,\mathbf{b};\\ \mathbf{c};\end{array}\mathbf{x},y\right] = (a)_{|\mathbf{k}|-l}\prod_{j=1}^{n}\frac{(b_j)_{k_j}}{(c_j)_{k_j}}\cdot\mathrm{H}_A^{(n,1)}\left[\begin{array}{r}a+|\mathbf{k}|-l,\mathbf{b}+\mathbf{k};\\ \mathbf{c}+\mathbf{k};\end{array}\mathbf{x},y\right], \tag{12}$$

hereinafter, $\mathbf{k} := (k_1, \dots, k_n)$ is an n-vector.

Using simple properties of the Pochhammer symbol

$$(a)_m(a+m)_k = (a)_{m+k},\ \ (a)_k = \frac{(-1)^k}{(1-a)_k},$$

we can represent the confluent hypergeometric function $\mathrm{H}_A^{(n,1)}$ as

$$\mathrm{H}_A^{(n,1)}\left[\begin{array}{r}a,\mathbf{b};\\ \mathbf{c};\end{array}\mathbf{x},y\right] = \sum_{k=1}^{\infty}\frac{(-1)^k}{(1-a)_k}\frac{y^k}{k!}F_A^{(n)}\left[\begin{array}{r}a-k,\mathbf{b};\\ \mathbf{c};\end{array}\mathbf{x}\right], \tag{13}$$

where $F_A^{(n)}$ is the Lauricella function defined in (9).

It is obvious that

$$\mathrm{H}_A^{(n,1)}\left[\begin{array}{r}a,\mathbf{b};\\ \mathbf{c};\end{array}\mathbf{x},0\right] = F_A^{(n)}\left[\begin{array}{r}a,\mathbf{b};\\ \mathbf{c};\end{array}\mathbf{x}\right]. \tag{14}$$

Theorem 1. Let a, b_k, c_k be real numbers, where $c_k \neq 0, -1, -2, \dots$ and $a > |\mathbf{b}| > 0$ and $c_k > b_k$. Then for $n = 1, 2, \dots$, the following limit correlation is true

$$\lim_{\varepsilon\to 0}\left\{\varepsilon^{-|\mathbf{b}|}\mathrm{H}_A^{(n,1)}\left[\begin{array}{r}a,\mathbf{b};\\ \mathbf{c};\end{array}1-\frac{z_1(\varepsilon)}{\varepsilon},\dots,1-\frac{z_n(\varepsilon)}{\varepsilon},\varepsilon y\right]\right\} = \frac{\Gamma\left(a-|\mathbf{b}|\right)}{\Gamma(a)}\prod_{k=1}^{n}\frac{|z_k(0)|^{-b_k}\,\Gamma\left(c_k\right)}{\Gamma\left(c_k-b_k\right)}, \tag{15}$$

where $|\mathbf{b}| := b_1 + \dots + b_n$; $z_k(\varepsilon)$ are arbitrary functions, and $z_k(0) \neq 0$; y is a real variable.

Proof. The proof of Theorem 1 follows from expansion (13), obvious equality (14) and limit correlation formula (10).

2 *System of differential equations satisfied by the confluent function* $\mathrm{H}_A^{(n,1)}$

We represent the confluent hypergeometric function $\mathrm{H}_A^{(n,1)}$, defined by the equality (11), in the form

$$\mathrm{H}_A^{(n,1)}\left[\begin{array}{r}a,\mathbf{b};\\ \mathbf{c};\end{array}\mathbf{x};y\right] = \sum_{|\mathbf{k}|+l=0}^{\infty}A\left(\mathbf{k};l\right)\prod_{j=1}^{n}x_j^{k_j}\cdot y^l, \tag{16}$$

where

$$A\left(\mathbf{k};l\right) = \frac{(a)_{|\mathbf{k}|-l}(b_1)_{k_1}\dots(b_n)_{k_n}}{k_1!\dots k_n!l!(c_1)_{k_1}\dots(c_n)_{k_n}}.$$

By virtue of (7) and (8), we have

$$f_j\left(\mathbf{k};l\right) = \frac{P_j\left(\mathbf{k};l\right)}{Q_j\left(\mathbf{k};l\right)},\ j=\overline{1,\,n};\ \ g\left(\mathbf{k};l\right) = \frac{1}{G\left(\mathbf{k};l\right)},$$

here

$$P_j(\mathbf{k};l)=(a+|\mathbf{k}|-l)(b_j+k_j),\ j=\overline{1,\,n};$$

$$Q_j(\mathbf{k};l)=(1+k_j)(c_j+k_j),\ j=\overline{1,\,n};$$

$$G=(1+l)(a-1+|\mathbf{k}|-l).$$

Series (16) satisfies a system of linear partial differential equations. Using differential operators

$$\delta_j\equiv x_j\frac{\partial}{\partial x_j},\ \ j=\overline{1,n};\ \ \delta'\equiv y\frac{\partial}{\partial y} \tag{17}$$

this system can be written in the form

$$\begin{cases}\left[Q_j(\delta_1,\ldots,\delta_n;\delta')\,x_j^{-1}-P_j(\delta_1,\ldots,\delta_n;\delta')\right]\omega=0,\ j=\overline{1,\,n},\\ \left[G(\delta_1,\ldots,\delta_n;\delta')\,y^{-1}-1\right]\omega=0.\end{cases} \tag{18}$$

Now, substituting differential operators (17) into (18), we get

$$\begin{cases}x_i(1-x_i)\,\omega_{x_ix_i}-x_i\sum\limits_{j=1,\,j\neq i}^{n}x_j\omega_{x_ix_j}+x_iy\omega_{x_iy}+[c_i-(a+1)\,x_i]\,\omega_{x_i}\\ \qquad -\,b_i\sum\limits_{j=1,\,j\neq i}^{n}x_j\omega_{x_j}+b_iy\omega_y-ab_i\omega=0,\ \ i=\overline{1,n},\\ y\omega_{yy}-\sum\limits_{j=1}^{n}x_j\omega_{x_jy}+(1-a)\,\omega_y+\omega=0,\end{cases} \tag{19}$$

where $\omega(\mathbf{x};y)=\mathrm{H}_A^{(n,1)}\left[\begin{array}{c}a,\mathbf{b};\\ \mathbf{c};\end{array}\mathbf{x},y\right]$.

Theorem 2. [13] System of differential equations (19) near the origin has 2^n linearly independent solutions:

$1:\{\,\mathrm{H}_A^{(n,1)}\left[\begin{array}{l}a,b_1,\ldots,b_n;\\ c_1,\ldots,c_n;\end{array}\mathbf{x};y\right],$

$$C_n^1:\begin{cases}x_1^{1-c_1}\mathrm{H}_A^{(n,1)}\left[\begin{array}{l}a+1-c_1,b_1+1-c_1,b_2,\ldots,b_n;\\ 2-c_1,c_2,\ldots,c_n;\end{array}\mathbf{x};y\right],\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ x_n^{1-c_n}\mathrm{H}_A^{(n,1)}\left[\begin{array}{l}a+1-c_n,b_1,\ldots,b_{n-1},b_n+1-c_n;\\ c_1,\ldots,c_{n-1},2-c_n;\end{array}\mathbf{x};y\right],\end{cases}$$

$$C_n^2:\begin{cases}x_1^{1-c_1}x_2^{1-c_2}\mathrm{H}_A^{(n,1)}\left[\begin{array}{l}a+2-c_1-c_2,b_1+1-c_1,b_2+1-c_2,b_3,\ldots,b_n;\\ 2-c_1,2-c_2,c_3,\ldots,c_n;\end{array}\mathbf{x};y\right],\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ x_1^{1-c_1}x_n^{1-c_n}\mathrm{H}_A^{(n,1)}\left[\begin{array}{l}a+2-c_1-c_n,b_1+1-c_1,b_2,...,b_{n-1},b_n+1-c_n;\\ 2-c_1,c_2,\ldots,c_{n-1},2-c_n;\end{array}\mathbf{x};y\right],\\ x_2^{1-c_2}x_3^{1-c_3}\mathrm{H}_A^{(n,1)}\left[\begin{array}{l}a+2-c_2-c_3,b_1,b_2+1-c_2,b_3+1-c_3,b_4,\ldots,b_n;\\ c_1,2-c_2,2-c_3,c_4,\ldots,c_n;\end{array}\mathbf{x};y\right],\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ x_{n-1}^{1-c_{n-1}}x_n^{1-c_n}\mathrm{H}_A^{(n,1)}\left[\begin{array}{l}a+2-c_{n-1}-c_n,b_1,\ldots,b_{n-2},b_{n-1}+1-c_{n-1},b_n+1-c_n;\\ c_1,\ldots,c_{n-2},2-c_{n-1},2-c_n;\end{array}\mathbf{x};y\right],\end{cases}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$1:\left\{x_1^{1-c_1}\dots x_n^{1-c_n}\mathrm{H}_A^{(n,1)}\left[\begin{array}{l}a+n-c_1-\dots-c_n,b_1+1-c_1,\dots,b_n+1-c_n;\\2-c_1,\dots,2-c_n;\end{array}\mathbf{x};y\right]\right.,$$

where $C_n^k=\dfrac{n!}{k!(n-k)!}$ are binomial coefficients.

When none of the numbers $c_1,\ c_2,\ \dots,\ c_n$ is equal to a negative integer, we obtain the general solution of system (19) by multiplying these 2^n partial solutions by arbitrary constants and then taking their sum.

It is easy to see that in the first group there is one solution $\left(C_n^0=1\right)$, in the second group there are $C_n^1=n$ solutions, the third group consists of $C_n^2=n(n-1)/2$ solutions, etc. So the system of hypergeometric equations (19) really has 2^n solutions.

However, within each group, the functions included in this group are symmetrical with respect to the numerical parameters. Therefore, for further purposes, it is enough to select one solution from each group, or more precisely, the solution that comes first in each group. So $n+1$ linearly independent solutions to the system of equations (19) will be identified by the formulas

$$\omega_0(\mathbf{x};y)=C_0\,\mathrm{H}_A^{(n,1)}\left[\begin{array}{r}a,b_1,\dots,b_n;\\c_1,\dots,c_n;\end{array}\mathbf{x};y\right],\tag{20}$$

$$\omega_i(\mathbf{x};y)=C_i\prod_{j=1}^{i}x_j^{1-c_j}\cdot\mathrm{H}_A^{(n,1)}\left[\begin{array}{r}a+i-|\mathbf{c}_i|,b_1+1-c_1,\dots,b_i+1-c_i,b_{i+1},\dots,b_n;\\2-c_1,\ \dots,\ 2-c_i,\ c_{i+1},\dots,c_n;\end{array}\mathbf{x};y\right],\tag{21}$$

where $C_0,\dots,C_n$ are arbitrary constants; $|\mathbf{c}_i|:=c_1+\dots+c_i,\ i=\overline{1,n}$.

Using the derivation formula (12), it is easy to verify that the functions defined in (20) and (21) really satisfy to the system of partial differential equations (19).

3 Particular solutions

Consider the multidimensional ultrahyperbolic equation

$$L(u)\equiv\sum_{j=1}^{n}\mathrm{sgn}(p-j)\left(\frac{\partial^2u}{\partial x_j^2}+\frac{2\alpha_j}{x_j}\frac{\partial u}{\partial x_j}\right)+\lambda u=0,\ \ p=\overline{1,n-1},\ \ n\geq 2\tag{22}$$

in the n-dimensional cone

$$\Omega=\left\{(x_1,\dots,x_n):x_1^2+\dots+x_p^2>x_{p+1}^2+\dots+x_n^2,\ p=\overline{1,n-1};\ x_j>0,\ j=\overline{1,n}\right\},$$

where α_j are constants ($0<2\alpha_j<1$, $j=\overline{1,n}$); λ is a real number;

$$\mathrm{sgn}(z):=\left\{\begin{array}{r}1,\ \text{if}\ z\geq 0,\\-1,\ \text{if}\ z<0.\end{array}\right.$$

Let $x:=(x_1,\dots,x_n)$ be any point and $\xi:=(\xi_1,\dots,\xi_n)$ be any fixed point of Ω. We search for a solution of equation (22) as follows:

$$u\left(x;\xi\right)=P(r)\omega\left(\sigma_p,\eta_p\right),\quad p=\overline{1,n-1},\tag{23}$$

where

$$P(r)=r_p^{-2\beta},\quad\beta=\frac{n-2}{2}+\sum_{j=1}^{n}\alpha_j;\tag{24}$$

ω is an unknown function, depending on $n+1$ variables

$$\sigma_p := (\sigma_{p1}, \sigma_{p2}, \ldots, \sigma_{pn}), \;\; \sigma_{pj} = -\mathrm{sgn}(p-j)\frac{4x_j\xi_j}{r_p^2}, \; j=\overline{1,n}, \tag{25}$$

$$\eta_p = \frac{1}{4}\lambda r_p^2, \;\; r_p^2 = \sum_{k=1}^{n}\mathrm{sgn}(p-k)(x_k-\xi_k)^2, \; p=\overline{1,n-1}.$$

First, we calculate the derivatives of $u(x;\xi)$ with respect to the variables $x_1,\ldots,x_n$:

$$\frac{\partial u}{\partial x_j} = \frac{\partial P}{\partial x_j}\omega + P\left(\sum_{k=1}^{n}\frac{\partial\omega}{\partial\sigma_k}\frac{\partial\sigma_k}{\partial x_i} + \frac{\partial\omega}{\partial\eta}\frac{\partial\eta}{\partial x_j}\right),$$

$$\frac{\partial^2 u}{\partial x_j^2} = P\sum_{k=1}^{n}\frac{\partial^2\omega}{\partial x_k^2}\left(\frac{\partial\sigma_k}{\partial x_j}\right)^2 + 2P\sum_{k=1}^{n}\left(\sum_{l=k+1}^{n}\frac{\partial^2\omega}{\partial\sigma_k\partial\sigma_l}\frac{\partial\sigma_l}{\partial x_j} + \frac{\partial^2\omega}{\partial\sigma_k\partial\eta}\frac{\partial\eta}{\partial x_j}\right)\frac{\partial\sigma_k}{\partial x_j}$$

$$+\sum_{k=1}^{n}\left[\left(2\frac{\partial P}{\partial x_j}\frac{\partial\sigma_k}{\partial x_j} + P\frac{\partial^2\sigma_k}{\partial x_j^2}\right)\frac{\partial\omega}{\partial\sigma_k} + \left(2\frac{\partial P}{\partial x_j}\frac{\partial\eta}{\partial x_j} + P\frac{\partial^2\eta}{\partial x_j^2}\right)\frac{\partial\omega}{\partial\eta}\right] + \frac{\partial^2 P}{\partial x_j^2}\omega.$$

Now substituting product (23) into equation (22), we obtain

$$\begin{aligned}\sum_{k=1}^{n}A_k\frac{\partial^2\omega}{\partial\sigma_k^2} + A_{n+1}\frac{\partial^2\omega}{\partial\eta^2} + \sum_{k=1}^{n-1}\sum_{l=k+1}^{n}B_{k,l}\frac{\partial^2\omega}{\partial\sigma_k\partial\sigma_l} + \\ + \sum_{k=1}^{n}B_{k,n+1}\frac{\partial^2\omega}{\partial\sigma_k\partial\eta} + \sum_{k=1}^{n}D_k\frac{\partial\omega}{\partial\sigma_k} + D_{n+1}\frac{\partial\omega}{\partial\eta} + E\omega = 0,\end{aligned} \tag{26}$$

where

$$A_k = P\sum_{j=1}^{n}\mathrm{sgn}(p-j)\left(\frac{\partial\sigma_k}{\partial x_j}\right)^2, \; A_{n+1} = P\sum_{j=1}^{n}\mathrm{sgn}(p-j)\left(\frac{\partial\eta}{\partial x_j}\right)^2,$$

$$B_{k,l} = 2P\sum_{j=1}^{n}\mathrm{sgn}(p-j)\frac{\partial\sigma_k}{\partial x_j}\frac{\partial\sigma_l}{\partial x_j}, \;\; B_{k,n+1} = 2P\sum_{j=1}^{n}\mathrm{sgn}(p-j)\frac{\partial\sigma_k}{\partial x_j}\frac{\partial\eta}{\partial x_j},$$

$$D_k = \sum_{j=1}^{n}\mathrm{sgn}(p-j)\left(P\frac{\partial^2\sigma_k}{\partial x_j^2} + 2\frac{\partial P}{\partial x_j}\frac{\partial\sigma_k}{\partial x_i} + 2P\frac{\alpha_j}{x_j}\frac{\partial\sigma_k}{\partial x_j}\right),$$

$$D_{n+1} = \sum_{j=1}^{n}\mathrm{sgn}(p-j)\left(P\frac{\partial^2\eta}{\partial x_j^2} + 2\frac{\partial P}{\partial x_j}\frac{\partial\eta}{\partial x_j} + 2P\frac{\alpha_j}{x_j}\frac{\partial\eta}{\partial x_j}\right),$$

$$E = \sum_{j=1}^{n}\mathrm{sgn}(p-j)\left(\frac{\partial^2 P}{\partial x_j^2} + \frac{2\alpha_j}{x_j}\frac{\partial P}{\partial x_j}\right) + \lambda P.$$

Let us calculate the derivatives appearing in these coefficients:

$$\frac{\partial\sigma_k}{\partial x_k} = -\mathrm{sgn}(p-k)\left(\frac{4\xi_k}{r^2} + \frac{2(x_k-\xi_k)}{r^2}\sigma_k\right), \;\; k=\overline{1,n}; \tag{27}$$

$$\frac{\partial\sigma_k}{\partial x_j} = -\mathrm{sgn}(p-j)\frac{2(x_j-\xi_j)}{r^2}\sigma_k, \;\; j\neq k, \;\; j,k=\overline{1,n}; \tag{28}$$

$$\frac{\partial^2\sigma_k}{\partial x_k^2}=\operatorname{sgn}(p-k)\left(\frac{4\xi_k}{x_k r^2}\sigma_k-\frac{6}{r^2}\sigma_k\right)+\frac{8\left(x_k-\xi_k\right)^2}{r^4}\sigma_k,\quad k=\overline{1,n}; \tag{29}$$

$$\frac{\partial^2\sigma_k}{\partial x_j^2}=-\operatorname{sgn}(p-j)\frac{2}{r^2}\sigma_k+\frac{8\left(x_j-\xi_j\right)^2}{r^4}\sigma_k,\quad j\neq k,\quad j,\,k=\overline{1,n}; \tag{30}$$

$$\frac{\partial\eta}{\partial x_j}=\frac{\lambda}{2}\operatorname{sgn}(p-j)\left(x_j-\xi_j\right),\ \frac{\partial^2\eta}{\partial x_j^2}=\frac{\lambda}{2}\operatorname{sgn}(p-j),\quad j=\overline{1,n}; \tag{31}$$

$$\frac{\partial P}{\partial x_j}=-2\beta r^{-2\beta-2}\operatorname{sgn}(p-j)\left(x_j-\xi_j\right),\quad j=\overline{1,n}; \tag{32}$$

$$\frac{\partial^2 P}{\partial x_j^2}=4\beta r^{-2\beta-2}\left[(1+\beta)\frac{\left(x_j-\xi_j\right)^2}{r^2}-\frac{1}{2}\operatorname{sgn}(p-j)\right],\quad j=\overline{1,n}. \tag{33}$$

Taking into account (27)–(33), the coefficients of equation (26) take the form

$$A_k=-\frac{4P(r)}{r^2}\frac{\xi_k}{x_k}\sigma_k\left(1-\sigma_k\right),\ B_{k,n+1}=\frac{4P(r)}{r^2}\frac{\xi_k}{x_k}\sigma_k\eta+\frac{\lambda^2}{2}P(r)\sigma_k,\ k=\overline{1,n}; \tag{34}$$

$$B_{kl}=\frac{4P(r)}{r^2}\left(\frac{\xi_k}{x_k}+\frac{\xi_l}{x_l}\right)\sigma_k\sigma_l,\ k<l,\ \ k,l=\overline{1,n};\ A_{n+1}=-\lambda P(r)\eta, \tag{35}$$

$$D_k=-\frac{4P(r)}{r^2}\left\{\left(2\alpha_k-\beta\sigma_k\right)\frac{\xi_k}{x_k}-\sigma_k\sum_{i=1}^{n}\frac{\xi_i}{x_i}\alpha_i\right\},\ k=\overline{1,n}; \tag{36}$$

$$D_{n+1}=\frac{4P(r)}{r^2}\eta\sum_{i=1}^{n}\frac{\xi_i}{x_i}\alpha_i-\lambda P(r)\beta,\ E=\frac{4\beta P(r)}{r^2}\sum_{i=1}^{n}\frac{\xi_i}{x_i}\alpha_i+\lambda P(r). \tag{37}$$

Substituting coefficients (34)–(37) into equation (26) and grouping similar terms, we obtain

$$\begin{cases}\sigma_i\left(1-\sigma_i\right)\dfrac{\partial^2\omega}{\partial\sigma_i^2}-\sigma_i\displaystyle\sum_{j=1,j\neq i}^{n}\sigma_j\frac{\partial^2\omega}{\partial\sigma_i\partial\sigma_j}+\sigma_i\eta\frac{\partial^2\omega}{\partial\sigma_i\partial\eta}+\left[2\alpha_i-\left(\beta+\alpha_i+1\right)\sigma_i\right]\frac{\partial\omega}{\partial\sigma_i}\\ \qquad-\alpha_i\displaystyle\sum_{j=1,j\neq i}^{n}\sigma_j\frac{\partial\omega}{\partial\sigma_j}+\alpha_i\eta\frac{\partial\omega}{\partial\eta}-\beta\alpha_i\omega=0,\ i=\overline{1,n},\\ \eta\dfrac{\partial^2\omega}{\partial\eta^2}-\displaystyle\sum_{j=1}^{n}\sigma_j\frac{\partial^2\omega}{\partial\sigma_j\partial\eta}+(1-\beta)\frac{\partial\omega}{\partial\eta}+\omega=0.\end{cases} \tag{38}$$

Thus, the multidimensional ultrahyperbolic equation (22) equivalently reduced to system (38).

Comparing system (38) with system (19) and, by virtue of (23), (20) and (21), we obtain particular solutions of equation (22):

$$q_{p0}\left(x;\xi\right)=C_{p0}\,r_p^{-2\beta}\mathrm{H}_A^{(n,1)}\left[\begin{matrix}\beta,\alpha_1,\ldots,\alpha_n;\\2\alpha_1,\ldots,2\alpha_n;\end{matrix}\ \sigma_p,\eta_p\right], \tag{39}$$

$$q_{pj}\left(x;\xi\right)=C_{pj}\,r_p^{-2\beta-2j+4\alpha_1+\ldots+4\alpha_j}\prod_{k=1}^{j}\left(x_k\xi_k\right)^{1-2\alpha_k}\times$$

$$\times\,\mathrm{H}_A^{(n,1)}\left[\begin{matrix}\beta+j-2\alpha_1-\ldots-2\alpha_j,1-\alpha_1,\ldots,1-\alpha_j,\alpha_{j+1},\ldots,\alpha_n;\\2-2\alpha_1,\ \ldots,\ \ 2-2\alpha_j,\ \ 2\alpha_{j+1},\ldots,2\alpha_n;\end{matrix}\ \sigma_p,\eta_p\right], \tag{40}$$

where $C_{p0},\ldots,C_{pn}$ are arbitrary constants; β, σ_p and η_p are defined in (25); $j=\overline{1,n}$, $p=\overline{1,n-1}$.

4 Some properties of particular solutions

It can be shown directly that the particular solutions $q_{pi}\left(x;\xi\right)$ defined in (39) and (40) satisfy equation (22) with respect to the variables x, but these functions with respect to the same variables do not satisfy the adjoint equation

$$L^{*}(u)\equiv\sum_{j=1}^{n}\operatorname{sgn}(p-j)\left(\frac{\partial^{2}u}{\partial x_{j}^{2}}-\frac{\partial}{\partial x_{j}}\left(\frac{2\alpha_{j}u}{x_{j}}\right)\right)+\lambda u=0,\quad x\in\Omega. \tag{41}$$

Let's introduce some notations for brevity

$$x^{(2\alpha)}:=\prod_{i=1}^{n}x_{i}^{2\alpha_{i}},\ \ \tilde{x}_{j}^{(2\alpha)}:=\prod_{i=1,\,i\neq j}^{n}x_{i}^{2\alpha_{i}},\ \ j=\overline{1,n}.$$

Lemma 1. If $q_{pk}\left(x;\xi\right)$ are particular solutions to equation (22) with respect to the variables x, then the following functions

$$\tilde{q}_{pk}\left(x;\xi\right)=x^{(2\alpha)}q_{pk}\left(x;\xi\right) \tag{42}$$

are satisfied equation (22) with respect to the variables ξ and adjoint equation (41) with respect to the variables x, where $k=\overline{0,n},\ p=\overline{1,n-1}$.

Proof. From the definition of variables ξ_p and η_p (see eq. (25)) it follows that each particular solution $q_{pk}\left(x;\xi\right)$ defined in (39) and (40) is symmetric with respect to the variables x and ξ. Therefore, the arbitrary solution of equation (22) with respect to the variables x is simultaneously the solution of the same equation with respect to the variables ξ and vice versa.

Now, assuming that the function $q_{pk}\left(x;\xi\right)$ satisfies equation $L\left(q_{pk}\right)=0$, we substitute the function $\tilde{q}_{pk}\left(x;\xi\right)$ defined in (42) into the adjoint equation $L^{*}\left(\tilde{q}_{pk}\right)=0$. First, we calculate the necessary partial derivatives

$$\frac{\partial\tilde{q}_{pk}}{\partial x_{j}}=\frac{\partial}{\partial x_{j}}\left(x^{(2\alpha)}q_{pk}\right)=2\alpha_{j}\tilde{x}_{j}^{(2\alpha)}x_{j}^{2\alpha_{j}-1}q_{pk}+x^{(2\alpha)}\frac{\partial q_{pk}}{\partial x_{j}},$$

$$\frac{\partial^{2}\tilde{q}_{pk}}{\partial x_{j}^{2}}=2\alpha_{j}\left(2\alpha_{j}-1\right)\tilde{x}_{j}^{(2\alpha)}x_{j}^{2\alpha_{j}-2}q_{pk}+4\alpha_{j}\tilde{x}_{j}^{(2\alpha)}x_{j}^{2\alpha_{j}-1}\frac{\partial q_{pk}}{\partial x_{j}}+x^{(2\alpha)}\frac{\partial^{2}q_{pk}}{\partial x_{j}^{2}},$$

$$\frac{\partial}{\partial x_{j}}\left(\frac{2\alpha_{j}\tilde{q}_{pk}}{x_{j}}\right)=2\alpha_{j}\left(2\alpha_{j}-1\right)\tilde{x}_{j}^{(2\alpha)}x_{j}^{2\alpha_{j}-2}q_{pk}+2\alpha_{j}\tilde{x}_{j}^{(2\alpha)}x_{j}^{2\alpha_{j}-1}\frac{\partial q_{pk}}{\partial x_{j}}$$

and substitute them into adjoint equation (41):

$$L^{*}\left(\tilde{q}_{pk}\right)=x^{(2\alpha)}L\left(q_{pk}\right)=0.$$

The last double relation completes the proof of Lemma 1.

Therefore, the following functions

$$q_{p0}\left(x;\xi\right)=C_{p0}\,r_{p}^{-2\beta}\prod_{j=1}^{n}x_{j}^{2\alpha_{j}}\cdot\mathrm{H}_{A}^{(n,1)}\left[\begin{array}{l}\beta,\alpha_{1},\ldots,\alpha_{n};\\ \ 2\alpha_{1},\ldots,2\alpha_{n};\end{array}\xi_{p};\eta_{p}\right], \tag{43}$$

$$q_{pk}\left(x;\xi\right)=C_{pk}\,r_{p}^{-2\beta-2k+4\alpha_{1}+\ldots+4\alpha_{k}}\prod_{j=1}^{n}x_{j}^{2\alpha_{j}}\cdot\prod_{j=1}^{k}\left(x_{j}\xi_{j}\right)^{1-2\alpha_{j}}\times$$

$$\times\,\mathrm{H}_{A}^{(n,1)}\left[\begin{array}{c}\beta+k-2\alpha_{1}-\ldots-2\alpha_{k},1-\alpha_{1},\ldots,1-\alpha_{k},\alpha_{k+1},\ldots,\alpha_{n};\\ 2-2\alpha_{1},\ \ldots,\ \ 2-2\alpha_{k},\ \ 2\alpha_{k+1},\ldots,2\alpha_{n};\end{array}\xi_{p};\eta_{p}\right], \tag{44}$$

are also partial solutions to equation (22).

Theorem 3. If $0 < 2\alpha_j < 1,$ then the particular solutions $q_{pk}\left(x;\xi\right)$ defined in (43) and (44) have a singularity of the order $\dfrac{1}{r_p^{n-2}}$ at $r_p \to 0$, where $k = \overline{0,n}, \quad j = \overline{1,n}, \quad p = \overline{1,n-1}.$

Proof. We consider the first particular solution $q_{p0}\left(x;\xi\right)$, defined in (43), the singularity of the remaining solutions is proved in a similar way.

By virtue of an equality $2\beta = n - 2 + 2\alpha$, where $\alpha := \alpha_1 + \ldots + \alpha_n$ (see eq. (24)), we can rewrite the particular solution $q_{p0}\left(x;\xi\right)$ in the form

$$q_0\left(x;\xi\right) = \frac{1}{r_p^{n-2}}\tilde{q}_0\left(x,\xi\right),$$

where

$$\tilde{q}_{p0} = C_{p0}\frac{x^{(2\alpha)}}{r_p^{2\alpha}}\mathrm{H}_A^{(n,1)}\left[\begin{array}{l}\beta, \alpha_1, \ldots, \alpha_n; \\ 2\alpha_1, \ldots, 2\alpha_n;\end{array} -\frac{4x_1\xi_1}{r_p^2}, \ldots, -\frac{4x_p\xi_p}{r_p^2}, \frac{4x_{p+1}\xi_{p+1}}{r_p^2}, \ldots, \frac{4x_n\xi_n}{r_p^2}, \frac{1}{4}\lambda r_p^2\right]. \quad (45)$$

Now we show that $\tilde{q}_{p0}\left(x,\xi\right)$ is bounded at $r_p \to 0$. On the right side (45) we make a replacement $x_j - \xi_j = \varepsilon t_j \;(j = \overline{1,n})$, where $t := (t_1, \ldots, t_n)$ are new variables and $\varepsilon \geq 0$, then

$$\tilde{q}_{p0}\left(x;\xi - \varepsilon t\right) = C_{p0}\frac{x^{(2\alpha)}\varepsilon^{-2\alpha}}{T_p^{2\alpha}}\mathrm{H}_A^{(n,1)}\left[\begin{array}{l}\beta, \alpha_1, \ldots, \alpha_n; \\ 2\alpha_1, \ldots, 2\alpha_n;\end{array} 1 - \frac{z_1(\varepsilon)}{\varepsilon^2}, \ldots, 1 - \frac{z_n(\varepsilon)}{\varepsilon^2}, \frac{1}{4}\lambda\varepsilon^2 T_p^2\right],$$

where

$$z_j(\varepsilon) = \frac{T_p^2\varepsilon^2 + \mathrm{sgn}(p-j)\cdot 4x_j\left(x_j - \varepsilon t_j\right)}{T_p^2}, \quad T_p^2 = \sum_{j=1}^{p}\mathrm{sgn}(p-j)t_j^2, \quad j = \overline{1,n}.$$

Using limit correlation (15), we have

$$\lim_{\varepsilon\to 0}\tilde{q}_{p0}\left(x,\xi - \varepsilon t\right) = C_{p0}\frac{\Gamma(\beta-\alpha)}{4^{2\alpha}\Gamma(\beta)}\prod_{j=1}^{n}\frac{\Gamma\left(2\alpha_j\right)}{\Gamma\left(\alpha_j\right)} < \infty.$$

Thus the function $\tilde{q}_{p0}\left(x;\xi\right)$ is bounded, hence the function $q_{p0}\left(x;\xi\right)$ has the singularity of the order $\dfrac{1}{r_p^{n-2}}$ at $r_p \to 0$.

Conclusion

In conclusion, we note that particular solutions satisfying the singular elliptic and ultrahyperbolic equations (2) and (3) (respectively, equations (4) and (22)) are always expressed in terms of the Lauricella function $F_A^{(n)}$ (respectively, the confluent hypergeometric function $\mathrm{H}_A^{(n,1)}$), the variables of which differ from each other only in signs, depending on the equation under consideration.

During the study, it became clear that solutions to second-order equations are expressed in terms of second-order hypergeometric functions, i.e. the order of the equation under consideration is equal to the order of the hypergeometric function through which particular (fundamental) solutions are expressed. This circumstance must be taken into account when constructing partial solutions of singular equations when their order exceeds two. For example, knowing that in [3] all 8 self-similar solutions to the equation

$$Lu = x^n y^m u_t - t^k y^m u_{xxx} - t^k x^n u_{yyy} = 0, \;\; m, n, k = const > 0$$

in the domain $D_1 = \{(x, y, t) : \ x > 0, \ y > 0, \ \ t > 0\}$ are written by third-order hypergeometric Kampé de Fériet function in two variables, we can guess that particular solutions of the equation

$$\prod_{j=1}^{n} x_j^{m_j} \cdot \frac{\partial u}{\partial t} - t^l \sum_{k=1}^{n} \left(\prod_{j=1, j\neq k}^{n} x_j^{m_j} \right) \frac{\partial^p u}{\partial x_k^p} = 0, \ \ l > 0, \ \ m_j > 0, \ j = \overline{1, n}$$

in the domain $D_2 = \{(\mathbf{x}, t) : x_1 > 0, \ldots, x_n > 0, \ t > 0\}$ are expressed through some confluent hypergeometric function of n variables with the order p.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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